Martingales and the Spine decomposition

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Contents

Galton Watson with immigration

Consider a discrete time Galton Watson process with immigration $(Z_n, n \geq 0)$. Assume that in addition to the law of a random integer ξ (offspring distribution) with generating function f, we are also given the law of a random integer ζ (immigration law) with generating function g.
The dynamics of the BGW model with immigration is given by the following rules:

- generation $n+1$ is made up of the offspring of individuals from generation n and of a random number ζ_{n+1} of immigrants, where the $(\zeta_i, i \geq 1)$ are independent and all distributed as ζ,
- conditional on Z_n, for any $1 \leq i \leq Z_n$, individual i from generation n begets a number ξ_i of offspring,
- the ξ_i are independent and all distributed as ξ.

It is important to remember that to each immigrant is given an independent GW descendant population with the same offspring distribution.

From the previous description we have that

$$\mathbb{E}_z\left(s^{Z_1}\right) = g(s)\left(f(s)\right)^z, \qquad z \in \mathbb{N}.$$

Iterating this expression, we get

$$\begin{aligned}\mathbb{E}_z\left(s^{Z_n}\right) &= (f_n(s))^z \prod_{j=1}^{n} g(f_{n-j}(s)) \\ &= (f_n(s))^z \prod_{k=0}^{n-1} g(f_k(s)), \qquad s \in [0,1], z \in \mathbb{N}, n \geq 1;\end{aligned} \tag{1}$$

where as usual f_n denotes the n-composition of f with itself.
The population at time n is formed by the descendants of the original z individuals after n generations, then the immigrants arriving at time $1 \leq k \leq n-1$, during $n-k$ generations, form families evolving as the original individuals.

Asymptotic behavior of the BGI

Theorem

The Galton Watson process with immigration has the following behavior

- **[Heathcote]** *If the mean* $0 < m = \mathbb{E}(\xi) < 1$, *then we have the dichotomy:*
  - $\mathbb{E}(\log^+(\zeta)) < \infty$, *then* Z_n *converges weakly as* $n \to \infty$.
  - $\mathbb{E}(\log^+(\zeta)) = \infty$, *then* Z_n *tends to* ∞ *as* $n \to \infty$, *in probability*
- **[Seneta]** *If the mean* $m = \mathbb{E}(\xi) > 1$, *then we have the dichotomy:*
  - $\mathbb{E}(\log^+(\zeta)) < \infty$, *then* $\lim_{n\to\infty} \frac{Z_n}{m^n}$ *exists and it is finite a.s.*
  - $\mathbb{E}(\log^+(\zeta)) = \infty$, *then* $\limsup_{n\to\infty} \frac{Z_n}{c^n} = \infty$ *for any positive constant* $c > 0$ *a.s.*

Take a GWB process $(Z_n, n \geq 0)$ with no-immigration, and branching generating function f. We know that if $m = \mathbb{E}(\xi_1) \in (0, \infty)$, then the process

$$W_n = \frac{Z_n}{m^n}, \qquad n \geq 0,$$

is a positive martingale. So, it is convergent a.s.
We define a new probability measure $\mathbb{P}^{\uparrow}$ as the Doob h-transform of $\mathbb{P}$ with density W, i.e.

$$\mathbb{E}_z^{\uparrow}(F(Z_0, \ldots, Z_n)) = \mathbb{E}_z\left(F(Z_0, \ldots, Z_n)\frac{W_n}{z}\right), \qquad n \geq 0.$$

The evolution under $\mathbb{P}^{\uparrow}$

Theorem

Under $\mathbb{P}^{\uparrow}$, the process $(Z_n - 1, n \geq 0)$ is a Branching Galton Watson process with immigration, BGWI, with branching mechanism determined by the generating function f, and immigration mechanism given by

$$\mathbb{P}(\zeta^{\uparrow} = k) = \frac{k\mathbb{P}(\zeta = k)}{m}, \qquad k \geq 1,$$

$$g(s) = \frac{1}{m} f'(s), \qquad s \in [0, 1].$$

$$\mathbb{E}_z^{\uparrow}(s^{Z_n - 1}) = (f_n(s))^{z-1} \prod_{k=0}^{n-1} \left[\frac{1}{m} f'\left(f_k(s)\right) \right]$$

