Branching processes: from theory to application

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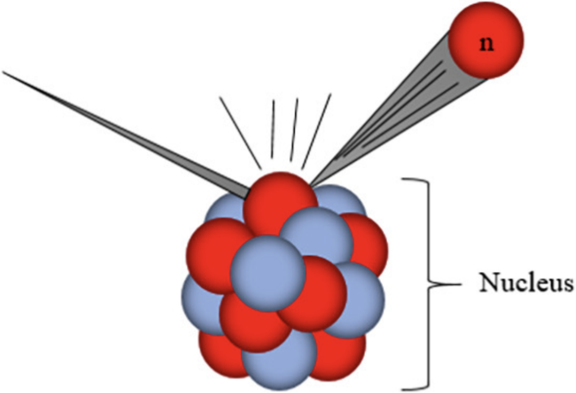
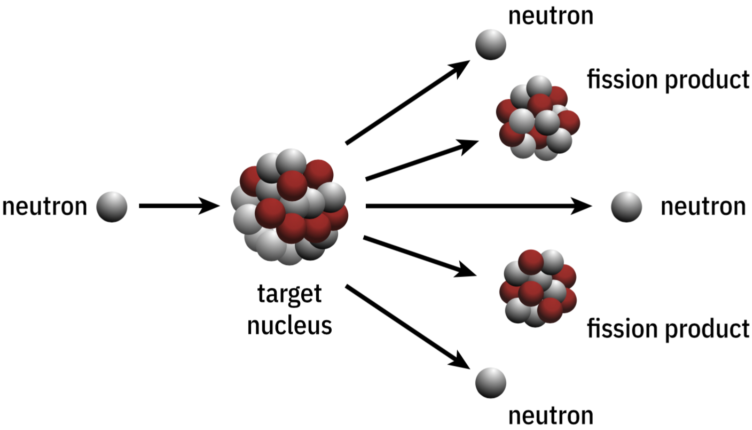
Motivating example: neutron transport



Motivating example: neutron transport



Motivating example: neutron transport

$$\begin{aligned}\frac{\partial \psi_t}{\partial t}(r,v) &= v\cdot\nabla\psi_t(r,v) - (\sigma_{\mathtt{s}}(r,v)+\sigma_{\mathtt{f}}(r,v))\psi_t(r,v)\\ &\quad + \sigma_{\mathtt{s}}(r,v)\int_V \psi_t(r,v')\pi_{\mathtt{s}}(r,v,v')\mathrm{d}v'\\ &\quad + \sigma_{\mathtt{f}}(r,v)\int_V \psi_t(r,v')\pi_{\mathtt{f}}(r,v,v')\mathrm{d}v',\\ &= (\mathtt{T}+\mathtt{S}+\mathtt{F})[\psi_t](r,v)\end{aligned}$$

where

$\sigma_{\mathtt{s}}(r,v)$: is the rate at which a neutron scatters,

$\sigma_{\mathtt{f}}(r,v)$: is the rate at which a fission event occurs,

$\pi_{\mathtt{s}}(r,v,v')$: is the probability a neutron with incoming velocity v scatters with new velocity v',

$\pi_{\mathtt{f}}(r,v,v')$: is the average number of neutrons produced in a fission event with new velocity v' from a neutron with incoming velocity v.

Motivating example: neutron transport

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\frac{\partial \psi_t}{\partial t}(r,v) &= v\cdot\nabla\psi_t(r,v) - (\sigma_{\mathtt{s}}(r,v)+\sigma_{\mathtt{f}}(r,v))\psi_t(r,v) \\
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Motivating example: neutron transport

$$\frac{\partial \psi_t}{\partial t}(r,v) = v\cdot\nabla\psi_t(r,v) - (\sigma_{\mathtt{s}}(r,v)+\sigma_{\mathtt{f}}(r,v))\psi_t(r,v)$$
$$+\sigma_{\mathtt{s}}(r,v)\int_V \psi_t(r,v')\pi_{\mathtt{s}}(r,v,v')\mathrm{d}v'$$
$$+\sigma_{\mathtt{f}}(r,v)\int_V \psi_t(r,v')\pi_{\mathtt{f}}(r,v,v')\mathrm{d}v',$$
$$= (\mathtt{T}+\mathtt{S}+\mathtt{F})[\psi_t](r,v)$$

