Neutron transport from a probabilistic perspective

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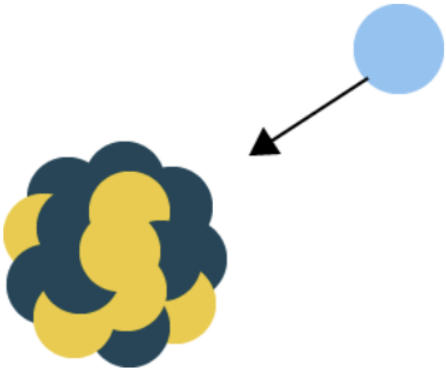
Contents

Neutron transport

- Neutron transport pertains to the study of the dynamics of neutrons and how they interact with surrounding material.
- Its origins date back to the 1800s when the Boltzmann equation was used to study the kinetic theory of gases.
- Large scale development of nuclear reactors during and after WWII caused an increase in focus on the mathematics of neutron transport.

Neutron transport



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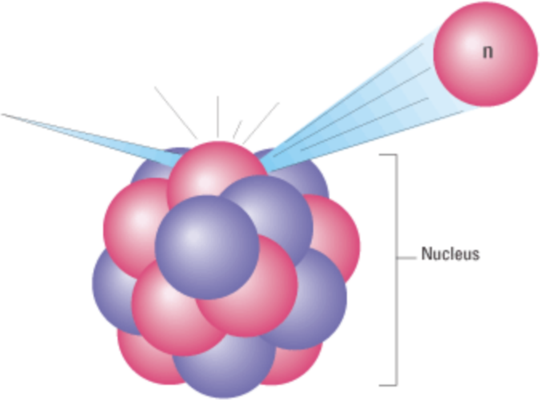

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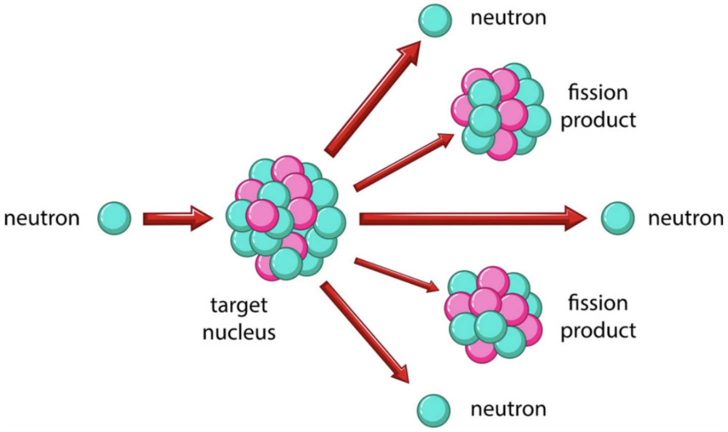

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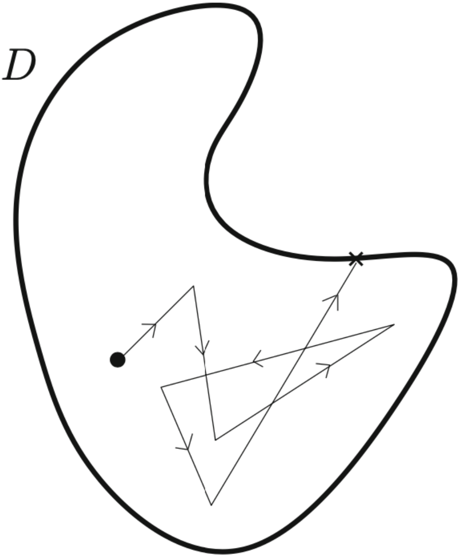

