Monte Carlo methods for branching processes

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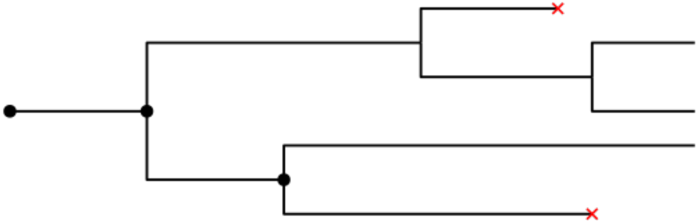
A binary branching process

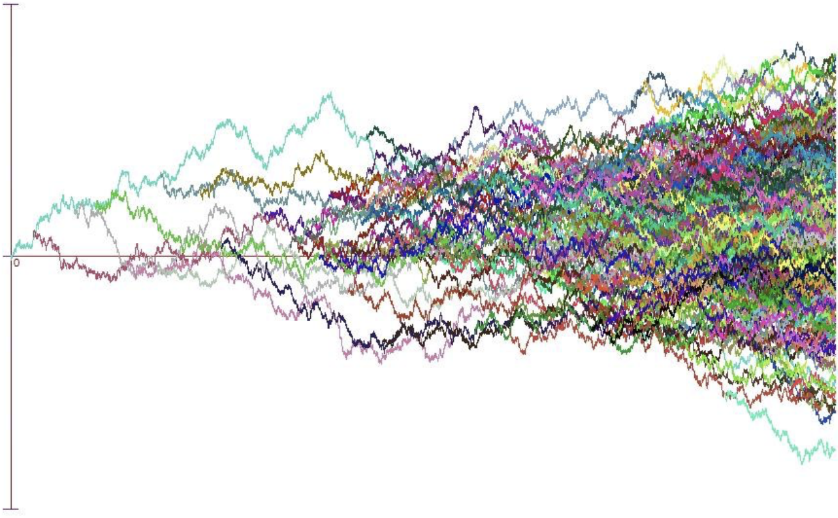
Consider a collection of particles $\{x_i(t) : i = 1, \ldots, N_t\}$ taking values in $E \cup \{\partial\}$ that evolve as follows.

- Particles move in E according to a continuous time Markov process $((\xi_t)_{t\geq 0}, P_x)$.
- When at $y \in E$, at rate $b(y)$ a particle is replaced by two new particles at positions $y_1, y_2 \in E$, i.e. branching occurs.
- When at $y \in E$, at rate $k(y)$ particles are sent to $\{\partial\}$, i.e. soft killing occurs.
- At time T_∂ a particle is sent to $\{\partial\}$, i.e. hard killing occurs.

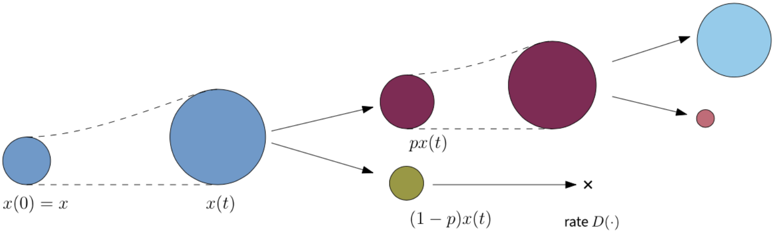
Example: GW process



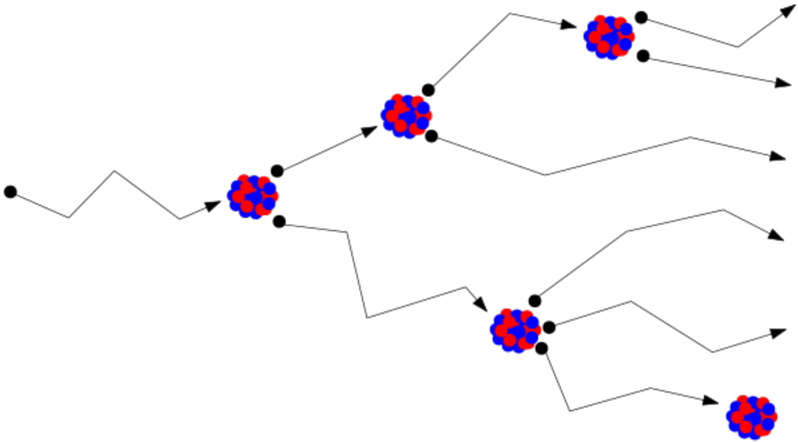
Example: BBM



Example: growth-fragmentation



Example: branching PDMP



A binary branching process

- The branching process is then defined via the atomic measures:

