Limit theorems for critical branching processes

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Contents

Recap

- The branching process is defined as

$$X_t := \sum_{i=1}^{N_t} \delta_{x_i(t)}.$$

- The law of $(X_t)_{t\geq 0}$ is characterised via the non-linear semigroup

$$v_t[g](x) := \mathbb{E}_{\delta_x}\Big[\prod_{i=1}^{N_t} g(x_i(t))\Big],$$

where

$$X_t[g] = \int_E g(y)X_t(\mathrm{d}y) = \sum_{i=1}^{N_t} g(x_i(t)).$$

- The mean semigroup is given by

$$\psi_t[g](x) := \mathbb{E}_{\delta_x}\left[X_t[g]\right].$$

Recap

- Recall the Perron Frobenius asymptotic

$$\psi_t[g](x) \sim \mathrm{e}^{\lambda_* t}\varphi(x)\eta[g], \qquad t \to \infty.$$

- When $\lambda_* = 0$, the expected population size remains constant but we have extinction almost surely.

- What happens to X_t if we condition on survival?

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Yaglom limit for BGW processes

- Suppose $(Z_n)_{n\geq 0}$ is a BGW process,

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i, \qquad \xi_i \sim^{\text{iid}} \xi.$$

- Assume $\mathbb{E}[\xi] = 1$ so that the process is critical.
- Further assume that $\sigma^2 := \mathbb{E}[\xi^2] - \mathbb{E}[\xi] < \infty$.

- Kolmogorow limit (Kolmogorov '38):

$$\lim_{n\to\infty} n\mathbb{P}(Z_n > 0) = \frac{2}{\sigma^2}$$

- Yaglom limit (Yaglom '48):

$$\lim_{n\to\infty} \mathbb{E}\left[\exp\left(-\theta\frac{Z_n}{n}\right) \Big| Z_n > 0\right] = \frac{1}{1+\theta\sigma^2/2}.$$

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Yaglom limit for BBM on a compact domain

- Let $D \subset \mathbb{R}^d$ be compact.
- Let $(X_t)_{t \geq 0}$ denote a branching process where, given their point of creation, particles move independently according to a diffusion with generator L. Particles are killed on ∂D and at rate $\beta > 0$, they branch into a random number of particles with distribution A.

- Let λ denote the first eigenvalue of $-L$ on D.
- Assume $m := \mathbb{E}[A] > 1$, $\mathbb{E}[A^2] < \infty$ and $\lambda = \beta(m-1)$.

- Kolmogorov result (Powell '19):

$$\lim_{t \to \infty} t\mathbb{P}_x(N_t > 0) = C_1(x).$$

- Yaglom limit (Powell '19):

$$\lim_{t \to \infty} \mathbb{E}_x \left[\exp\left(-\frac{\theta}{t} \sum_{i=1}^{N_t} f(X_t^i) \right) \middle| N_t > 0 \right] = \frac{1}{1 + \theta C_2(f)}.$$

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General setting

Define

$$\mathcal{V}[g](x) := \mathcal{E}_x\left[\sum_{\substack{i,j=1\\ i\neq j}}^{N} g(y_i)g(y_j)\right], \quad x \in E, f \in B^+(E).$$

and

$$\Sigma = \eta[\beta\mathcal{V}[\varphi]].$$

Theorem (Kolmogorov survival probability)

Under certain assumptions, we have

$$\lim_{t\to\infty}\sup_{x\in E}\left|\frac{t\mathbb{P}_{\delta_x}(N_t > 0)}{\varphi(x)} - \frac{2}{\Sigma}\right| = 0,$$

i.e. $\mathbb{P}_{\delta_x}(N_t > 0) \sim \dfrac{C_1(x)}{t}$.

General setting

Theorem (Yaglom limit)

Under the same assumptions, for each $f \in B^+(E)$,

$$\left(\frac{X_t[f]}{t} \,\middle|\, N_t > 0\right) \to Y, \quad \text{as } t \to \infty,$$

in distribution, where Y is an exponential random variable with mean $\eta[f]\Sigma/2$.

