Convergence to the Brownian CRT for critical branching Markov processes

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Properties of Enhanced Branching Structures
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Based on joint work with Ellen Powell (Durham University)

Branching Markov process

- Particles evolve in a Polish space E according to a càdlàg (sub)Markov process $(\xi, \mathbf{P})$.

- When at $x \in E$, at rate $\gamma(x)$, the parent is replaced by new particles that are created according to the point process $(\mathcal{Z}, \mathcal{P}_x)$, where

$$\mathcal{Z} = \sum_{i=1}^{N} \delta_{x_i}.$$

- If particles die, they are sent to a cemetery state $\dagger \notin E$.

Branching Markov process

- Let $\{x_i(t) : i = 1, \dots, N_t\}$ denote the configuration of the system at time $t \geq 0$.

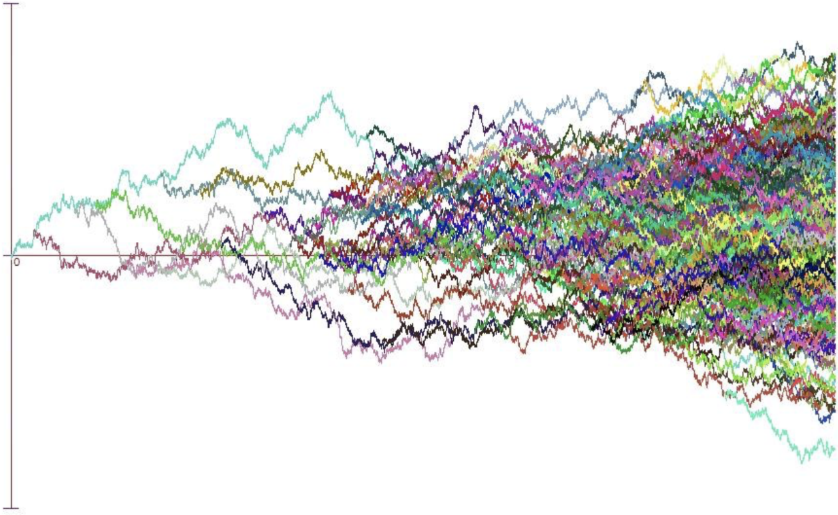
- The branching process is defined as

$$X_t := \sum_{i=1}^{N_t} \delta_{x_i(t)}.$$

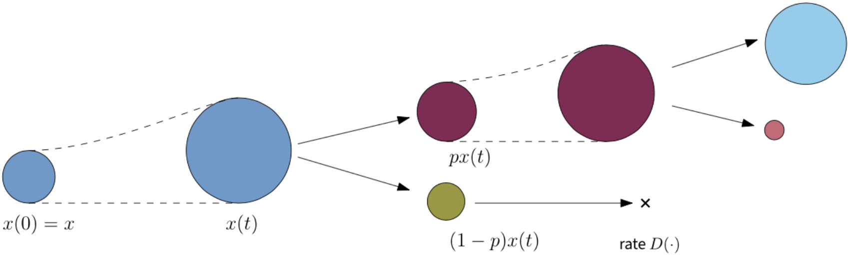
- For a function f and measure μ, we will use the notation

$$\langle f, \mu \rangle := \int_E f(y)\mu(\mathrm{d}y).$$

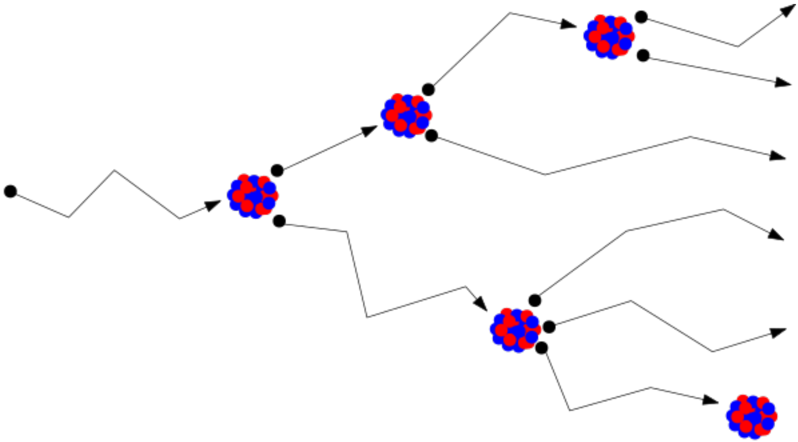
Example: BBM



Example: growth-fragmentation



Example: neutron transport



Criticality

For $x \in E, g \in L^\infty_+(E)$ and $t \geq 0$, define

$$\psi_t[g](x) = \mathbb{E}_{\delta_x}[\langle g, X_t\rangle] = \mathbb{E}_{\delta_x}\left[\sum_{i=1}^{N_t} g(x_i(t))\right].$$

