Conditioned exit measures of branching Markov processes

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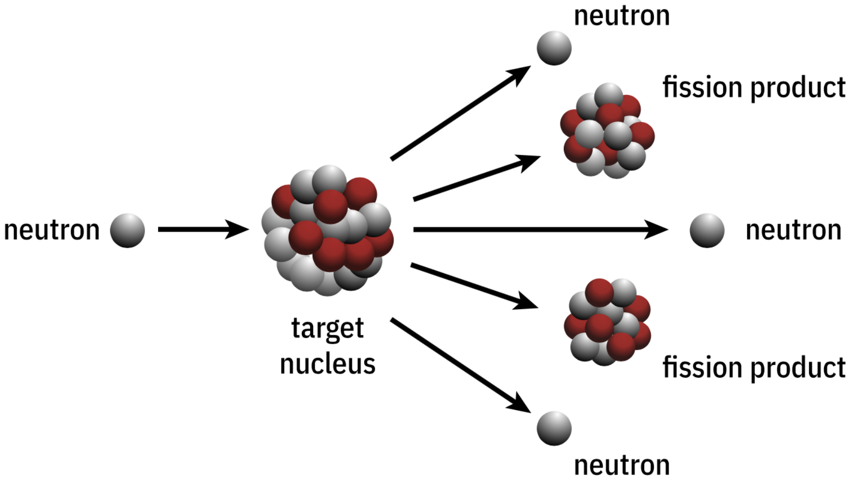
Ancestral lines in population models with interaction
BOKU University
24-26 September 2025

Joint work with Christopher Dean (Warwick) and Simon Harris (Auckland) and Andreas Kyprianou (Warwick).

Contents

Rare events in neutron transport



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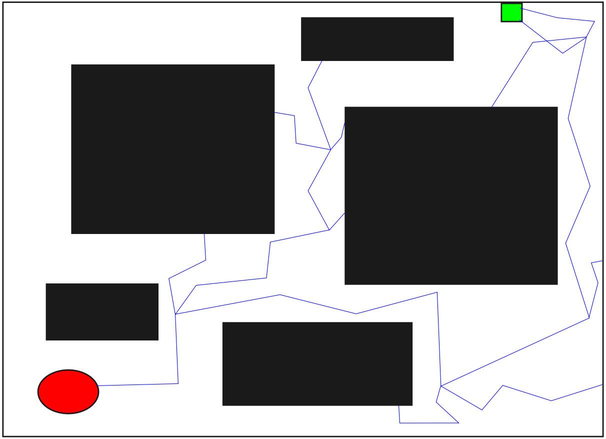

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- Something else...

Adaptive Multilevel Splitting

1. Fix $N \geq 1$ and an importance function $\Phi : E \to [0,1]$.
2. Simulate N independent copies of $(\xi, \mathbf{P}_x)$ until each particle is either absorbed or reaches the detector.
3. Compute $M_i := \sup_{0 \leq s \leq \tau_i} \Phi(\xi^i_s)$ for $i = 1, \ldots, N$.
4. Sort into increasing order: $M_{(1)} \leq \cdots \leq M_{(N)}$ and set $L := M_{(1)}$.
5. If $L < 1$, delete $\xi^{(1)}$ and uniformly choose one of the remaining trajectories.
6. Clone the corresponding trajectory until the first time it enters $\{\Phi \geq L\}$ and then simulate an independent copy of ξ until absorption.
7. Return to 3.