By induction, it is proved that

$$\mathbb{E}_z^{\uparrow}(s^{Z_n - 1}) = \mathbb{E}_z \left(\frac{Z_n}{zm^n} s^{Z_n - 1} \right) = (f_n(s))^{z-1} \frac{f_n'(s)}{m^n}, \qquad n \geq 1.$$

Spine decomposition under $\mathbb{P}^{\uparrow}$

- Start with a initial particle v_0, give it a random number $\zeta^{\uparrow}$ of children with size biased distribution

$$\mathbb{P}(\zeta^{\uparrow} = k) = \frac{k\mathbb{P}(\zeta = k)}{m},$$

  $k \geq 1$;
- Pick one of these children at random, v_1;
- Give to the other children independent populations with branching mechanism f, and to the particle v_1 give a random number of children with distribution $\zeta^{\uparrow}$;
- Again pick at random an individual, v_2, give to the other individuals independent populations with branching mechanism f, and to v_2 give a size biased number of children;
- Continue

This algorithm gives a population with a tagged particle, labeled $(v_0, v_1, \ldots, v_n, \ldots)$, which is inmortal. This tagged particle is the so-called spine. The law of the total population is $\mathbb{P}^{\uparrow}$.

Spine decomposition

R. LYONS, R. PEMANTLE AND Y. PERES

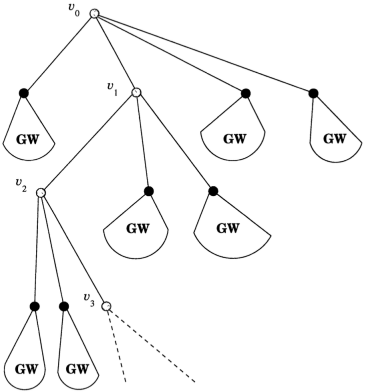

FIG. 1. *Schematic representation of size-biased Galton–Watson trees.*

If $m \leq 1$, the genealogy of the siblings of the spine gets extinct eventually a.s.

Unified approach of Pemantle et al. (1995)

THEOREM A (Supercritical processes [Kesten and Stigum (1966)]). *Suppose that $1 < m < \infty$ and let W be the limit of the martingale Z_n/m^n. The following are equivalent*:

(i) $\mathbf{P}[W = 0] = q$,
(ii) $\mathbf{E}[W] = 1$,
(iii) $\mathbf{E}[L \log^+ L] < \infty$.

THEOREM B (Subcritical processes [Heathcote, Seneta and Vere-Jones (1967)]). *The sequence $\{\mathbf{P}[Z_n > 0]/m^n\}$ is decreasing. If $m < 1$, then the following are equivalent*:

(i) $\lim_{n\to\infty} \mathbf{P}[Z_n > 0]/m^n > 0$,
(ii) $\sup \mathbf{E}[Z_n | Z_n > 0] < \infty$,
(iii) $\mathbf{E}[L \log^+ L] < \infty$.

Unified approach of Pemantle et al. (1995)

THEOREM C (Critical processes [Kesten, Ney and Spitzer (1966)]). *Suppose that $m = 1$ and let $\sigma^2 := \mathrm{Var}(L) = \mathbf{E}[L^2] - 1 \leq \infty$. Then we have:*

(i) *Kolmogorov's estimate*:

$$\lim_{n \to \infty} n\mathbf{P}[Z_n > 0] = \frac{2}{\sigma^2}.$$

(ii) *Yaglom's limit law: If $\sigma < \infty$, then the conditional distribution of Z_n/n given $Z_n > 0$ converges as $n \to \infty$ to an exponential law with mean $\sigma^2/2$. If $\sigma = \infty$, then this conditional distribution converges to infinity.*