$$X_t := \sum_{i=1}^{N_t} \delta_{x_i(t)}, \qquad t \geq 0.$$

- Define its (linear) expectation semigroup,

$$\psi_t[g](x) := \mathbb{E}_{\delta_x}\left[\langle g, X_t \rangle\right] := \mathbb{E}_{\delta_x}\left[\sum_{i=1}^{N_t} g(x_i(t))\right], \quad t \geq 0, x \in E.$$

Motivation

Particularly interested in branching processes for which there exists

- $\lambda_* \in \mathbb{R}$,
- a positive function $\varphi \in L^\infty_+(E)$,
- a probability measure $\tilde{\varphi}$ on E

such that for all $g \in L^\infty_+(E)$, $x \in E$, we have

$$\psi_t[g](x) \sim \mathrm{e}^{\lambda_* t}\varphi(x)\langle \tilde{\varphi}, g\rangle, \quad t \to \infty.$$

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Aim: find efficient ways to estimate λ_*, φ and $\tilde{\varphi}$ (or other quantities).

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Monte Carlo methods: branching process

Recall the Perron Frobenius asymptotic,

$$\psi_t[g] \sim \mathrm{e}^{\lambda_* t}\langle \tilde{\varphi}, g\rangle \varphi, \quad t \to \infty.$$

Manipulation of this allows us to estimate the eigen-elements, e.g.

$$\lambda_* = \lim_{t\to\infty} \frac{1}{t} \log \psi_t[\mathbf{1}](x) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_{\delta_x}[N_t].$$

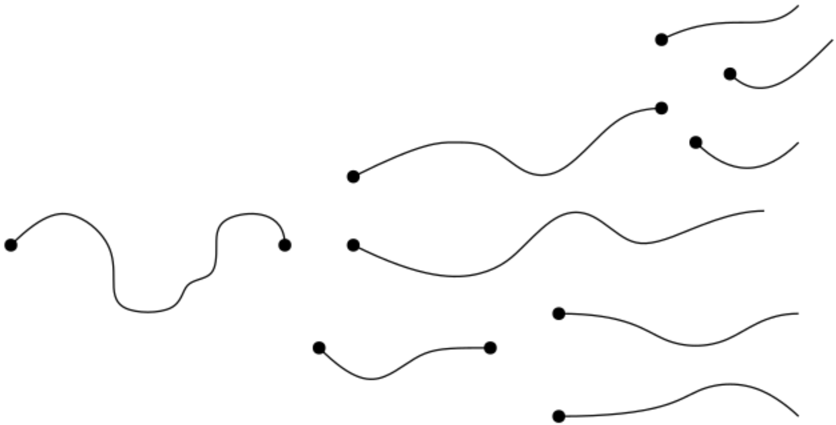
Many-to-one

Many-to-one lemma

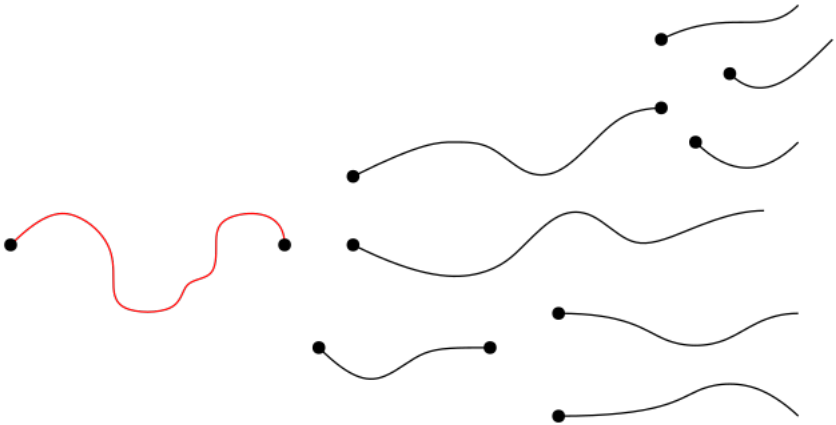
There exists a Markov process Y such that for $x \in E$, $t \geq 0$, $g \in L_+^\infty(E)$, we have

$$\mathbb{E}_{\delta_x}[\langle g, X_t \rangle] = \mathbf{E}_x \left[\mathrm{e}^{\int_0^t b(Y_s) - k(Y_s) \mathrm{d}s} g(Y_t) \mathbf{1}_{t < \tau_\partial} \right], \quad t \geq 0, x \in E.$$