(PF) There exists $\varphi \in L^+_\infty(E)$ and a probability measure $\tilde{\varphi}$ on E such that, for all $g \in L^+_\infty(E)$ and $t \geq 0$, we have

$$\psi_t[\varphi] = \varphi \qquad \text{and} \qquad \langle \psi_t[g], \tilde{\varphi}\rangle = \langle g, \tilde{\varphi}\rangle.$$

Moreover there exists $\varepsilon > 0$ such that

$$\sup_{\|g\|\leq 1} \|\varphi^{-1}\psi_t[g] - \langle g, \tilde{\varphi}\rangle\|_\infty = O(e^{-\varepsilon t}) \text{ as } t \to \infty.$$

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(PF) There exists $\lambda \in \mathbb{R}$, $\varphi \in L^{+}_{\infty}(E)$ and a probability measure $\tilde{\varphi}$ on E such that, for all $g \in L^{+}_{\infty}(E)$ and $t \geq 0$, we have

$$\langle \psi_t[g], \tilde{\varphi}\rangle = \mathrm{e}^{\lambda t}\langle g, \tilde{\varphi}\rangle \text{ and } \psi_t[\varphi] = \mathrm{e}^{\lambda t}\varphi.$$

Moreover there exists $\varepsilon > 0$ such that

$$\sup_{\|g\|\leq 1} \|\varphi^{-1}\mathrm{e}^{-\lambda t}\psi_t[g] - \langle g, \tilde{\varphi}\rangle\| = O(e^{-\varepsilon t}) \text{ as } t \to \infty.$$

Existing results

Under (PF), $\sup_{x \in E} \mathcal{E}_x[N^2] < \infty$ and some additional assumptions, we have the following results.

- Kolmogorov survival probability:

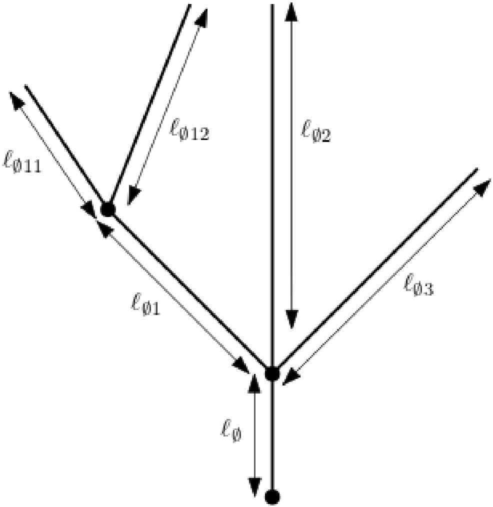
$$\mathbb{P}_{\delta_x}(N_t > 0) \sim \frac{2\varphi(x)}{\Sigma t}, \quad t \to \infty.$$

- Yaglom limit:

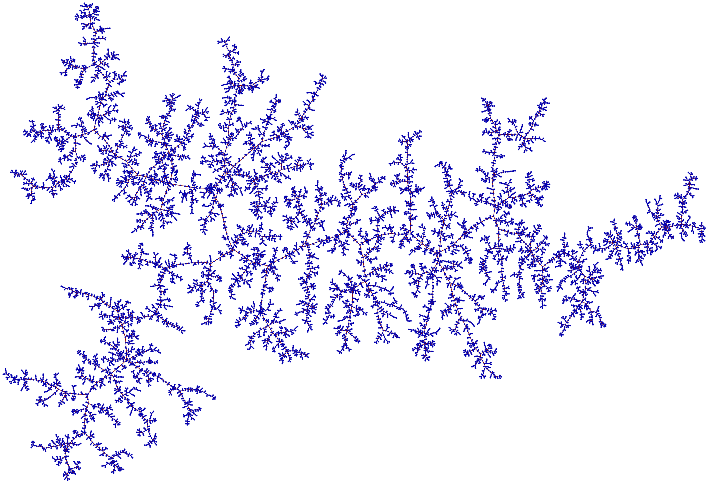
$$\mathrm{Law}\left(\frac{\langle f, X_t \rangle}{t} \middle| N_t > 0 \right) \to \mathrm{Law}(Y), \quad t \to \infty,$$

where $Y \sim Exp(\Sigma \langle f, \tilde{\varphi} \rangle / 2)$.