Theorem

Let $\mathbb{P}^{\uparrow}$ as the Doob h-transform of $\mathbb{P}$ with density W, i.e.

$$\mathbb{E}_z^{\uparrow}\left(F(Z_0,\ldots,Z_n)\right) = \mathbb{E}_z\left(F(Z_0,\ldots,Z_n)\frac{W_n}{z}\right), \qquad n \geq 0.$$

Assume $m \leq 1$. We have, for any F continuous and bounded

$$\mathbb{E}_z^{\uparrow}\left(F(Z_0,\ldots,Z_n)\right) = \lim_{k\to\infty} \mathbb{E}_z^{\uparrow}\left(F(Z_0,\ldots,Z_n) | Z_{n+k} \neq 0\right).$$

- *If $m < 1$, the process $(Z_n^{\uparrow}, n \geq 0)$, is positive recurrent if and only if $\mathbb{E}(\zeta \log^{+}(\zeta)) < \infty$, with ζ the size of a typical family. In this case, the process has as invariant distribution the size biased version of the limit law of $Z_n | Z_n \neq 0$ as $n \to \infty$.*
- *If $m = 1$, $(Z_n^{\uparrow}, n \geq 0)$, is transient. If the variance, σ^2, of ζ is finite, then*

$$lim_{n\to\infty}\mathbb{P}_1^{\uparrow}\left(2Z_n/\sigma > x\right) = \int_x^{\infty} ze^{-z}dz, \qquad x \geq 0.$$

The basic facts from Galton Watson processes were taken from the survey by Amaury Lambert

- POPULATION DYNAMICS AND RANDOM GENEALOGIES
  Stochastic Models, 24:45–163, 2008
  DOI: 10.1080/15326340802437728

Contents

Recap

- Particles move in E according to a Markov process $(\xi, \mathbf{P}_x)$. The associated semigroup is
$$\mathtt{P}_t[f](x) = \mathbf{E}_x[f(\xi_t)\mathbb{1}_{(t<\zeta)}]$$

- When at $x \in E$, at rate $\gamma(x)$, the particle is killed and sent to the cemetery state $\dagger \notin E$.

- At this point, new particles are created according to the point process $(\mathcal{Z}, \mathcal{P}_x)$, where
$$\mathcal{Z} = \sum_{i=1}^{N} \delta_{x_i}.$$

For convenience, we define $m[f](x) = \mathcal{E}_x\left[\sum_{i=1}^{N} f(x_i)\right]$.

Recap

- The branching process is defined as

$$X_t := \sum_{i=1}^{N_t} \delta_{x_i(t)}.$$

- The law of $(X_t)_{t\geq 0}$ is characterised via the non-linear semigroup

$$v_t[g](x) := \mathbb{E}_{\delta_x}\left[\mathrm{e}^{-X_t[g]}\right],$$

where

$$X_t[g] = \int_E g(y) X_t(\mathrm{d}y) = \sum_{i=1}^{N_t} g(x_i(t)).$$

Recap

We are also interested in the mean (linear) semigroup

$$\psi_t[g](x) := \mathbb{E}_{\delta_x}\left[X_t[g]\right].$$

Many-to-one lemma

There exists a Markov process $(\hat{\xi}, \hat{\mathbf{P}})$ taking values in $E \cup \{\dagger\}$ such that

$$\psi_t[g](x) = \hat{\mathbf{E}}_x\left[\mathrm{e}^{\int_0^t B(\hat{\xi}_s)\mathrm{d}s} g(\hat{\xi}_t)\mathbf{1}_{(t<\tau)}\right],$$

where $B(x) = \gamma(x)(m[1](x) - 1)$.

- Recall that $m[f](x) = \mathcal{E}_x[\mathcal{Z}[f]]$.

- Recall also that $(\hat{\xi}_t)_{t\geq 0}$ evolves according to ξ and at rate $\gamma(x)m[1](x)$ jumps to a new location in $A \subset E$ with probability $m[\mathbf{1}_A](x)/m[1](x)$.