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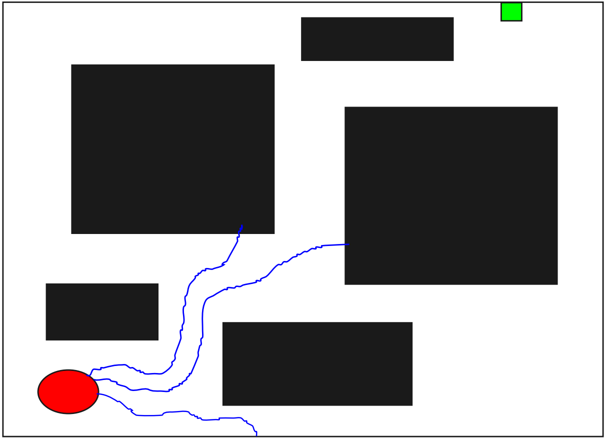

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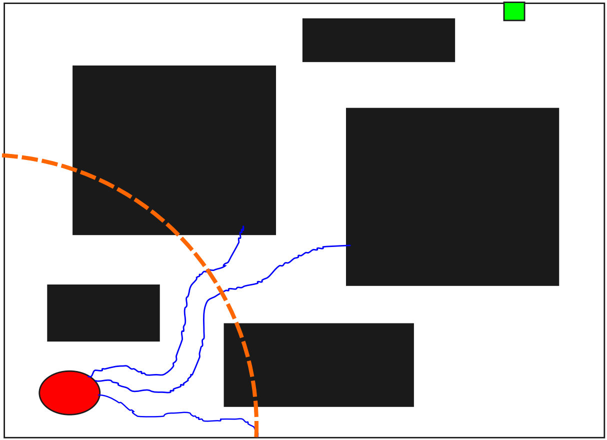

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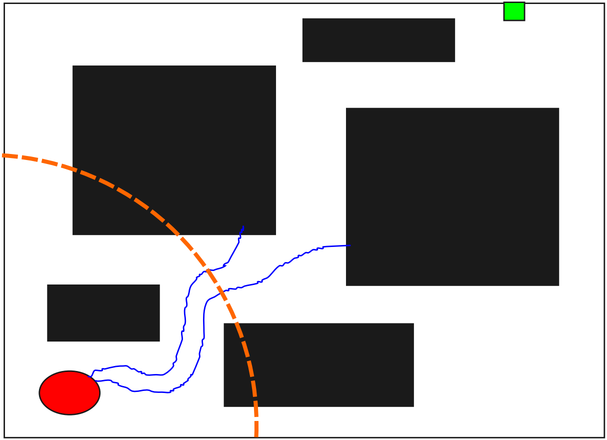

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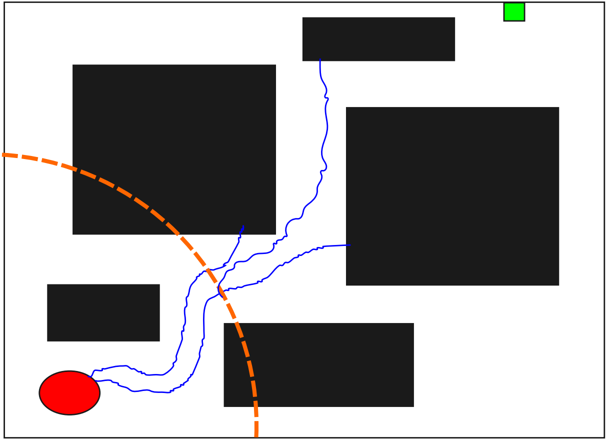

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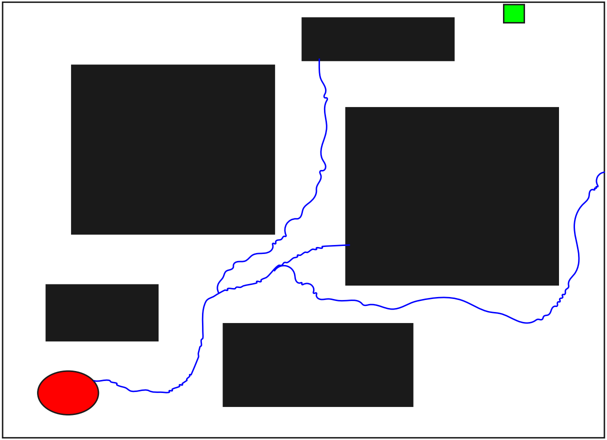