Genealogical tree



Brownian CRT



Main result

- Let X be a critical MBP with finite variance offspring distribution and T the associated genealogical tree.
- Let d denote the genealogical distance on T.
- Let ν denote the total occupation time of the depth-first exploration of T.
- Let $\mathcal{T}_{n,x} = (T, \frac{1}{n}d, \frac{1}{n^2}\nu, \mathbf{r})$ under $\mathbb{P}_{\delta_x}(\cdot | N_n > 0)$.

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Theorem 1

Under some mild regularity and ergodicity assumptions

$$\mathcal{T}_{n,x} \xrightarrow{d} \mathcal{T}_{\mathbf{e}}, \quad \mathbb{R} \ni n \to \infty,$$

with respect to the Gromov-Hausdorff-weak topology where $\mathcal{T}_{\mathbf{e}}$ denotes the Brownian CRT generated from a Brownian excursion with speed $\sigma^2(f)$.

Literature

- Aldous (1991) The Continuum Random Tree I
- Aldous (1993) The Continuum Random Tree III
- Le Gall & Duquesne (2002) Random trees, Lévy processes, and spatial branching processes
- Miermont (2008) Invariance principles for spatial multitype Galton-Watson trees
- Powell (2019) An invariance principle for branching diffusions in bounded domains
- Foutel-Rodier (2024) A moment approach for the convergence of spatial branching processes to the continuum random tree

Proof strategy

- First show convergence in the Gromov-weak topology: sample finitely many points in each tree according to its mass measure and compare their distance matrices.
- Upgrade Gromov-weak convergence to Gromov-Hausdorff-weak via a tightness condition.
- The ideal strategy would be to work directly with the (scaled) genealogical distance.
- Instead, we construct a martingale that allows us to construct a proxy for the genealogical distance.

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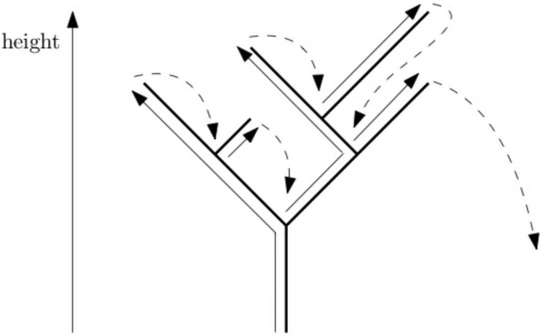
Outline of the rest of the talk

- Step 1: Definitions of $\mathcal{T}_{n,x}$ and $\mathcal{T}_{\mathbf{e}}$.
- Step 2: DFE martingale.
- Step 3: DFE martingale $\approx$ height function.
- Step 4: Gromov-weak convergence.
- Step 5: Gromov-Hausdorff-weak convergence.

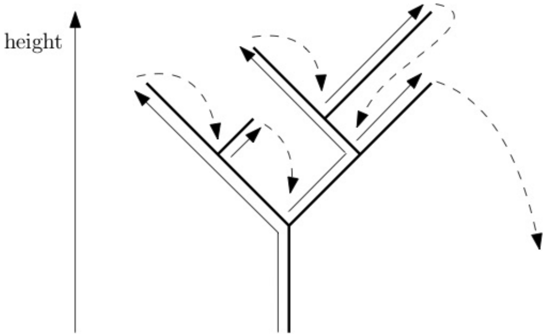
Step 1: Definitions of $\mathcal{T}_{n,x}$ and $\mathcal{T}_{\mathbf{e}}$

Depth-first exploration



- v_t = label of particle being explored in the DFE
- h_t = the "height" of v_t in the tree
- ζ_t = the position of v_t in E
- D_t = set of discovered particles that are yet to be explored