- The quantity $B(x) = \gamma(x)(m[1](x) - 1)$ "keeps track" of the mass in the branching process, i.e.

$$\mathbb{E}_{\delta_x}[N_t] = \hat{\mathbf{E}}_x\left[\mathrm{e}^{\int_0^t B(\hat{\xi}_s)\mathrm{d}s}\mathbf{1}_{t<\tau}\right].$$

- If $\sup_{x\in E} B(x) < 0$, then we can interpret $|B|$ as a killing rate:

$$\hat{\mathbf{P}}_x(t < T|\sigma(\hat{\xi}_s, s \leq t)) = \mathrm{e}^{-\int_0^t |B(\xi_s)|\mathrm{d}s}.$$

Contents

Stability

Last week aim: to provide sufficient conditions for

- $\lambda_* \in \mathbb{R}$,
- a positive function $\varphi \in B^+(E)$,
- a probability measure η on E

such that

$$\psi_t[\varphi] = \mathrm{e}^{\lambda_* t}\varphi, \quad \eta[\psi_t[g]] = \mathrm{e}^{\lambda_* t}\eta[g],$$

and

$$\psi_t[g](x) \sim \mathrm{e}^{\lambda_* t}\varphi(x)\eta[g], \qquad \text{as } t \to \infty.$$

Stability

$$\psi_t[g](x) \sim \mathrm{e}^{\lambda_* t}\varphi(x)\eta[g], \qquad \text{as } t \to \infty.$$

- Subcritical: if $\lambda_* < 0$, the average mass decays at rate $-\lambda_*$.
- Critical: if $\lambda_* = 0$, the average mass remains constant.
- Supercritical: if $\lambda_* > 0$, the average mass in the system grows at rate λ_*.

Example: branching Markov chains

- Consider the case where $(\xi_t, t \geq 0)$ is a continuous time Markov chain on $E = \{1, \dots, n\}$ with transition matrix $(P_{i,j}(t))_{i,j \in E}$

- At rate γ, particles produce two offspring locally.

- What is the long-term average behaviour of the branching process?

- The key to answering this is the many-to-one:

$$\psi_t[g](i) = \mathrm{e}^{\gamma t} \mathbf{E}_i[g(\xi_t)]$$

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Example: branching Markov chains

Perron Frobenius theorem

Let A be a non-negative, irreducible square matrix. Then the following hold.

- There is a simple positive real eigenvalue λ and such that all other eigenvalues have absolute value less than or equal to λ.
- The (unique up to scaling) left- and right-eigenvectors, φ and η resp., corresponding to λ are positive.
- $\lim_{n\to\infty} A^n/\lambda^n = \varphi\eta^T$ where the left and right eigenvectors for A are normalized so that $\eta^T\varphi = 1$.

Example: branching Markov chains

Assuming the chain is irreducible, Perron Frobenius theory tells us that there exist $\lambda_c \leq 0$ and vectors φ, η such that

$$P(t)\varphi = \mathrm{e}^{\lambda_c t}\varphi, \quad \eta^T P(t) = \mathrm{e}^{\lambda_c t}\eta^T,$$

and

$$P_{i,j}(t) \sim \mathrm{e}^{\lambda_c t}\varphi(i)\eta(j) + o(\mathrm{e}^{\lambda_c t}), \quad t \to \infty.$$

Example: branching Markov chains

Using the fact that $\psi_t = \mathrm{e}^{\gamma t} P(t)$, we have

- $P(t)\varphi = \mathrm{e}^{\lambda_c t}\varphi \quad \Longrightarrow \quad \psi_t[\varphi] = \mathrm{e}^{(\gamma+\lambda_c)t}\varphi$;
- $\eta^T P(t) = \mathrm{e}^{\lambda_c t}\eta^T \quad \Longrightarrow \quad \eta^T \psi_t = \mathrm{e}^{(\gamma+\lambda_c)t}\eta^T$;
- $P_{i,j}(t) \sim \mathrm{e}^{\lambda_c t}\varphi(i)\eta(j) \quad \Longrightarrow \quad \psi_t[g](i) \sim \mathrm{e}^{(\gamma+\lambda_c)t}\varphi(i)\eta^T g$.