Kolmogorov survival probability

- Note that

$$\mathbb{P}_{\delta_x}(N_t > 0) = 1 - \mathbb{P}_{\delta_x}(N_t = 0) = 1 - \mathtt{v}_t[\mathbf{0}](x) = 1 - \mathbb{E}_{\delta_x}\left[0^{N_t}\right]$$

- Recall that

$$\mathtt{v}_t[f](x) = \hat{\mathtt{P}}_t[f](x) + \int_0^t \mathtt{P}_s\left[\mathtt{G}[\mathtt{v}_{t-s}[f]]\right](x)\mathrm{d}s,$$

where $\hat{\mathtt{P}}_t[f](x) = \mathbb{E}_x[f(\xi_{t\wedge\tau_\partial})]$.

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Kolmogorov survival probability

For $f \in B_1^+(E)$ and $x \in E$, set

$$\mathtt{u}_t[f](x) = 1 - \mathtt{v}_t[f](x), \qquad t \geq 0$$

and

$$\mathtt{A}[f](x) = \gamma(x)\mathcal{E}_x\left[\prod_{i=1}^{N}(1 - f(x_i)) - 1 + \sum_{i=1}^{N} f(x_i)\right].$$

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Lemma

For all $g \in B_1^+(E)$, $x \in E$ and $t \geq 0$, $\mathtt{u}_t[g](x)$ satisfies

$$\mathtt{u}_t[g](x) = \psi_t[1 - g](x) - \int_0^t \psi_s\left[\mathtt{A}[\mathtt{u}_{t-s}[g]]\right](x)\mathrm{d}s.$$

Yaglom limit

Three possible approaches:

- Spine decomposition
- Method of moments
- Laplace transforms/non-linear semigroup

Yaglom limit

- Know that exponential distribution is characterised by its moments.

- Let us consider the moments of $t^{-1}X_t[f]$ under $\mathbb{P}_{\delta_x}(\cdot|N_t > 0)$:

$$\mathbb{E}_{\delta_x}\left[\left(\frac{X_t[f]}{t}\right)^k \middle| N_t > 0\right] = \frac{\frac{1}{t^{k-1}}\mathbb{E}_{\delta_x}[X_t[f]^k \mathbf{1}_{N_t>0}]}{t\mathbb{P}_{\delta_x}(N_t > 0)}$$

- Recall that

$$\mathbb{E}_{\delta_x}[X_t[f]^k] = (-1)^k \frac{\partial^k}{\partial\theta^k}\mathbb{E}_{\delta_x}[\mathrm{e}^{-\theta X_t[f]}]\Big|_{\theta=0}.$$

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Yaglom limit

Recall

$$\mathtt{u}_t[g](x) = \psi_t[1-g](x) - \int_0^t \psi_s\left[\mathtt{A}[\mathtt{u}_{t-s}[g]]\right](x)\mathrm{d}s,$$

so that

$$\mathtt{u}_t[\mathrm{e}^{-\theta g}](x) = \psi_t[1-\mathrm{e}^{-\theta g}](x) - \int_0^t \psi_s\left[\mathtt{A}[\mathtt{u}_{t-s}[\mathrm{e}^{-\theta g}]]\right](x)\mathrm{d}s,$$

where

$$\mathtt{A}[f](x) = \gamma(x)\mathcal{E}_x\left[\prod_{i=1}^N (1-f(x_i)) - 1 + \sum_{i=1}^N f(x_i)\right].$$