Depth-first exploration



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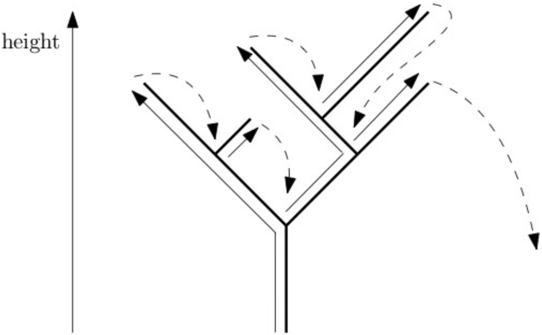

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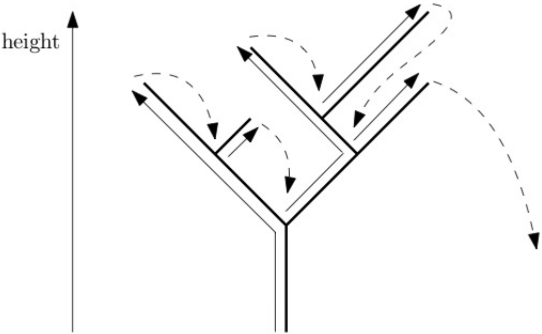

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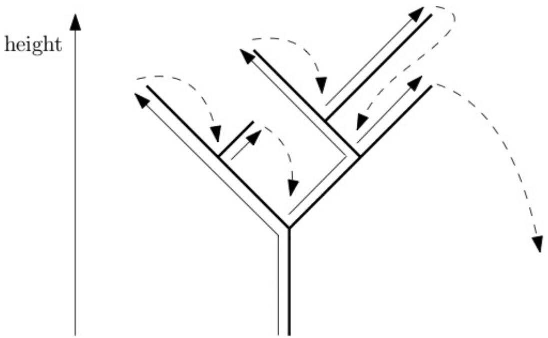

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Definition of $\mathcal{T}_{n,x}$

- Let $L = \int_0^\infty N_s ds$ denote the length of the tree.
- For $s, t \in [0, L)$ set

$$d(s,t) = h_s + h_t - 2 \inf_{u \in [s \wedge t, s \vee t]} h_u$$

- Define (T, d) to be $([0, L)/\sim, d)$, where $s \sim t \Leftrightarrow d(s,t) = 0$.
- Equip (T, d) with the measure ν, the push forward of Lebesgue measure on $[0, L)$ under the equivalence relation $\sim$.

Definition of $\mathcal{T}_{\mathbf{e}}$

- Let $\mathbf{e} = (\mathbf{e}_s)_{s \le \tau}$ denote a Brownian excursion with speed $\sigma^2(f)$ conditioned to have maximum height at least 1.

- For $s, t \in [0, \tau)$ set

$$d_e(s,t) = \mathbf{e}_s + \mathbf{e}_t - 2 \inf_{u \in [s \wedge t, s \vee t]} \mathbf{e}_u.$$

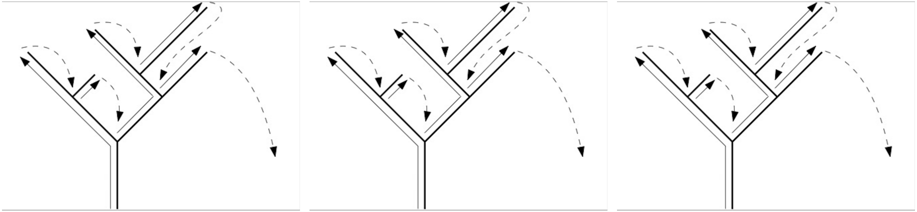
- Define (T_e, d_e) to be $([0,\tau)/\sim_e, d_e)$, where $s \sim_e t \Leftrightarrow d_e(s,t) = 0$.

- Equip (T_e, d_e) with the measure ν_e, the push forward of Lebesgue measure on $[0, \tau)$ under the equivalence relation $\sim_e$.