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Stability in the general case

- Again, the key will be the many-to-one formula:

$$\psi_t[f](x) = \hat{\mathbf{E}}_x \left[\mathrm{e}^{\int_0^t B(\hat{\xi}_s)\mathrm{d}s} g(\hat{\xi}_t) \mathbf{1}_{t<\tau} \right].$$

- Since $\gamma, m[1] \in B^+(E)$, it follows that $\bar{B} := \sup_{x\in E} B(x) < \infty$.

- Hence, we may define

$$\psi_t^\dagger[f](x) := \mathrm{e}^{-\bar{B}t}\psi_t[f](x) = \hat{\mathbf{E}}_x \left[\mathrm{e}^{\int_0^t (B(\hat{\xi}_s)-\bar{B})\mathrm{d}s} g(\hat{\xi}_t) \mathbf{1}_{t<\tau} \right]$$
$$=: \hat{\mathbf{E}}_x \left[g(\hat{\xi}_t) \mathbf{1}_{t<\kappa} \right]$$

κ is the random time

$$\mathbb{P}(\kappa > t | \widehat{\xi}) = \exp - \int_0^t (\bar{B} - B(\hat{\xi}_s))\mathrm{d}s.$$

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QSDs

- Let $(Y_t)_{t\geq 0}$ be a time-homogeneous Markov process on $E \cup \{\dagger\}$ with probabilities $(\mathbf{P}^{\dagger}_x, x \in E)$ and semigroup $(\psi^{\dagger}_t)_{t\geq 0}$.
- Assume that $\kappa := \inf\{t > 0 : Y_t = \dagger\} < \infty$, $\mathbf{P}^{\dagger}_x$-almost surely for all $x \in E$.
- Assume further that for all $x \in E$, $\mathbf{P}^{\dagger}_x(t < \kappa) > 0$.

QSDs

Definition

A **limit quasi-stationary distribution** (QSD) is a probability measure η on E such that

$$\eta = \lim_{t\to\infty} \mathbf{P}^{\dagger}_{\mu}(Y_t \in \cdot | t < \kappa)$$

for some initial probability measure μ on E.

QSDs

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Proposition

A probability measure η is a QSD if and only if, for any $t \geq 0$,

$$\eta = \mathbf{P}^{\dagger}_{\eta}(Y_t \in \cdot | t < \kappa).$$

QSDs

Assumption A

There exists a probability measure ν on E such that

(A1) there exists $t_0, c_1 > 0$ such that for all $x \in E$,

$$\mathbf{P}_x^{\dagger}(Y_{t_0} \in \cdot | t_0 < \kappa) \geq c_1 \nu(\cdot);$$

(A2) there exists $c_2 > 0$ such that for all $x \in E$ and $t \geq 0$,

$$\mathbf{P}_{\nu}^{\dagger}(t < \kappa) \geq c_2 \mathbf{P}_x^{\dagger}(t < \kappa).$$

QSDs

Theorem (Champagnat, Villemonais)

Under Assumption A, there exists a constant $\lambda_c < 0$, a function $\varphi \in B^+(E)$ and a probability measure η on E such that

$$\psi_t^\dagger[\varphi] = \mathrm{e}^{\lambda_c t}\varphi, \qquad \eta[\psi_t^\dagger[g]] = \mathrm{e}^{\lambda_c t}\eta[g].$$

Moreover, there exist constants $C, \varepsilon > 0$ such that

$$\sup_{x\in E, g\in B_1^+(E)} |\mathrm{e}^{-\lambda_c t}\varphi(x)^{-1}\psi_t^\dagger[g] - \eta[g]| \leq C\mathrm{e}^{-\epsilon t}.$$

QSDs

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Since $\psi_t = \mathrm{e}^{\bar{B}t}\psi_t^\dagger$, the same conclusion then holds for ψ_t with λ_c replaced by $\lambda_* = \lambda_c + \bar{B}$.