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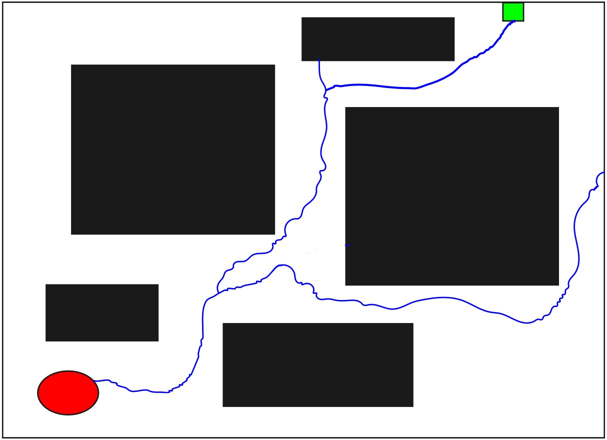

Adaptive Multilevel Splitting

- After M resampling events, the probability of hitting D can be approximated by

$$\hat{p}(N, M) = \left(\frac{N-1}{N}\right)^M \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_D(\xi^i_{\tau^i})$$

- Always have N particles.
- We can view AMS as approximation of the Markov process conditioned to hit D.
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Adaptive Multilevel Splitting

- Cérou, F., & Guyader, A. (2007). Adaptive multilevel splitting for rare event analysis. Stochastic Analysis and Applications, 25(2), 417-443.
- Cérou, F., Delyon, B., Guyader, A., & Rousset, M. (2019). On the asymptotic normality of adaptive multilevel splitting. SIAM/ASA Journal on Uncertainty Quantification, 7(1), 1-30.
- Cérou, F., Guyader, A., & Rousset, M. (2019). Adaptive multilevel splitting: Historical perspective and recent results. Chaos: An Interdisciplinary Journal of Nonlinear Science, 29(4).

Back to the problem with branching

- For initial work on this in the context of neutron transport, see Louvin (2017) Development of an adaptive variance reduction technique for Monte Carlo particle transport, PhD thesis, https://theses.hal.science/tel-01661422.

- Problem: how to choose a good importance function.

- Aim: study the branching process conditioned to hit D in order to inform a branching version of the AMS algorithm.

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Contents

Branching Markov process

- Particles evolve in a Polish space E according to a (sub)Markov process $(\xi, \mathbf{P}_x)$, whose associated semigroup is given by

$$\mathtt{P}_t[f](x) = \mathbf{E}_x[f(\xi_t)\mathbf{1}_{(t<\zeta)}].$$

- When at $x \in E$, at rate $\beta(x)$, the parent is replaced by new particles that are created according to the point process $(\mathcal{Z}, \mathcal{P}_x)$, where

$$\mathcal{Z} = \sum_{i=1}^{N} \delta_{x_i}.$$

- If particles die, they are sent to a cemetery state $\dagger \notin E$.

Branching Markov process

- Let $\{x_i(t) : i = 1, \dots, N_t\}$ denote the configuration of the system at time $t \geq 0$.