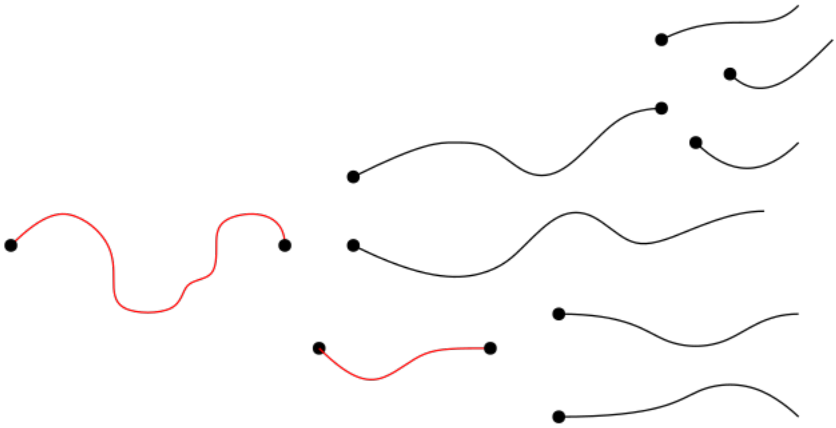
Many-to-one



Many-to-one



Many-to-one



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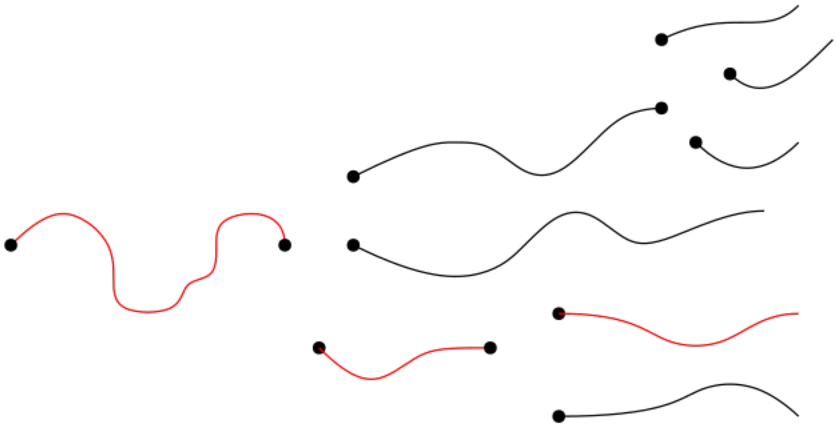


Monte Carlo methods: many-to-one

The many-to-one states that there exists a process $(Y_t)_{t\geq 0}$ on E such that

$$\mathbb{E}_{\delta_x}[\langle g, X_t\rangle] = \mathbf{E}_x\left[\mathrm{e}^{\int_0^t b(Y_s)-k(Y_s)\mathrm{d}s} g(Y_t)\mathbf{1}_{t<\tau_\partial}\right], \quad t \geq 0, x \in E.$$

Thus, we can replace the branching process by the single weighted trajectory, e.g.

$$\lambda_* = \lim_{t\to\infty} \frac{1}{t}\log \psi_t[\mathbf{1}](x) = \lim_{t\to\infty} \frac{1}{t}\log \mathbf{E}_x\left[\mathrm{e}^{\int_0^t b(Y_s)-k(Y_s)\mathrm{d}s}\mathbf{1}_{t<\tau_\partial}\right].$$

Monte Carlo methods: Fleming Viot

Let $(Y, \mathbf{P}^\dagger)$ be a sub-Markov process.

Simulate $N \geq 1$ independent copies of $(Y, \mathbf{P}^\dagger)$ until one of the particles is absorbed.