Contents

Spine decomposition

- The branching property and the fact that

$$\mathbb{E}_{\delta_x}[X_t[\varphi]] = \mathrm{e}^{\lambda_* t}\varphi(x),$$

imply that

$$W_t^1 := \mathrm{e}^{-\lambda_* t}\frac{X_t[\varphi]}{\varphi(x)}, \quad t \geq 0,$$

is a unit mean $\mathbb{P}_{\delta_x}$-martingale.

- Thus, we can define the change of measure

$$\left.\frac{\mathbb{P}^{\varphi}_{\delta_x}}{\mathbb{P}_{\delta_x}}\right|_{\mathcal{F}_t} := W_t^1, \quad t \geq 0, x \in E,$$

i.e. $\mathbb{P}^{\varphi}_{\delta_x}(A) = \mathbb{E}_{\delta_x}[\mathbf{1}_A W_t^1]$.

Spine decomposition

Under $\mathbb{P}^{\varphi}$, the branching process X can be constructed as follows.

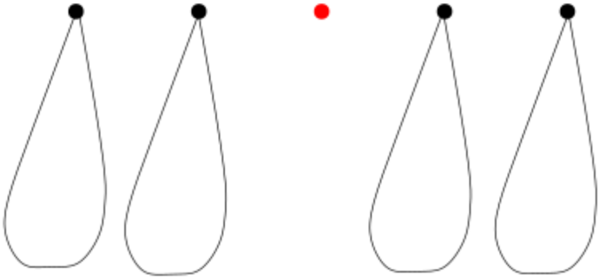
Spine decomposition

Under $\mathbb{P}^{\varphi}$, the branching process X can be constructed as follows.

1. From the initial configuration $\mu = \sum_{i=1}^{n} \delta_{x_i}$, the i^*-th individual is selected with probability $\varphi(x_{i^*})/\mu[\varphi]$ and marked the *spine*.

Spine decomposition

2. The individuals $j \neq i^*$ in the initial configuration each issue independent copies of $(X, \mathbb{P}_{\delta_{x_j}})$ respectively.



Spine decomposition

3. The marked individual, "spine", issues a single particle whose motion is determined by the semigroup

$$\mathtt{S}_t[f](x) := \mathbf{E}_x\left[\mathrm{e}^{\int_0^t B(\xi_s)\left(\frac{m[\varphi(\hat{\xi}_s)]}{\varphi(\hat{\xi}_s)}-1\right)\mathrm{d}s} \frac{\varphi(\xi_t)}{\varphi(x)} f(\xi_t)\right] \qquad x \in E,\ f \in B^+(E).$$

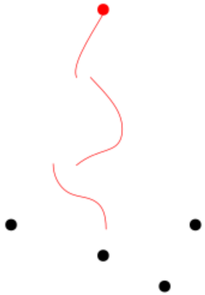
Spine decomposition

4. When at $x \in E$, the spine undergoes branching at rate

$$\rho(x) := B(x)\frac{m[\varphi](x)}{\varphi(x)}$$

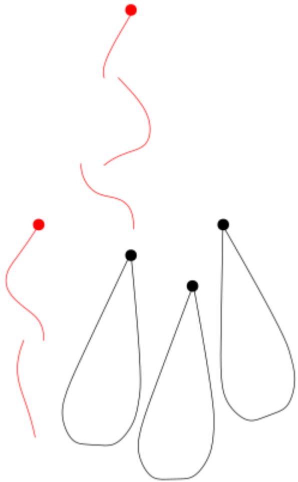
at which point, it produces particles according $(\mathcal{Z}, \mathcal{P}_x^\varphi)$, φ-size biasing, where

$$\frac{\mathrm{d}\mathcal{P}_x^\varphi}{\mathrm{d}\mathcal{P}_x} = \frac{\mathcal{Z}[\varphi]}{m[\varphi](x)}.$$