Yaglom limit

Differentiating k times with respect to θ and setting $\theta = 0$ gives

$$\psi_t^{(k)}[f](x) = \psi_t[f^k](x) + \int_0^t \psi_s \left[\beta\eta_{t-s}^{(k-1)}[f]\right](x)\,\mathrm{d}s, \qquad t \geq 0, \tag{1}$$

where

$$\eta_{t-s}^{(k-1)}[f](x) = \mathcal{E}_x \left[\sum_{[k_1,\ldots,k_N]_k^2} \binom{k}{k_1,\ldots,k_N} \prod_{j:k_j>0} \psi_{t-s}^{(k_j)}[f](x_j) \right],$$

and $[k_1,\ldots,k_N]_k^2$ is the set of all non-negative N-tuples $(k_1,\ldots,k_N)$ such that $\sum_{i=1}^{N} k_i = k$ and at least two of the k_i are strictly positive.

Yaglom limit

- Proceed by induction. Base case: $\psi_t[f](x) \sim \varphi(x)\eta[f]$.

- Inductive step:

$$\psi_t^{(k+1)}[f](x) = \psi_t[f^{k+1}](x) + \int_0^t \psi_s\left[\beta\eta_{t-s}^{(k)}[f]\right](x)\,\mathrm{d}s.$$

- Recall that from (H1), $\psi_t[f](x) \to \varphi(x)\eta[f]$ so that, for $k \geq 2$,

$$\lim_{t\to\infty} t^{-k}\psi_t[f^{k+1}](x) = 0.$$

Yaglom limit

Hence

$$\lim_{t\to\infty} t^{-k}\psi_t^{(k+1)}[f](x)$$

$$= \lim_{t\to\infty} t^{-k}\int_0^t \psi_s\left[\mathcal{E}_\cdot\left[\sum_{[k_1,\dots,k_N]^2_{k+1}} \binom{k+1}{k_1,\dots,k_N}\prod_{j:k_j>0}\psi_{t-s}^{(k_j)}[f](x_j)\right]\right](x)\mathrm{d}s$$

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$$= \frac{1}{t}\int_0^t \psi_s\left[\mathcal{E}_\cdot\left[\sum_{[k_1,\dots,k_N]_{k+1}^2}\binom{k+1}{k_1,\dots,k_N}\frac{(t-s)^{k+1-\#\{j:k_j>0\}}}{t^{k-1}}\prod_{j:k_j>0}\frac{\psi_{t-s}^{(k_j)}[f](x_j)}{(t-s)^{k_j-1}}\right]\right](x)\mathrm{d}s$$

Yaglom

- The inductive proof yields

$$\psi_t^{(k)} \sim t^{k-1}\varphi(x)k!\eta[f]^k(\Sigma/2)^{k-1}, \qquad t \to \infty.$$

- Then, we have

$$\begin{aligned}\mathbb{E}_{\delta_x}\left[\left(\frac{X_t[f]}{t}\right)^k \middle| N_t > 0\right] &= \frac{\frac{1}{t^{k-1}}\mathbb{E}_{\delta_x}[X_t[f]^k \mathbf{1}_{N_t>0}]}{t\mathbb{P}_{\delta_x}(N_t > 0)} \\ &= \frac{\varphi(x)k!\eta[f]^k(\Sigma/2)^{k-1}}{\varphi(x)(2/\Sigma)} \\ &= k!\eta[f]^k(\Sigma/2)^k.\end{aligned}$$

Why the 2nd moments?

Probabilistic explanation: asymptotically, two children of the MRCA, each with at least 1 descendant alive at time t.

Recall the operator

$$
\begin{aligned}
\mathtt{A}[h](x) &= \gamma(x)\mathcal{E}_x\left[1 - \prod_{i=1}^{N}(1 - h(x_i)) - \sum_{i=1}^{N} h(x_i)\right] \\
&= \gamma(x)\mathcal{E}_x\left[\sum_{i\neq j} h(x_i)h(x_j) - \dots\right] \\
&= \mathcal{V}[h](x) + h.o.t
\end{aligned}
$$

Why the exponential distribution?