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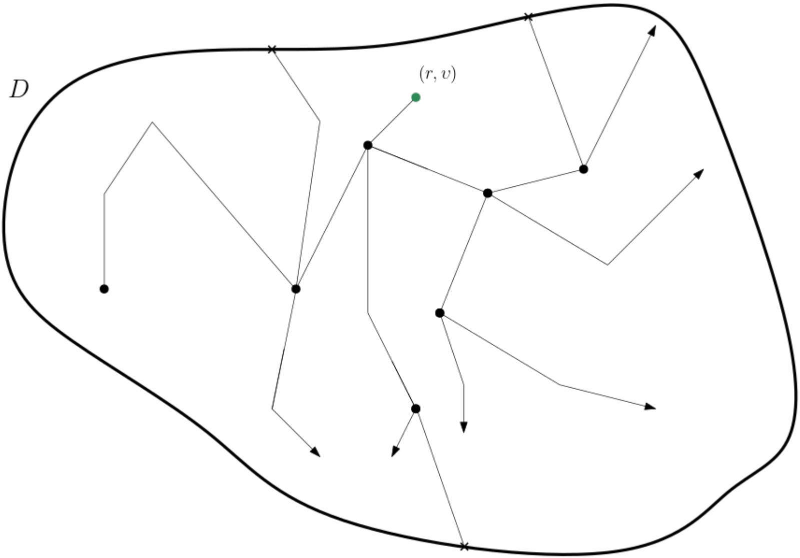
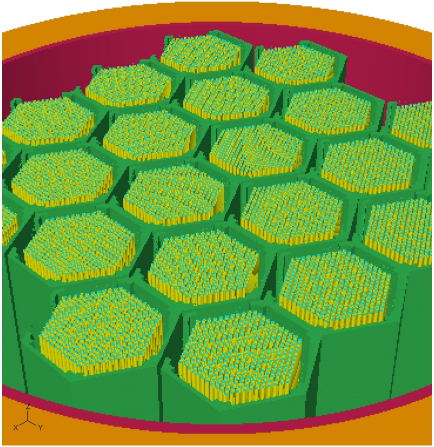
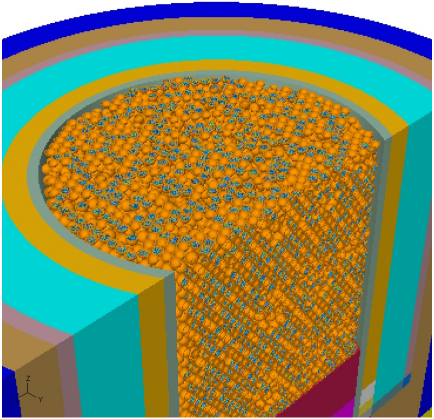


Neutron transport

- Reactor physics
  - Understanding neutron interactions with the material.
  - Mathematical modelling, simulations, thermal hydraulics, ...
- Shielding
  - What proportion of neutrons will reach a certain area of the reactor?
  - Rare event analysis, large deviations, sensitivity analysis, ...
- Decommissioning
  - Infer neutron sources from boundary readings.
  - Inverse problems, constrained optimisation, non-linear programming, ...
- Criticality
  - Asymptotic growth rate of number of neutrons and their spatial distribution.
  - PDEs, branching processes, stability analysis, ...

Neutron transport





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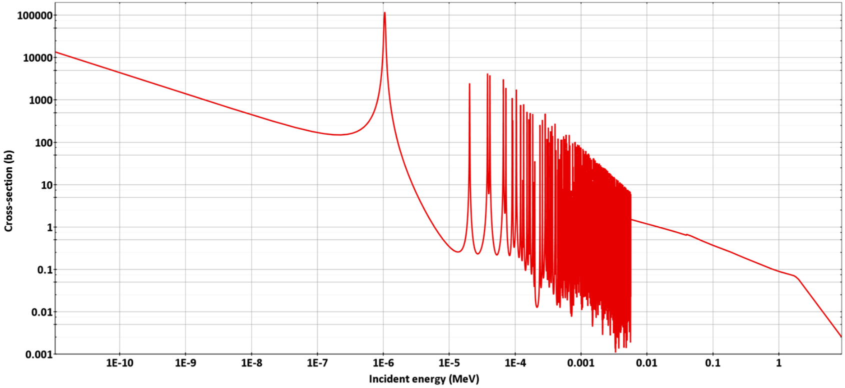

Neutron transport

- We want to use mathematics to come up with clever ways of analysing and simulating these systems.
- For the rest of the talk, we will focus on criticality problems.
- Our aim will be to use tools from probability theory to understand the asymptotic behaviour of neutrons undergoing nuclear fission.

Contents

Notation

- Let $D \subset \mathbb{R}^3$ be a open and bounded.
- Let $V := \{v \in \mathbb{R}^3 : v_{min} \le |v| \le v_{max}\}$, $0 < v_{min} \le v_{max} < \infty$.
- If a particle has label $(r, v) \in D \times V$, we mean that the particle is at position r with velocity v.
- Let $\psi_t(r, v)$ denote the density of neutrons at position r with velocity v at time $t \ge 0$.

Notation

- Let $\sigma_s(r, v)$ denote the rate at which a scattering event occurs for a particle with configuration (r, v). More precisely,

$$P_{(r_0,v_0)}(T_s > t) = \mathrm{e}^{-\int_0^t \sigma_s(r_0+v_0u, v_0)\mathrm{d}u}.$$

- Given a scattering event occurs at r with incoming velocity v, $\pi_{\mathtt{s}}(r, v, v')$ is the probability that the outgoing velocity is v'.