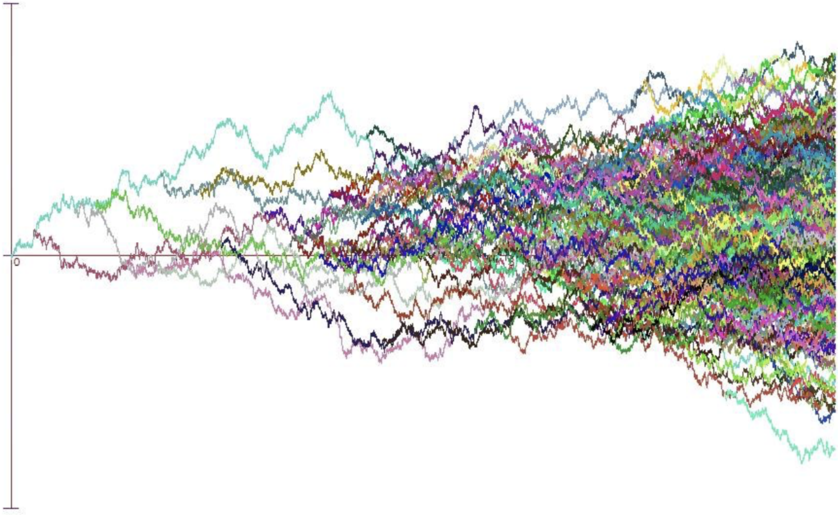
- The branching process is defined as

$$X_t := \sum_{i=1}^{N_t} \delta_{x_i(t)}.$$

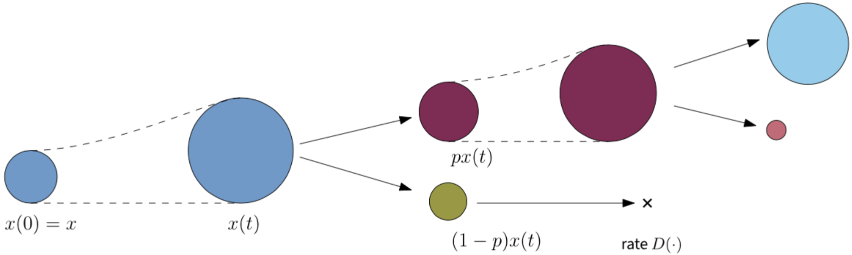
- For a function f and measure μ, we will use the notation

$$\mu[f] := \int_D f(y)\mu(\mathrm{d}y).$$

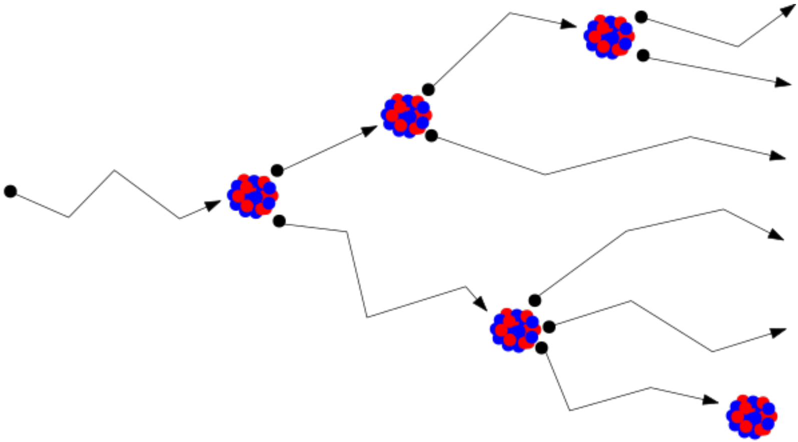
Example: BBM



Example: growth-fragmentation



Example: neutron transport



Assumptions

(A1) The Markov process admits a càdlàg modification.

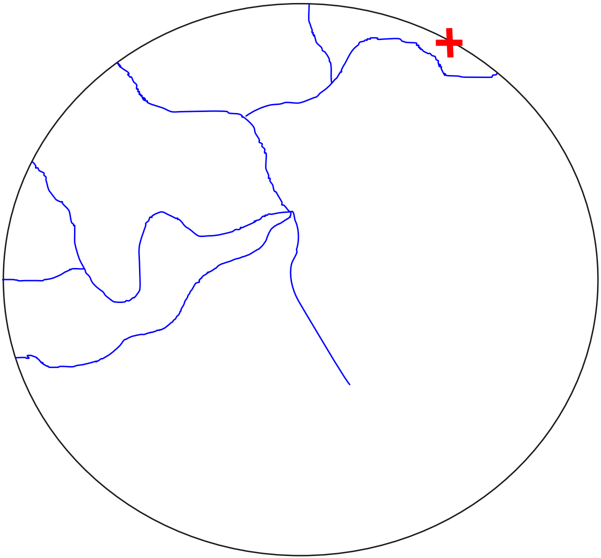
(A2) $\sup_{x\in D} \mathcal{E}_x[\mathcal{Z}[1]] < \infty$ and $\sup_{x\in D} \beta(x) < \infty$.