- There are asymptotically two children of the MRCA, each with at least 1 descendant alive at time t.
- Distribution of the time of the MRCA of the particles alive at time t is uniform.
- Therefore, under $\mathbb{P}_{\delta_x}(\cdot|N_t > 0)$,

$$\frac{X_t}{t} \approx U\left(\frac{X_{Ut}^{(1)}}{Ut} + \frac{X_{Ut}^{(2)}}{Ut}\right).$$

Contents

Many-to-few

Recall the moment evolution equation:

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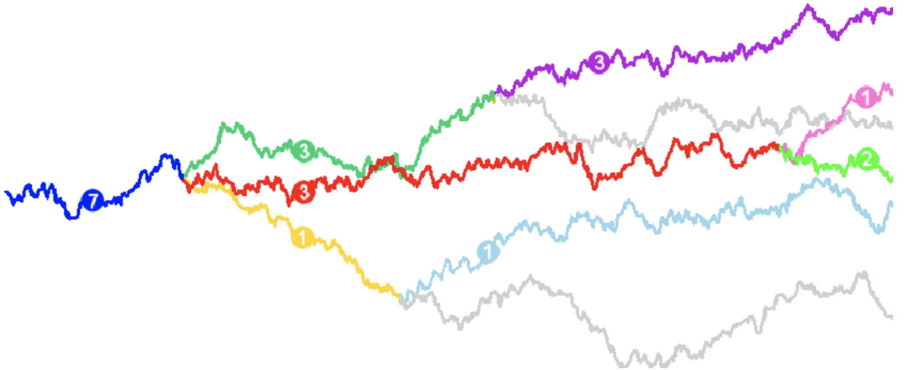

Genealogical structure: ancestral trees

- Let $(X, \mathbb{P})$ denote a Markov branching process.
- Let $T > 0$. On the event $\{N_T \geq k\}$, choose k distinct particles $U_1, \ldots, U_k$ uniformly from those alive at time T.
- What does the ancestral tree formed from these k particles look like?

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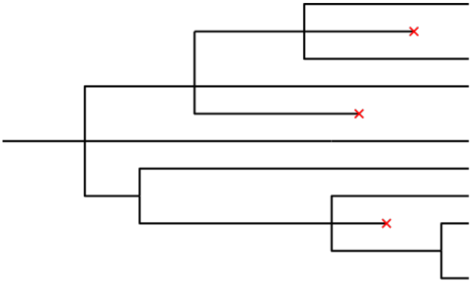
Ancestral trees



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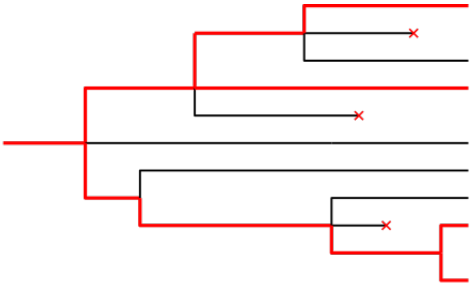

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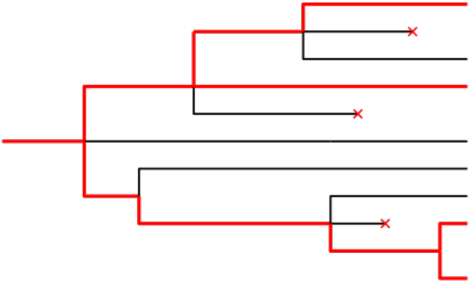

Equivalently, define the equivalence relation $\sim_t$ on $\{1, \ldots, k\}$ by

$$i \sim_t j \quad \Leftrightarrow \quad U_i \text{ and } U_j \text{ share a common ancestor alive at time } t.$$

Let $\pi_t^{k,T}$ denote the random partition of $\{1, \ldots, k\}$ corresponding to this equivalence relation.