- $\sigma_{\mathtt{f}}(r, v)$ denotes the rate at which a fission event occurs.

- Given a fission event occurs at r with incoming velocity v, $\pi_{\mathtt{f}}(r, v, v')$ is the average number of neutrons produced in a fission event with outgoing new velocity v'.

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Neutron transport equation

$$
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\frac{\partial \psi_t}{\partial t}(r,v) &= v\cdot\nabla\psi_t(r,v)\\
&\quad + \sigma_{\mathtt{s}}(r,v)\left(\int_V \psi_t(r,v')\pi_{\mathtt{s}}(r,v,v')\mathrm{d}v' - \psi_t(r,v)\right)\\
&\quad + \sigma_{\mathtt{f}}(r,v)\left(\int_V \psi_t(r,v')\pi_{\mathtt{f}}(r,v,v')\mathrm{d}v' - \psi_t(r,v)\right),\\
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We also impose the following boundary and initial conditions:

- $\psi_t(r,v) = 0, \quad r \in \partial D, \mathbf{n}_r \cdot v > 0,$
- $\psi_0(r,v) = g(r,v)$

Eigenvalue problem

Want to fine $\lambda \in \mathbb{R}$ and a function $\varphi : D \times V \to [0, \infty)$ such that

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Eigenvalue problem

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- λ tells us about the asymptotic growth rate (per unit time) of the number of neutrons.
- φ tells us which regions are favourable.

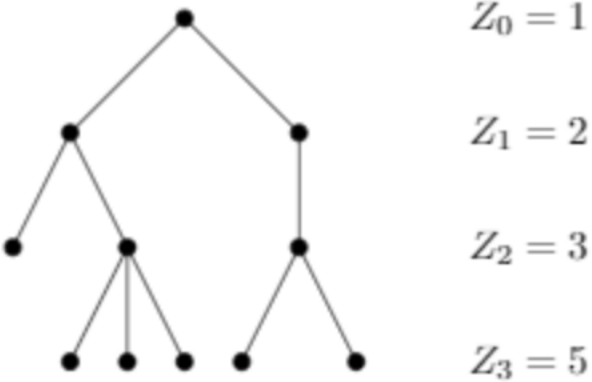
Contents

BGW process

- The simplest example of a branching process is the Bienaymé-Galton-Watson (BGW) process.
- The BGW process is a stochastic process $(Z_n)_{n\geq 0}$ where Z_n represents the number of individuals in the n-th generation of a randomly evolving population that undergoes asexual reproduction.
- The process evolves according to the recurrence formula: $Z_0 = 1$ and for $n \geq 0$,

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i^{(n)}, \qquad \xi_i^{(n)} \overset{i.i.d.}{\sim} \xi.$$

BGW process

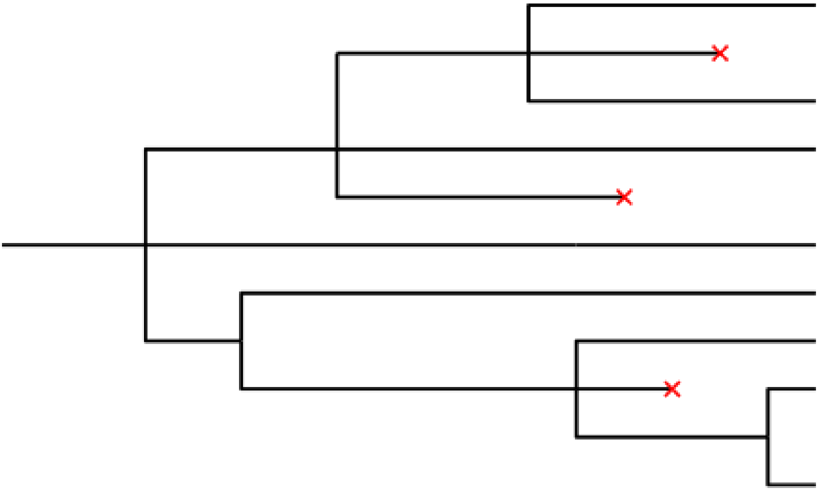


Continuous time BGW process

- The continuous time BGW process is a stochastic process $(Z_t)_{t\geq 0}$, where Z_t represents the number of individuals alive at time t in a randomly evolving population that undergoes asexual reproduction.