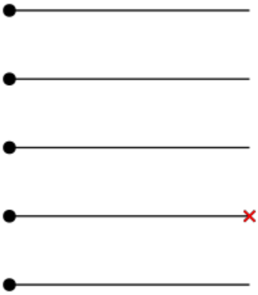


Figure: Fleming Viot particle system

When this happens, duplicate one of the remaining $N-1$ particles and return to the previous step.

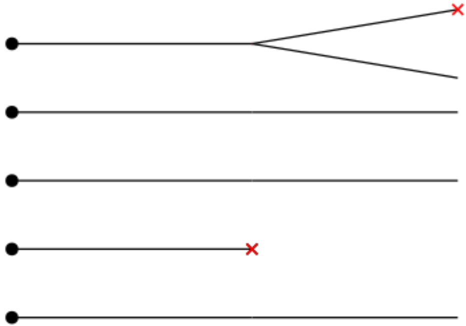


Figure: Fleming Viot particle system

Monte Carlo methods: Fleming Viot

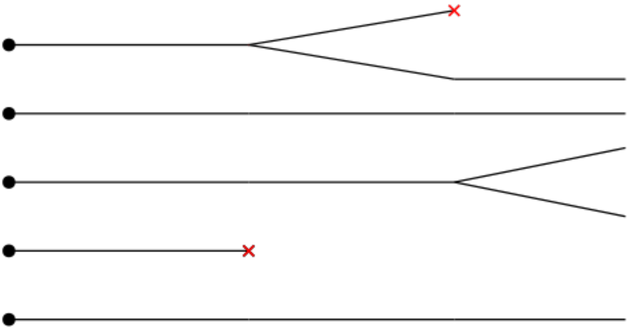


Figure: Fleming Viot particle system

Monte Carlo methods: Fleming Viot

- Let A_t denote the number of resampling events up to time t.
- Then

$$\mathbf{E}_x^{\dagger}[g(Y_t)] = \mathbb{E}\left[\left(\frac{N-1}{N}\right)^{A_t} \frac{1}{N}\sum_{i=1}^{N} f(Y_t^i)\right].$$

Monte Carlo methods: Fleming Viot

- With $\beta := \sup_{x \in E}(b(x) - k(x))$, consider

$$\mathrm{e}^{-\beta t}\psi_t[g](x) = \mathbf{E}_x\left[\mathrm{e}^{\int_0^t (b(Y_s)-k(Y_s)-\beta)\mathrm{d}s} g(Y_t)\mathbf{1}_{\{t<\tau_\partial\}}\right] =: \mathbf{E}_x^\dagger[g(Y_t)].$$

- In this case, we have

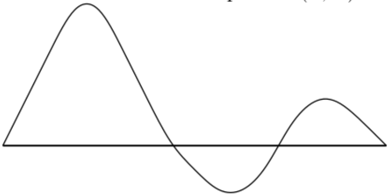
$$\mathbb{E}_{\delta_x}[\langle g, X_t\rangle] = \mathrm{e}^{\beta t}\mathbb{E}\left[\left(\frac{N-1}{N}\right)^{A_t}\frac{1}{N}\sum_{i=1}^N f(X_t^i)\right]$$

and

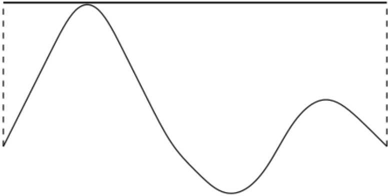
$$\lambda_* = \beta + \lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\left(\frac{N-1}{N}\right)^{A_t}\right].$$

Monte Carlo methods: Fleming Viot

Profile of the branching and killing for the process $(Y, \mathbf{P})$



Profile of the branching and killing for the process $(Y, \mathbf{P}^{\dagger})$



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A binary branching model with Moran interactions (BBMMI)

- $\mathcal{P}_f(\mathbb{N})$ is the collection of finite subsets of $\mathbb{N}$.

- Fix $N_0 \geq 2$. We consider a particle system $(S_t, (X_t^i)_{i \in S_t})_{t \geq 0}$, where
  - $S_t \in \mathcal{P}_f(\mathbb{N})$ is the set enumerating the particles in the system at time $t \geq 0$,
  - X_t^i denotes the position of the i-th particle in the system at time $t \geq 0$.