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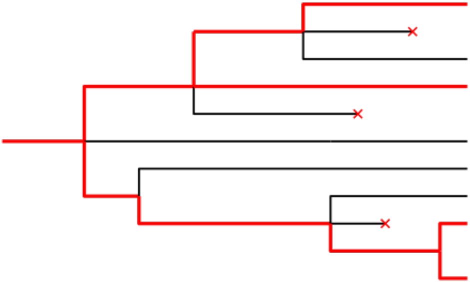

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What is the law of $(\pi_t^{k,T})_{t \geq 0}$ conditional on $N_T \geq k$?

Literature

- O'Connell, The genealogy of branching processes and the age of our most recent common ancestor.
- Lambert, Coalescence times for the branching process.
- Harris & Roberts, The many-to-few lemma and multiple spines.
- Harris, Johnston & Roberts, The coalescent structure of continuous-time Galton-Watson trees.
- Harris, Horton, Kyprianou & Powell, Many-to-few for non-local branching Markov process.
- Johnston, The genealogy of Galton-Watson trees.
- Zubkov, Limiting distributions of the distance to the closest common ancestor.
- Athreya, Boenkost, Durrett, Foutel-Rodier, Le, Palau, Pardo, Schertzer, Schweinsberg, Tourniaire, . . .

Genealogical structure: convergence to the BCRT

- Aim is to look at the scaling limit of the continuous planar tree associated with a MBP.
- Ulam Harris notation:

$$\Omega = \bigcup_{n=0}^{\infty} \mathbb{N}^n.$$

- The label $\emptyset$ denotes the initial ancestor.
- Labels are of the form $u = \emptyset u_1 u_2 \ldots u_n$, e.g. label $\emptyset 215$ means the particle is the 5th child of the 1st child of the 2nd child of the initial ancestor.

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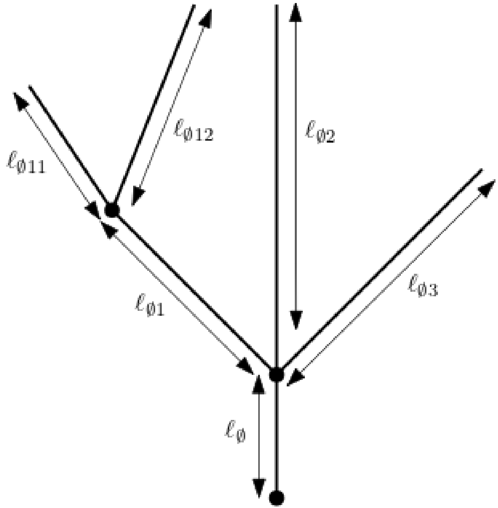
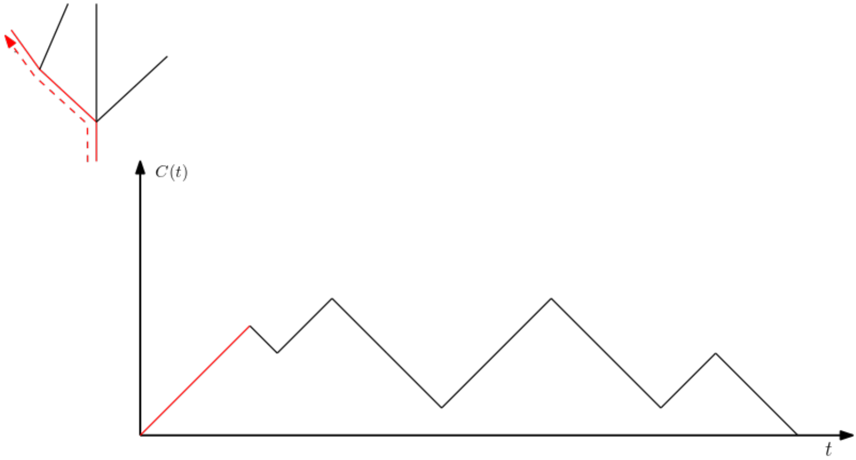