(A3) The branching process is subcritical, i.e. there exists $\lambda < 0$ such that

$$\psi_t[f](x) := \mathbb{E}_{\delta_x}[X_t[f]] \sim \mathrm{e}^{\lambda t} C(x, f), \quad t \to \infty.$$

Contents

A single point



Aside: conditioning on survival

- Consider instead the problem of conditioning X to survive.
- It is well-known that the law $\mathbb{P}^{\uparrow}_{\delta_x}$ defined via

$$\mathbb{P}^{\uparrow}_{\delta_x}(A) := \lim_{s\to\infty} \mathbb{P}_{\delta_x}(A | N_{t+s} > 0), \quad A \in \mathcal{F}_t,$$

is related to $\mathbb{P}_{\delta_x}$ via a martingale change of measure:

$$\left.\frac{\mathrm{d}\mathbb{P}^{\uparrow}_{\delta_x}}{\mathrm{d}\mathbb{P}_{\delta_x}}\right|_{\mathcal{F}_t} = W_t,$$

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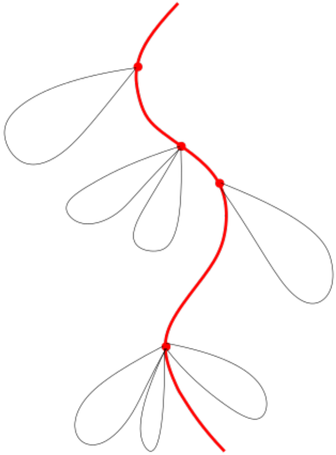
Aside: conditioning on survival

Moreover, it is known that $(X, \mathbb{P}^{\uparrow}_{\delta_x})$ can be identified as a single 'spine' that is conditioned to survive, dressed with independent copies of $(X, \mathbb{P})$.



Exit measures

- We will work with a modification of the branching process X. Let $(X_t^{\partial E}, t \geq 0)$ denote the process that is equal to X in E but whose particles become frozen upon exiting E.

- We let $X_\infty^{\partial E}$ denote the exit measure from E.

- Dynkin, E. B. (2001). Branching exit Markov systems and superprocesses. Annals of probability, 1833-1858.

Finding a martingale

- Define
$$M_\infty := X_\infty^{\partial E}[\mathbf{1}_{D_1}].$$

- For $t \geq 0$, $x \in E$, define
$$M_t := \mathbb{E}_{\delta_x}[M_\infty | \mathcal{F}_t] =: X_t[v_1], \quad t \geq 0, x \in E,$$
where $v_1(y) = \mathbb{E}_{\delta_y}[X_\infty^{\partial E}[\mathbf{1}_{D_1}]]$.

- Then, $X_t[v_1]/\mu[v_1]$ is a unit mean martingale under $\mathbb{P}_\mu$ and we can thus define the change of measure
$$\left.\frac{\mathrm{d}\mathbb{Q}_\mu}{\mathrm{d}\mathbb{P}_\mu}\right|_{\mathcal{F}_t} := \frac{X_t[v_1]}{\mu[v_1]}$$

Spine decomposition

Under $\mathbb{Q}$, the process $X^{\partial E}$ can be constructed as follows.