Spine decomposition

5. Given $\mathcal{Z}$ from the previous step, μ is redefined as $\mu = \mathcal{Z}$ and Step 1 is repeated.



Spine decomposition

- From the many-to-one lemma,

$$\mathbb{E}_{\delta_x}[X_t[\varphi]] = \hat{\mathbf{E}}_x \left[\mathrm{e}^{\int_0^t \gamma(\hat{\xi}_s)\mathrm{d}s} \varphi(\hat{\xi}_t) \right] = \mathrm{e}^{\lambda_* t} \varphi(x).$$

- It follows that

$$W_t^2 := \mathrm{e}^{-\lambda_* t + \int_0^t \gamma(\hat{\xi}_s)\mathrm{d}s} \frac{\varphi(\hat{\xi}_t)}{\varphi(x)}, \quad t \geq 0.$$

is a unit mean $\hat{\mathbf{P}}_x$-martingale.

- Thus, we can define a second change of measure

$$\left. \frac{\mathrm{d}\mathbf{P}_x^{\varphi}}{\mathrm{d}\hat{\mathbf{P}}_x} \right|_{\mathcal{G}_t} := W_t^2, \quad t \geq 0, x \in E.$$

Spine decomposition

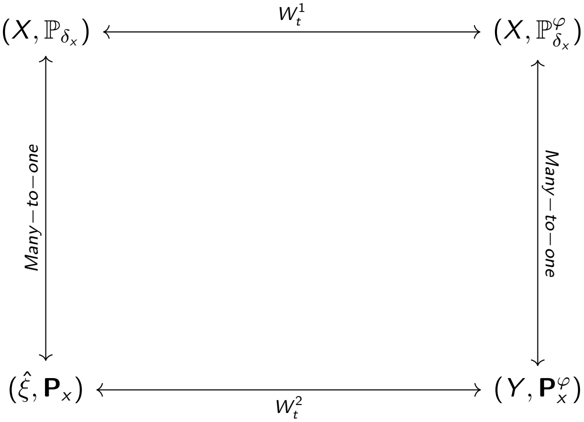
Ergodicity of the spine

The spine process is equal in law to $(\hat{\xi}, \mathbf{P}^{\varphi})$. The semigroup $(\mathrm{P}^{\varphi}_t,\ t \geq 0)$ associated to $(\hat{\xi}, \mathbf{P}^{\varphi})$ is conservative, and satisfies

$$\mathrm{P}^{\varphi}_t[f](x) = \frac{\mathrm{e}^{-\lambda_* t}}{\varphi(x)} \psi_t[\varphi f], \qquad t \geq 0,\ f \in B^+(E),$$

with stationary distribution

$$\varphi(x)\eta(\mathrm{d}x), \qquad x \in E.$$


$(X, \mathbb{P}_{\delta_x})$
W_t^1
$(X, \mathbb{P}_{\delta_x}^{\varphi})$
Many—to—one
Many—to—one
$(\hat{\xi}, \mathbf{P}_x)$
W_t^2
$(Y, \mathbf{P}_x^{\varphi})$

The limit martingale

Theorem

Assume (A) and that for some $k \geq 2$, $\sup_{x \in E} \mathcal{E}_x[\mathcal{Z}[1]^k] < \infty$. *We have the following*

- *If* $\lambda^* > 0$, *then* W *is* L_2*-convergent (and hence has a non-trivial limit);*
- *If* $\lambda^* < 0$, *then* $W_\infty = 0$ *almost surely;*
- *If* $\lambda^* = 0$, *and for all* t *large enough,* $x \in E$, $\mathbf{P}^\dagger_x(t < \kappa) < 1$, *then* $W_\infty = 0$ *almost surely.*

Let $\zeta = \inf\{t > 0 : X_t[1] = 0\}$. *We have that* $\zeta < \infty$ *a.s. if and only if* $\lambda^* \leq 0$.

Thank you!

¡Muchas gracias!

Merci beaucoup!