Convergence to the Brownian CRT



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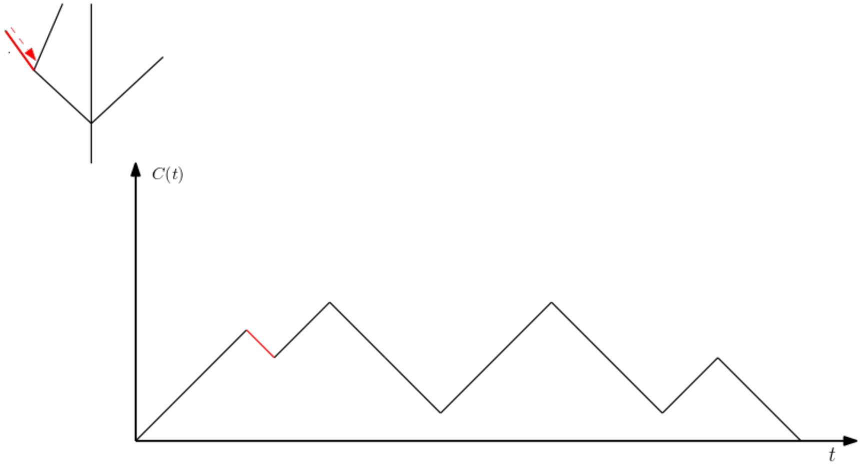

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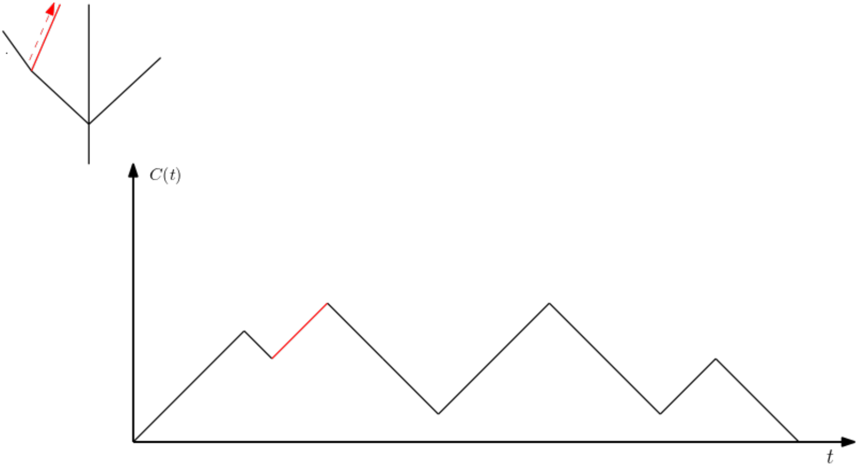

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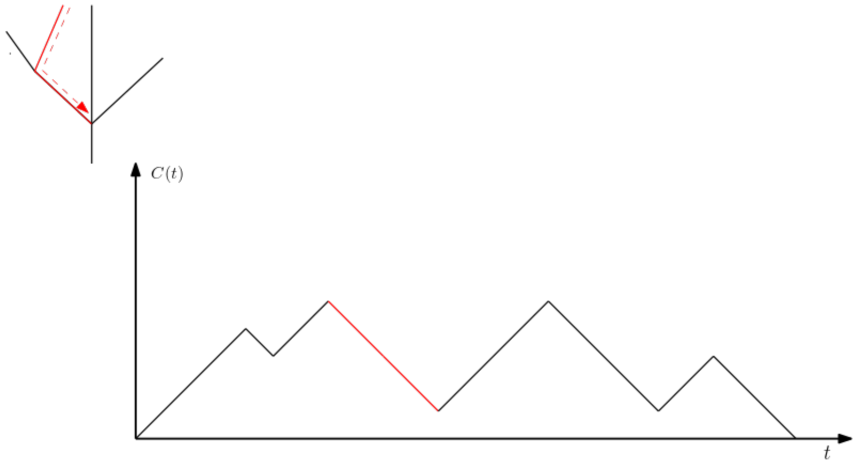