Spine decomposition

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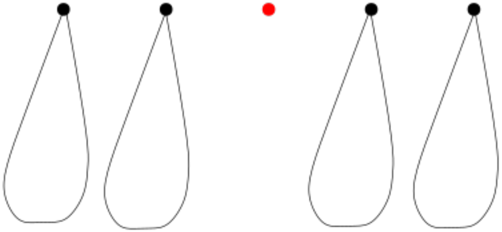
1. From the initial configuration $\mu = \sum_{i=1}^{n} \delta_{x_i}$, the i^*-th individual is selected with probability $v_1(x_{i^*})/\mu[v_1]$ and marked the *spine*.

2. The individuals $j \neq i^*$ in the initial configuration each issue independent copies of $(X^{\partial E}, \mathbb{P}_{\delta_{x_j}})$ respectively.



Spine decomposition

3. The marked individual, "spine", issues a single particle whose motion is determined by the semigroup

$$S_t[g](x) = \mathbf{E}_x \left[\mathrm{e}^{\int_0^t \beta(\xi_s)\left(\frac{\mathcal{E}_{\xi_s}[\mathcal{Z}[v_1]]}{v_1(\xi_s)} - 1\right)\mathrm{d}s} \frac{v_1(\xi_t)}{v_1(x)} g(\xi_t) \right]$$

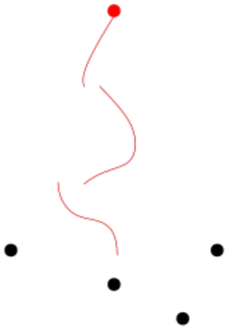
Spine decomposition

4. When at $x \in E$, the spine undergoes branching at rate

$$\rho(x) := \frac{\beta(x)\mathcal{E}_x[\mathcal{Z}[v_1]]}{v_1(x)}$$

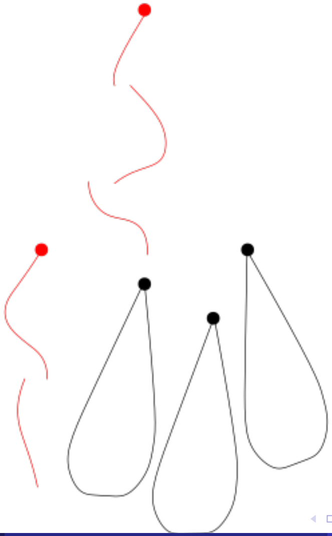
at which point, it produces particles according $(\mathcal{Z}, \mathcal{P}_x^1)$, where

$$\frac{\mathrm{d}\mathcal{P}_x^1}{\mathrm{d}\mathcal{P}_x} = \frac{\mathcal{Z}[v_1]}{\mathcal{E}_x[\mathcal{Z}[v_1]]}.$$



Spine decomposition

5. Given $\mathcal{Z}$ from the previous step, μ is redefined as $\mu = \mathcal{Z}$ and Step 1 is repeated.



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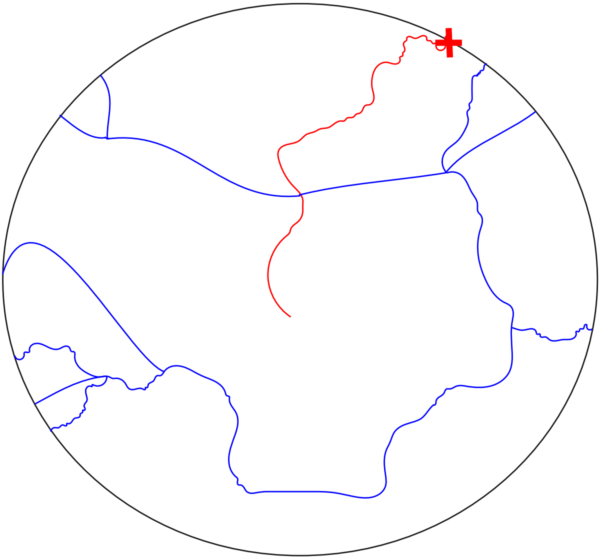