Step 2: The DFE martingale

A sequence of trees

We extend the DFE to a sequence of i.i.d. trees:

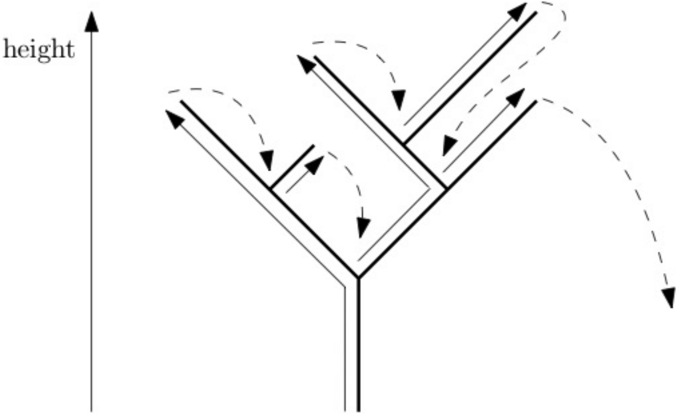


- The DFE is then given by (I_t, v_t) where I_t is the index of the tree being visited and v_t is the label of the particle being visited in this tree.
- We abuse notation and write h_t, ζ_t and D_t, as before.
- Let $\mathbb{P}^{\star}_{\delta_x}$ denote the law of an i.i.d. sequence of branching processes started with a single particle at x.

The DFE martingale

Set

$$M_t = \varphi(\zeta_t) + \sum_{w \in D_t} \varphi(X_w(b_w)) - (I_t - 1)\varphi(x).$$



The DFE martingale

Lemma 1

$(M_t)_{t\geq 0}$ is a local martingale under $\mathbb{P}^\star_{\delta_x}$ with predictable quadratic variation given by

$$\langle M \rangle_t = \int_0^t f(\zeta_s) ds,$$

where

$$f(x) = \Gamma(\varphi)(x) + \gamma(x)\mathcal{E}_x\left[\left(\sum_{i=1}^N \varphi(x_i) - \varphi(x)\right)^2\right]$$

and Γ is the Carré du Champ operator.

- If $\varphi \in \mathcal{D}(L)$ then $\Gamma(\varphi)(x) = L(\varphi^2) - 2\varphi L(\varphi)$.
- In the case of branching diffusions, we have

$$\Gamma(\varphi) = 2\sum_{i,j=1}^d a_{ij}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}.$$

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The DFE martingale

- Intuition: from (PF), $\langle \varphi, X_t \rangle$ is a martingale. One can think of M as being the same object but explored in depth-first order rather than breadth-first order.

- Proof idea: instead, write

$$M_t = m_t + (M_t - m_t),$$

where m is a local martingale for the single particle motion.

The DFE Martingale

Theorem 2

We have

$$\left(\frac{1}{n} M_{n^2 t}\right)_{t\geq 0} \xrightarrow{d} \left(B_{\sigma^2(f)t}\right)_{t\geq 0}$$

as $n \to \infty$, where B is a standard one-dimensional Brownian motion, and with f defined in Lemma 1,

$$\sigma^2(f) := \frac{\langle \tilde{\varphi}, f \rangle}{\langle \tilde{\varphi}, 1 \rangle}. \tag{1}$$

- Proved by verifying the conditions of a general martingale CLT.
- The jumps become negligible under $1/n$ scaling.
- Show that

$$\frac{\langle M \rangle_t}{t} = \frac{1}{t}\int_0^t f(\zeta_s)ds \to \sigma^2(f).$$

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The DFE martingale

Corollary 1

Let $(\hat{M}_t)_{t\geq 0}$ be the martingale $(M_t)_{t\geq 0}$ but restricted to the exploration of a single critical tree. Then,

$$\mathcal{L}\left(\frac{1}{n}\hat{M}_{n^2\cdot}\,\middle|\,\sup \hat{M} > n\right) \to \mathcal{L}(\mathbf{e}\,|\,\sup(\mathbf{e}) > 1), \quad n \to \infty,$$

where $\mathbf{e}$ is a Brownian excursion run at speed $\sigma^2(f)$.