- Let $b_i : (E \cup \partial)^{\mathcal{P}_f(\mathbb{N})} \to [0, \infty)$ and $\kappa_i : (E \cup \partial)^{\mathcal{P}_f(\mathbb{N})} \to [0, \infty)$, be bounded measurable functions, $i \in \mathbb{N}$.

- Let $p_i : (E \cup \partial)^{\mathcal{P}_f(\mathbb{N})} \to [0, 1]$ and $q_i : (E \cup \partial)^{\mathcal{P}_f(\mathbb{N})} \to [0, 1]$ be measurable functions for $i \in \mathbb{N}$.

1. The particle $X_0^i, i \in S_0$, evolves as an independent copy of Y.

2. Set

$$\tau_b^i = \inf\{t > 0 : \int_0^t b_i(X_s^i, i \in S_0)\mathrm{d}s > \mathbf{e}_i^b\},$$

$$\tau_\kappa^i = \inf\{t > 0 : \int_0^t \kappa_i(X_s^i, i \in S_0)\mathrm{d}s > \mathbf{e}_i^\kappa\},$$

$$\tau_\partial^i = \inf\{t > 0 : X_t^i \in \partial\}.$$

3. Further set $\tau_1 = \inf_{i \in S_0}(\tau_b^i \wedge \tau_\kappa^i \wedge \tau_\partial^i)$ and let i_0 denote the index of the (unique) particle such that $\tau_1 = \tau_b^{i_0} \wedge \tau_\kappa^{i_0} \wedge \tau_\partial^{i_0}$.

BBMMI: Evolution of the system

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BBMMI: Evolution of the system

4. If $\tau_1 \in \{\tau_\partial^{i_0}, \tau_\kappa^{i_0}\}$, we say a **killing** event occurred.
$\rightsquigarrow$ Remove particle i_0 from the system
$\rightsquigarrow$ With probability $p^{i_0}(X^i_{\tau_1}, i \in S_0)$ a **resampling** event occurs: another particle is chosen uniformly at random and is duplicated.

5. If $\tau_1 = \tau_b^{i_0}$, we say a **branching** event occurred.
$\rightsquigarrow$ Duplicate particle i_0.
$\rightsquigarrow$ With probability $q_{i_0}(X^i_{\tau_1}, i \in S_0)$ a **selection** event occurs: one of the particles is selected uniformly at random and removed from the system.

Repeat to generate a sequence $\tau_0 := 0 < \tau_1 < \tau_2 < \cdots < \tau_n < \cdots$ of resampling/selection times.

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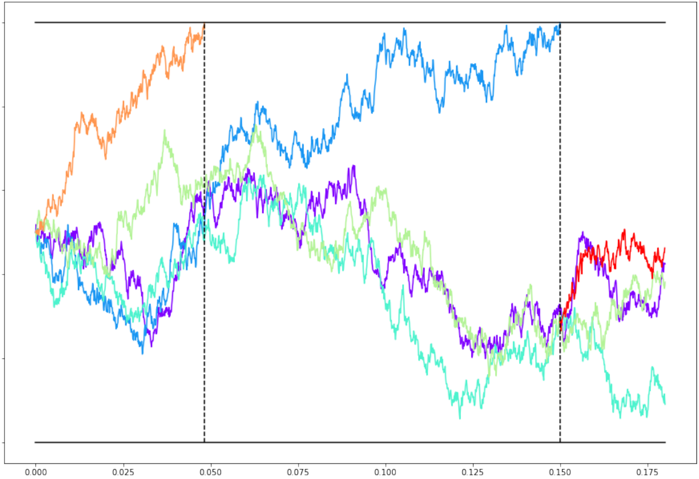
BBMMI: Example

Example: branching Brownian motion killed at 0 and 1.