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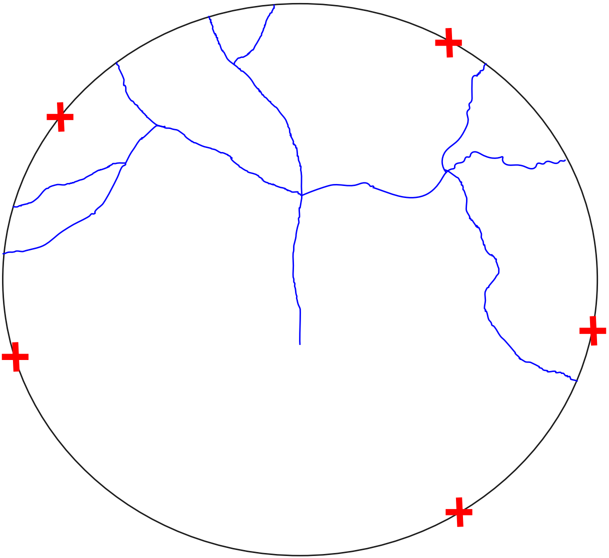
Multiple exit points



Constructing the martingale

- For $A \subset \{1, \ldots, k\}$ and $n \geq 1$, define

$$\mathcal{I}_{A,n} := \{\mathbf{i} \in [n]^A : i_j \neq i_\ell \text{ for all } j \neq \ell\}.$$

- Then set

$$M_\infty^A := \sum_{\mathbf{i} \in \mathcal{I}_{A, N_\infty^{\partial E}}} \prod_{j \in A} \mathbf{1}_{D_j}(x_{i_j}).$$

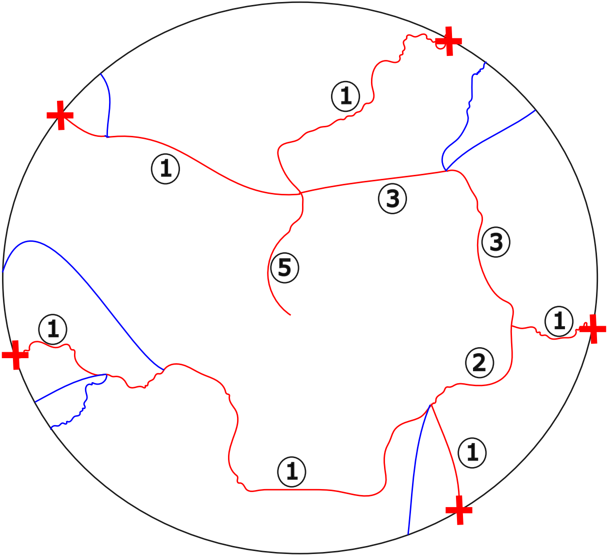
- As before, we set

$$M_t^A = \mathbb{E}[M_\infty^A | \mathcal{F}_t] = X_t[v^A],$$

where $v^A(y) = \mathbb{E}_{\delta_y}[M_\infty^A]$ and define the change of measure

$$\left.\frac{\mathrm{d}\mathbb{Q}_\mu^A}{\mathrm{d}\mathbb{P}_\mu}\right|_{\mathcal{F}_t} := \frac{M_t^A}{\mu[v^A]}.$$

Multiple spines



Comments: martingale for multiple spines

To understand why this is the correct description, let us look at M_t^A in more detail.

- Let $\mathcal{P}(A)$ denote the set of partitions of A.