- With each individual, we associate an independent $Exp(1)$ random variable (clock). When an individual's clock rings, they give birth to a random number of individuals $\overset{i.i.d.}{\sim} \xi$.

BGW process



Neutron branching process

- Let $\{(r_i(t), v_i(t)) : i = 1, \dots, N_t\}$ denote the collection of particles alive at time $t \geq 0$ in $D \times V$.

- Define
$$\phi_t[g](r, v) = \mathbb{E}_{\delta_{(r,v)}}\left[\sum_{i=1}^{N_t} g(r_i(t), v_i(t))\right].$$

- E.g., taking $g = \mathbf{1}_{D\times V}$, $\phi_t[g](r, v) = \mathbb{E}_{\delta_{(r,v)}}[N_t]$.
Similarly taking $g = \mathbf{1}_{D_\epsilon \times V}$, $\phi_t[g](r, v) = \mathbb{E}_{\delta_{(r,v)}}[\#\text{ particles in } D_\epsilon]$.

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Neutron branching process

Theorem

$\phi_t[g](r, v)$ "solves" the neutron transport equation with initial condition $\psi_0 = g$, i.e.

$$\frac{\partial \phi_t}{\partial t}(r, v) = (T + S + F)\phi_t[g](r, v).$$

Neutron branching process

The proof uses generalisations of the following facts:

- If $T_i \sim Exp(\lambda_i)$, $i = 1, 2$ are independent, then $\min\{T_1, T_2\} \sim Exp(\lambda_1 + \lambda_2)$.
- $P(\min\{T_1, T_2\} = T_i) = \lambda_i/(\lambda_1 + \lambda_2)$.
- $\mathbb{E}[X] = \sum_i \mathbb{E}[X|A_i]\mathbb{P}(A_i)$.

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In our case,

- $T_1 =$ first scatter time, $T_2 =$ first fission time.
- X is replaced by $\sum_{i=1}^{N_t} f(r_i(t), v_i(t))$.
- $A_1 = \{\min\{T_1, T_2\} > t\}$.
- $A_2 = \{\min\{T_1, T_2\} \leq t, \min\{T_1, T_2\} = T_1\}$.
- $A_3 = \{\min\{T_1, T_2\} \leq t, \min\{T_1, T_2\} = T_2\}$.

Neutron branching process

- By showing ϕ_t "solves" the NTE, we've justified working with the branching process to inform us about solutions to the NTE.
- Why is this useful?
- Returning to the BGW process, if $m = E[\xi]$, we can see that $E[Z_n] = m^n$.
- From this, we have three cases: $m > 1$, $m = 1$ and $m < 1$.
- The continuous time version is $E[Z_t] = \mathrm{e}^{(m-1)t}$ and we have the same three cases.

Neutron branching process

Theorem

There exists $\lambda \in \mathbb{R}$ and a function $\varphi : D \times V \to [0, \infty)$ such that

$$\phi_t[\varphi](r, v) = \mathrm{e}^{\lambda t}\varphi(r, v)$$

and, for any bounded function g, there exists a constant $C_g > 0$ such that

$$\phi_t[g](r, v) \sim C_g \mathrm{e}^{\lambda t}\varphi(r, v), \qquad t \to \infty.$$

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Monte Carlo

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- Monte Carlo was first suggested by Stanisław Ulam who was attempting to calculate the probability of winning a game of solitaire.
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Monte Carlo for neutron transport

- Recall the asymptotic

$$\phi_t[g](r,v) \sim C_g \mathrm{e}^{\lambda t}\varphi(r,v).$$

- Setting $g = \mathbf{1}_{D\times V}$, we have $C_g = 1$ and hence

$$\mathbb{E}_{\delta_{(r,v)}}[N_t] \sim \mathrm{e}^{\lambda t}\varphi(r,v).$$

- Manipulating this gives

$$\lambda = \lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_{\delta_{(r,v)}}[N_t] \approx \frac{1}{T}\log\frac{1}{N}\sum_{i=1}^{N} N_T^{(i)}.$$

and

$$\varphi(r,v) = \lim_{t\to\infty}\mathrm{e}^{-\lambda t}\mathbb{E}_{\delta_{(r,v)}}[N_t] \approx \mathrm{e}^{-\hat{\lambda} t}\frac{1}{N}\sum_{i=1}^{N} N_T^{(i)}.$$