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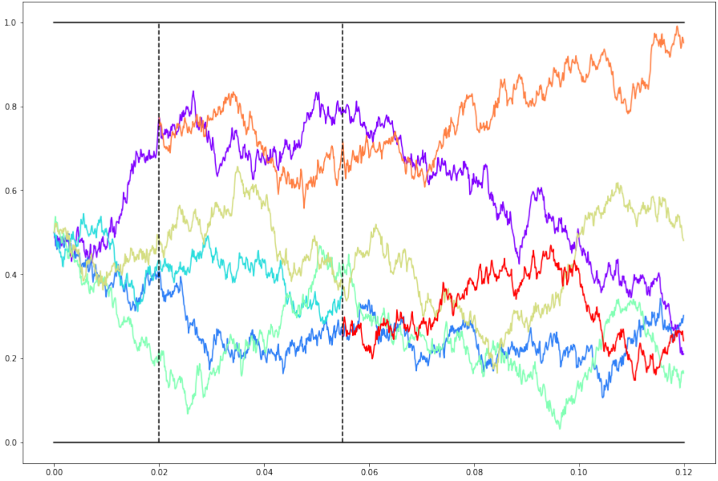

BBMMI: Notation

- Define
  - $\rho_n = n$-th resampling time.
  - $\sigma_n = n$-th selection time.

- Define
  - $A_t := \sup\{n \geq 0 : \rho_n \leq t\}$, the number of resampling events up to time t,
  - $B_t := \sup\{n \geq 0 : \sigma_n \leq t\}$, the number of selection events up to time t,

- Define $\Pi_t^A := \prod_{i=1}^{A_t} \left(\frac{N_{\rho_n} - 1}{N_{\rho_n}} \right)$ and $\Pi_t^B := \prod_{i=1}^{B_t} \left(\frac{N_{\sigma_n} + 1}{N_{\sigma_n}} \right)$

BBMMI: Main result

- ψ_t = semigroup of the branching process/many-to-one
- m_t = empirical distribution of the BBMMI at time t.

Theorem (Cox, H. & Villemonais '24)

Under certain assumptions, for all time $T \geq 0$ and all bounded measurable functions $f : E \to \mathbb{R}$, the BBMMI satisfies

$$\psi_T f(x) = \mathbf{E}_x \left[\Pi_T^A \, \Pi_T^B m_T(f) \right]. \tag{1}$$

Moreover,

$$\left\| \frac{\psi_T f(x)}{\psi_T \mathbf{1}_E(x)} - \frac{m_T(f)}{m_T(\mathbf{1}_E)} \mathbf{1}_{m_T \neq 0} \right\|_2 \leq \frac{C_T}{\sqrt{N_0}} \frac{\|f\|_\infty}{m_0 \psi_T \mathbf{1}_E / N_0}, \tag{2}$$

where C_T is a constant depending on $\|b\|_\infty$ and T.

BBMMI: The assumptions

(B1) For any $x \in E$ and $t \geq 0$, $\mathbb{P}_x(\tau_\partial = t) = 0$ and $\mathbb{P}_x(\tau_\partial > t) > 0$.

(B2) For all $s \in \mathcal{P}_f(\mathbb{N})$, $(x_i)_{i \in s}$ and $i_0 \in s$,

$$b_{i_0}(x_i, i \in s) - \kappa_{i_0}(x_i, i \in s) = b(x_{i_0}) - k(x_{i_0}).$$

(B3) For all $s \in \mathcal{P}_f(\mathbb{N})$, $(x_i)_{i \in s}$ and $i_0 \in s$,

$$p_{i_0}(x_i, i \in s) = 0 \text{ whenever } |s| = 1.$$

(B4) The sequence, $(\tau_n)_{n \geq 1}$, of event times satisfies $\tau_n \to \infty$ as $n \to \infty$.

BBMMI: Some remarks

- $p = q = 0 \rightsquigarrow$ binary branching process.
  Then $\Pi_T^A = \Pi_T^B = 1$ a.s., and (1) is the classical many-to-one formula.

- When $p = q = 1$, we obtain an extension of the FV particle system

- It is possible to let p and q depend on time.

- The above dynamics allow one to constrain the size of the process to remain between two bounds $0 \leq N_{min} \leq N_{max}$, $N_{min} \neq 1$, by choosing

$$p_{i_0}(x_i, i \in s) = \mathbf{1}_{\{|s| = N_{min}\}} \quad and \quad q_{i_0}(x_i, i \in s) = \mathbf{1}_{\{|s| = N_{max}\}}.$$

We call this the N_{min}-N_{max} model.