- Then
$$M_t^A = \mathbb{E}_{\delta_x}[M_\infty^A | \mathcal{F}_t] = \sum_{\sigma \in \mathcal{P}(A)} \sum_{\mathbf{i} \in \mathcal{I}_{\sigma, N_t}} \prod_{j=1}^{|\sigma|} \mathbb{E}_{\delta_{x_{i_j}(t)}}[M_\infty^{\sigma_j}]$$

Comments: importance functions for AMS

- Recall that

$$M_t^A = \mathbb{E}_{\delta_x}[M_\infty^A|\mathcal{F}_t] = \sum_{\sigma\in\mathcal{P}(A)} \sum_{\mathbf{i}\in\mathcal{I}_{\sigma,N_t}} \prod_{j=1}^{|\sigma|} \mathbb{E}_{\delta_{x_{i_j}(t)}}[M_\infty^{\sigma_j}]$$

- Taking expectations yields

$$\begin{aligned} v^A(x) &= \mathbb{E}_{\delta_x}[M_\infty^A] \\ &= \int_0^\infty \psi_s \left[\gamma\mathcal{E}.\left[\sum_{\sigma\in\mathcal{P}^*(A)} \sum_{\mathbf{i}\in\mathcal{I}_{\sigma,N_t}} \prod_{j=1}^{|\sigma|} v^{\sigma_j}(x_{i_j})\right]\right](x)\mathrm{d}s, \end{aligned}$$

where $\mathcal{P}^*(A)$ is the set of partitions of A with at least two blocks.

Comments: exiting via sets

- Suppose now we want to look at the process conditioned to exit the boundary via sets, rather than points.
- The construction of the martingale and change of measure remain the same, however, it is not the case that the resulting process yields the original process, conditioned on exiting in such a way.
- In this case, a additional change of measure is required to recover the conditioned process.

Comments: (super)critical case

- In the case that $\mathbb{E}_{\delta_x}[X_t[f]] \sim \mathrm{e}^{\lambda t} C(x, f)$ for $\lambda \geq 0$ then the exit measure will have infinite mass.
- In this case, the same arguments can be applied by considering the exit measure up to a fixed time horizon.

Comments: completing the picture ($k = 1$)

$$
\begin{tikzcd}
(X, \mathbb{P}_{\delta_x}) \arrow[rr, leftrightarrow, "M_t^1"] & & (X, \mathbb{Q}_{\delta_x}) \arrow[d, leftrightarrow, "Spine"] \\
& & (Y, \mathbf{Q}_x)
\end{tikzcd}
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\begin{tikzcd}
(X, \mathbb{P}_{\delta_x}) \arrow[rrr, leftrightarrow, "M_t^1"] \arrow[d, leftrightarrow, "Many-to-one"'] & & & (X, \mathbb{Q}^1_{\delta_x}) \arrow[d, leftrightarrow, "Spine"] \\
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$$
\begin{tikzcd}
(X, \mathbb{P}_{\delta_x}) \arrow[r, leftrightarrow, "M_t^1"] \arrow[d, leftrightarrow, "Many-to-one"'] & (X, \mathbb{Q}^1_{\delta_x}) \arrow[d, leftrightarrow, "Spine"] \\
(Y, \mathbf{P}_x) \arrow[r, leftrightarrow, "\widetilde{M}_t^1"'] & (Y, \mathbf{Q}_x)
\end{tikzcd}
$$

Other questions

- In practice, we can't compute the optimal importance function so what are good choices?
- Fluctuation theory and explicit formula for the asymptotic variance.
- Genealogical structure.

Some references

- Harris, S. C. & Roberts, M. I., (2017) The many-to-few lemma and multiple spines
- Harris, S. C., Horton, E., Kyprianou, A. E., & Powell, E. (2024). Many-to-few for non-local branching Markov process. Electronic Journal of Probability, 29, 1-26. Ann. Inst. H. Poincaré Probab. Statist. 53(1): 226-242.
- Salisbury, T. S., & Verzani, J. (1999). On the conditioned exit measures of super Brownian motion. Probability theory and related fields, 115, 237-285.
- Salisbury, T. S., & Verzani, J. (2000). Non-degenerate conditionings of the exit measures of super Brownian motion. Stochastic processes and their applications, 87(1), 25-52.

Thank you!