Step 3: $M \approx h$

Comparison of M and h

- Let $\mathbb{P}^1_{\delta_x}$ denote the law of T under $\mathbb{P}_{\delta_x}$ along with $t^1 \sim Unif[0, L)$.
- For a particle v, let $D(v)$ denote the set of younger siblings of the ancestors of v and set

$$\hat{M}(v) = \sum_{w \in D(v)} \varphi(X_w(b_w)).$$

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Lemma 2

For any $\eta > 0$,

$$\lim_{R\to\infty} \lim_{n\to\infty} \mathbb{P}^1_{\delta_x}\left(\sup_{\substack{v \prec v_{t^1} \\ R \le h(v) \le h_{t^1}}} \left| \frac{\hat{M}(v)}{h(v)} - \frac{\Sigma}{2} \right| > \eta \;\middle|\; N_n > 0 \right) = 0.$$

Step 4: Gromov-weak convergence

Gromov-weak convergence

- Let $\mathbb{P}^k_{\delta_x}$ denote the law of a tree T under $\mathbb{P}_{\delta_x}$ together with k random variables $t^1, \dots, t^k$ chosen conditionally independently and uniformly at random from $[0, L)$.
- For $i, j = 1, \dots, k$, define

$$(D_n^h)_{ij} = \tfrac{1}{n}(h_{t^i} + h_{t^j} - 2h^{ij}),$$
$$(D_n^M)_{ij} = \tfrac{1}{n}(\hat{M}(v_{t^i}) + \hat{M}(v_{t^j}) - 2\hat{M}(v_{t^i} \wedge v_{t^j})).$$

Proposition 1

Let $k \geq 1$ and $\varepsilon > 0$. Then

$$\mathbb{P}^k_{\delta_x}\left(\left\|\tfrac{1}{2}\Sigma D_n^h - D_n^M\right\| > \varepsilon | N_n > 0\right) \to 0$$

as $n \to \infty$.

Gromov-weak convergence

- The law of D_n^h is exactly the law of the distances in $\mathcal{T}_{n,x}$ between k points sampled from $(\nu/n^2)^{\otimes k}$ (normalised).
- Proposition 1: D_n^h and D_n^M are close (up to a constant).
- Corollary 1 says that
  - D_n^M and the distances between k points in $\mathcal{T}_{\mathbf{e}}$ are close in law,
  - $|\nu/n^2| = L/n^2$ converges to the length of a Brownian excursion run at speed $\sigma^2(f)$ and conditioned to reach height (at least) 1.
- Putting the pieces together implies that Theorem 1 holds with respect to the Gromov-weak topology.

Step 5: Gromov-Hausdorff-weak convergence

Upgrade to Gromov-Hausdorff-weak

- Greven, Pfaffelhuber & Winter (2009) Convergence in distribution of random metric measure spaces (λ-coalescent measure trees)
- Depperschmidt, Greven & Pfaffelhuber (2011) Marked metric measure spaces
- Foutel-Rodier (2024) Vague convergence and method of moments for random metric measure spaces

Upgrade to Gromov-Hausdorff-weak

For $r > 0$ and $w \in \mathcal{N}_r$, let $B_\delta(w, r)$ be the collection of all vertices with distance (in (T, d)) at most δ from (w, r).

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Lemma 3

For any $\delta > 0$,

$$\lim_{\eta \to 0} \limsup_{n \to \infty} \mathbb{P}_{\delta_x}(\exists w : \tfrac{1}{n^2}\nu(B_{4\delta n}(w, r)) < \eta | N_n > 0) = 0.$$

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$\square$

Assumptions

- γ and $x \mapsto \mathcal{E}_x[N^2]$ are both in $L^+_\infty(E)$.

- For all $x \in E$, the MBP becomes extinct $\mathbb{P}_{\delta_x}$-almost surely, i.e. setting $\zeta := \inf\{t > 0 : N_t = 0\}$, we have
$$\mathbb{P}_{\delta_x}(\zeta < \infty) = 1.$$

- There exist constants $C, M \in (0, \infty)$ such that for all $g \in L^+_\infty(E)$, setting $\mathcal{V}_M[g](x) := \mathcal{E}_x[\sum_{i \neq j} g(x_i)g(x_j)\mathbf{1}_{\{N \leq M\}}]$,
$$\langle \gamma \mathcal{V}_M[g], \tilde{\varphi} \rangle \geq C \langle g, \tilde{\varphi} \rangle^2.$$

Assumptions

- There exists a bounded function $L\varphi^2$ such that

$$\varphi^2(\xi_t) - \varphi^2(x) - \int_0^t L\varphi^2(\xi_s)ds$$

  is a $\mathbf{P}_x$-martingale.

- From the fact that φ is an eigenfunction, it follows that there exists a bounded function $L\varphi$ such that the above holds for φ.

- Then $\Gamma(\varphi) = L\varphi^2 - 2\varphi L\varphi$.

Thank you for your attention!