Refining the result

- Assume that we are in the setting of the N_{min}-N_{max} version of the BBMMI.
- Assume that we have the following Perron Frobenius asymptotic for the underlying branching process:

$$\sup_{x\in E,\|f\|\leq 1} \left|\varphi(x)^{-1}\mathrm{e}^{-\lambda_* t}\psi_t[f](x) - \langle\tilde{\varphi}, f\rangle\right| \leq C\mathrm{e}^{-\gamma t}, \qquad t \geq 0.$$

- Finally, we assume that there exists $t_0 > 0$ such that $\inf_{x\in E} \psi_{t_0}[\mathbf{1}_E](x) > 0$.

Corollary (Cox, H. & Villemonais '25)

Under these additional assumptions, there exists a constant $C_f > 0$ such that for all $T > 0$,

$$\left\| \frac{\psi_T f(x)}{\psi_T \mathbf{1}_E(x)} - \frac{m_T(f)}{m_T(\mathbf{1}_E)}\mathbf{1}_{m_T\neq 0} \right\|_2 \leq \frac{C_f}{\sqrt{N_{min}}}. \tag{3}$$

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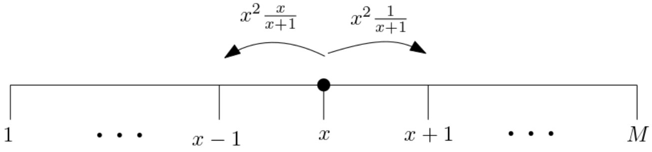
A branching birth and death process

$E = \{1, \ldots, M\}$ for some $M \geq 1$. When at $x \in E$, one of the following things may occur:

- the particle jumps with rate x^2 to state
  - $\max\{1, x-1\}$ with probability $x/(x+1)$,
  - $\min\{x+1, M\}$ with probability $1/(x+1)$.



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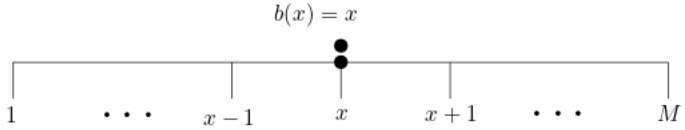
  - $\max\{1, x-1\}$ with probability $x/(x+1)$,
  - $\min\{x+1, M\}$ with probability $1/(x+1)$.
- at rate $b(x) = x$, a new particle is produced at the same site, which will continue to evolve independently according to the same dynamics.



N_{min}-N_{max} vs FV

- We now consider the problem of numerically approximating $\tilde{\varphi}_M$, the **left eigenmeasure** of the process.
- Fix $T > 0$ and $N_0 = N_{min} = N_{max}$.
- $\mathcal{X}_N^M$ = empirical distribution of the N_{min}–N_{max} particle system.
  $\mathcal{Y}_N^M$ = the empirical distribution FV particle systems.
- The estimator is $\hat{\theta}_N$, the empirical distribution of the relevant particle system.
- We compare three quantities:
  - the bias of the estimator, $|\mathbb{E}(\hat{\theta}_N(f)) - \tilde{\varphi}_M(f)|$,
  - the standard deviation of the estimator, $\mathrm{Std}(\hat{\theta}_N)$,
  - the number of interaction events, $(A_T + B_T)/T$.

N_{min}-N_{max} vs FV

$N = 100$		$\lvert\mathbb{E}(\hat{\theta}_N) - \tilde{\varphi}_M(f)\rvert$	$\mathrm{Std}(\hat{\theta}_N)$	$(A_T + B_T)/T$
$M = 10$	N_{min}–N_{max}	0.01	0.12	144
	FV	0.02	0.18	857
$M = 100$	N_{min}–N_{max}	0.01	0.12	144
	FV	0.10	0.39	9866
$M = 1000$	N_{min}–N_{max}	0.01	0.12	144
	FV	0.20	0.50	99873
$M = +\infty$	N_{min}–N_{max}	0.01	0.12	144
	FV	$*$	$*$	$*$

N_{min}-N_{max} vs FV

"Theorem"

The BBMMI wins!

Future work

- Extending the results to more general branching processes
- Scaling limits and genealogical structure
- Central limit theorem
- Branching processes with infinite branching rate
- Adaptive simulation schemes
- ...

Merci !