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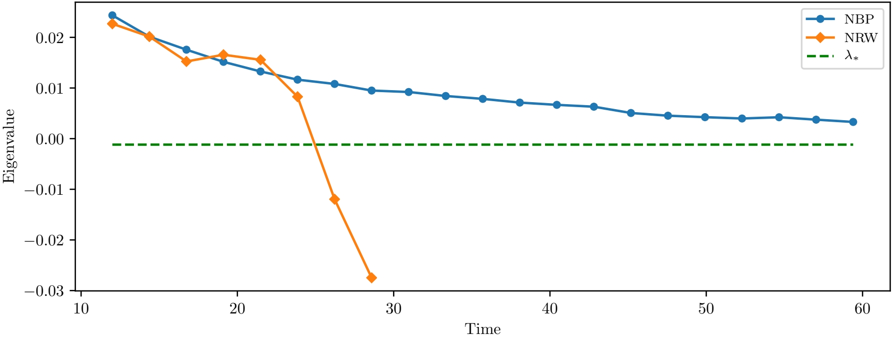
Monte Carlo for neutron transport

- $D = (-L, L)$, $V = \{-1, +1\}$.

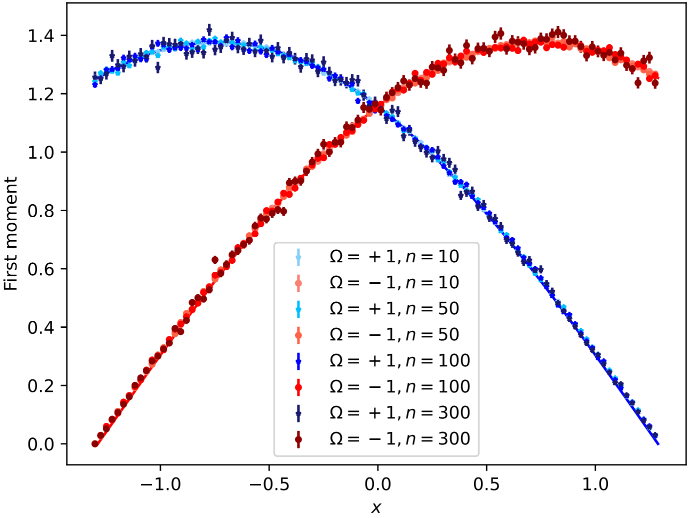
- Fission occurs at a constant rate:
  - 0 offspring w.p. p_0,
  - 2 offspring w.p. p_2, isotropic velocities.

- Scattering occurs at a constant rate and is isotropic.

Monte Carlo for neutron transport



Simulations



Other considerations

What about a left eigenfunction?

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Theorem

There exists $\lambda \in \mathbb{R}$ and functions φ, $\tilde{\varphi} : D \times V \to [0, \infty)$ such that

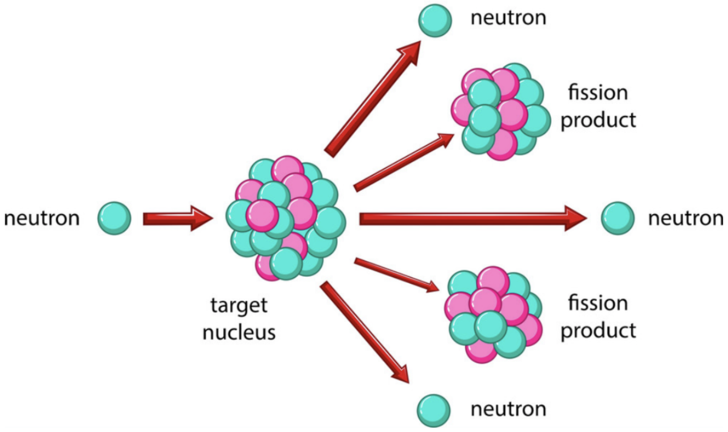
$$\phi_t[\varphi](r, v) = \mathrm{e}^{\lambda t}\varphi(r, v), \qquad \langle \tilde{\varphi}, \phi_t[g] \rangle = \mathrm{e}^{\lambda t}\langle \tilde{\varphi}, g \rangle,$$

and, for any bounded function g,

$$\phi_t[g](r, v) \sim \mathrm{e}^{\lambda t}\varphi(r, v)\langle g, \tilde{\varphi} \rangle, \qquad t \to \infty.$$

Here, $\langle f, g \rangle = \int_{D \times V} f(r, v)g(r, v)drdv$.

Other considerations



Other considerations

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- How does that apply in this setting? Why might that be useful?

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Other (related) application areas

- Radiotherapy
- Cosmic rays
- Fusion
- ...

Thank you!