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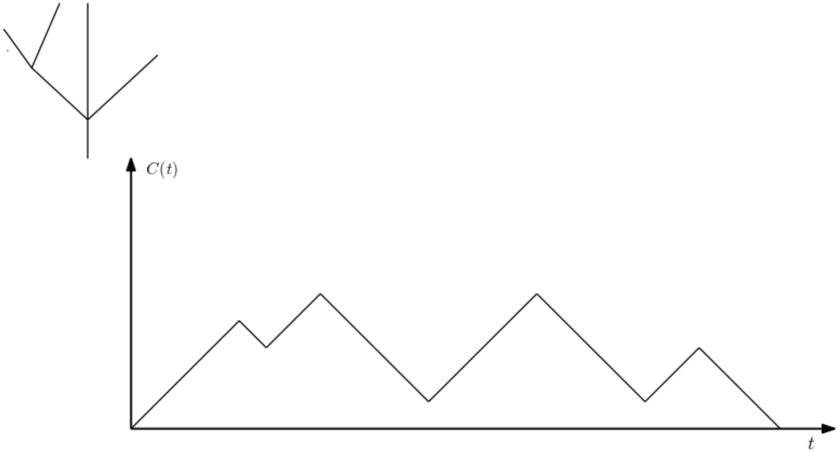
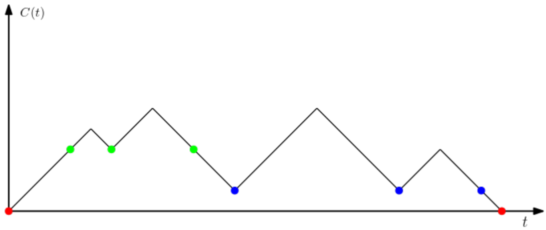
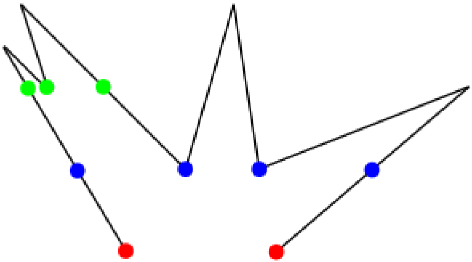


Convergence to the CRT





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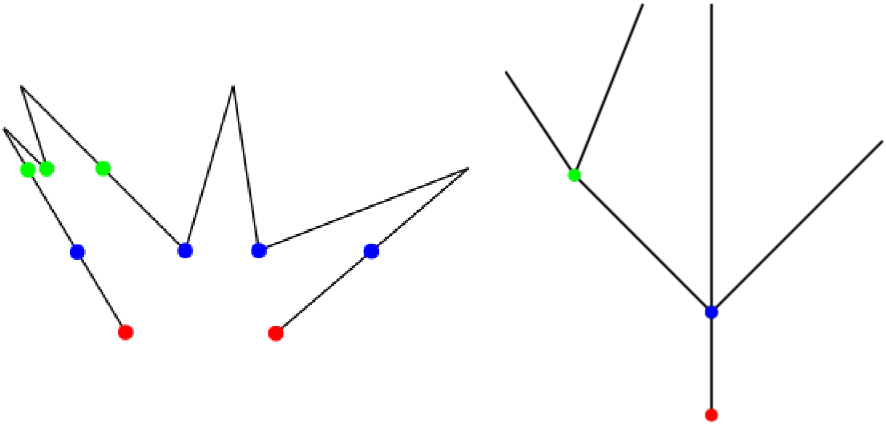
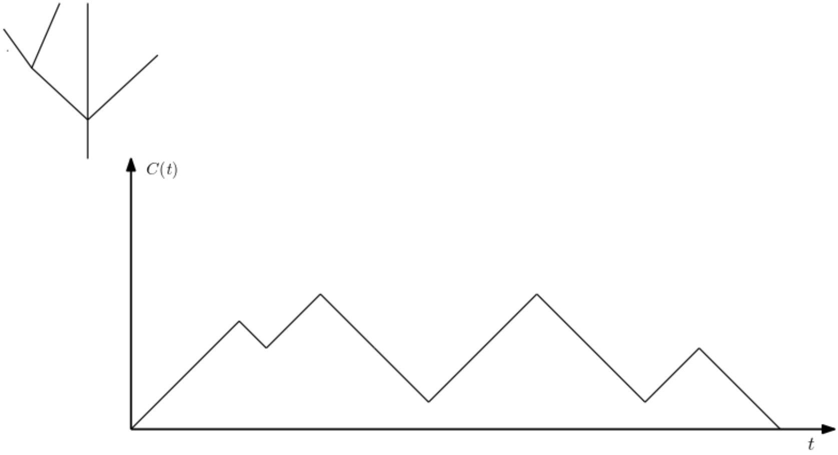
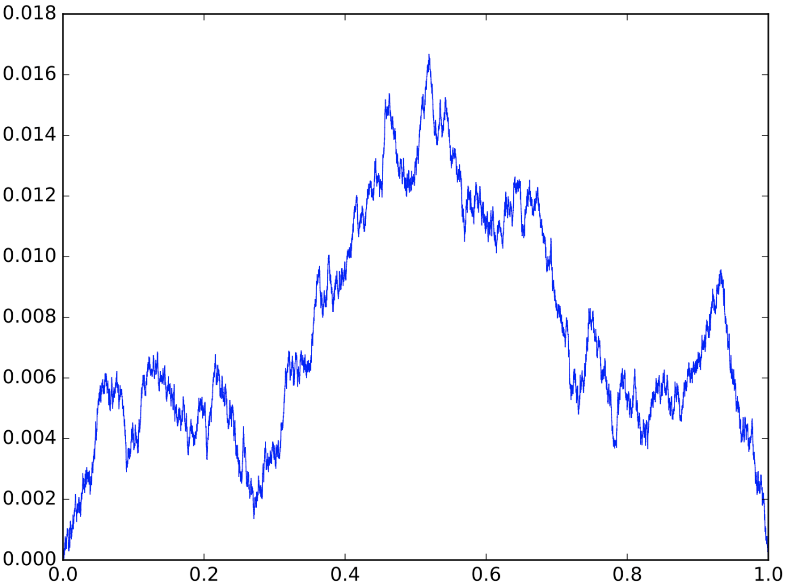


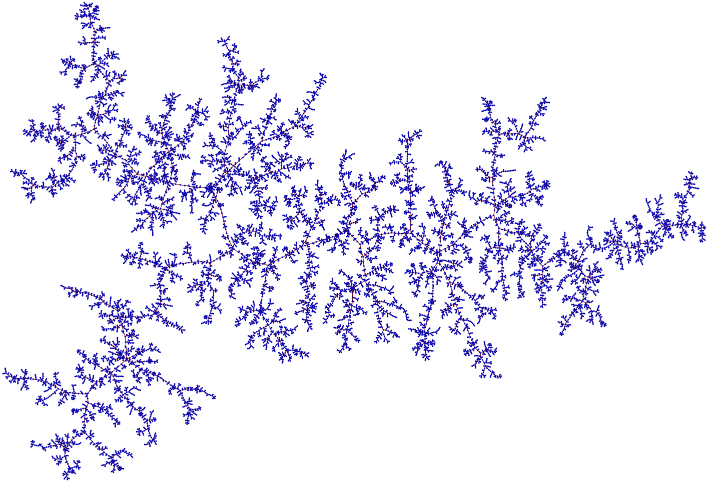
Convergence to the Brownian CRT



Convergence to the Brownian CRT



Convergence to the Brownian CRT



Thank you!