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Motivating example: neutron transport

We also impose the following boundary and initial conditions:

- $\psi_t(r, v) = 0, \quad r \in \partial D, \mathbf{n}_r \cdot v > 0,$
- $\psi_0(r, v) = g(r, v)$

Motivating example: neutron transport

Aim

Find $\lambda \in \mathbb{R}$, $\varphi : D \times V \to [0, \infty)$ and a probability measure on η such that

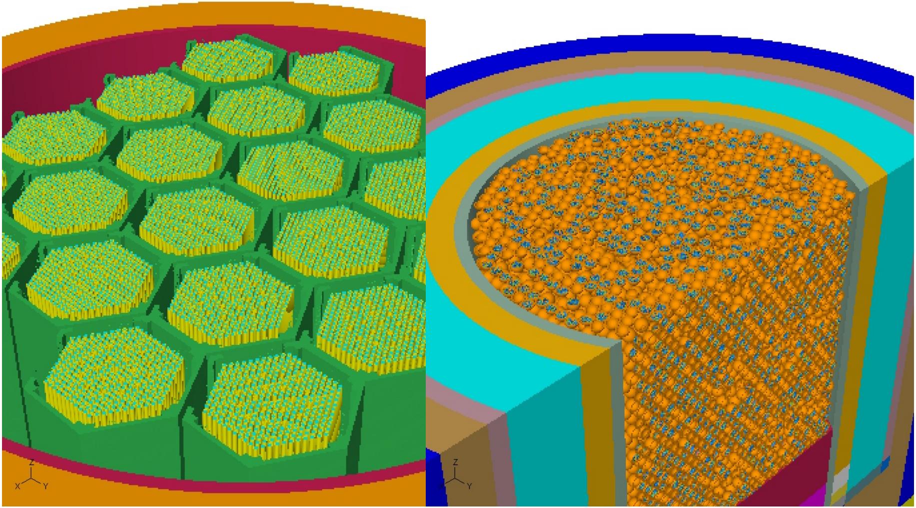
$$(\mathtt{T} + \mathtt{S} + \mathtt{F})\varphi = \lambda\varphi,$$

and

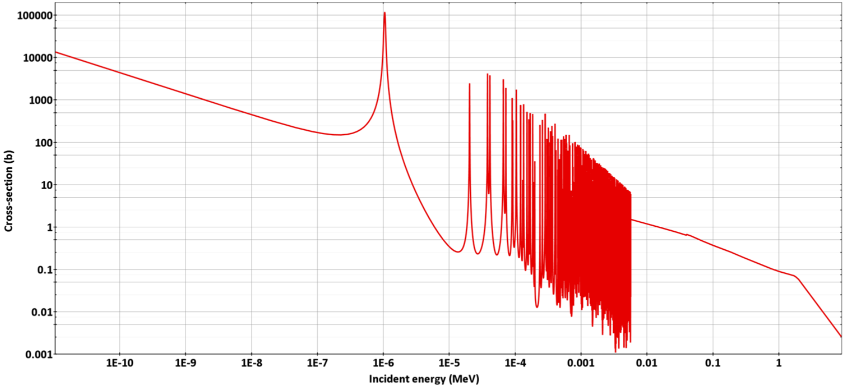
$$\langle \eta, (\mathtt{T} + \mathtt{S} + \mathtt{F})g \rangle = \lambda \langle \eta, g \rangle,$$

for suitable test functions $g : D \times V \to [0, \infty)$.

Motivating example: neutron transport



Motivating example: neutron transport



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Branching Markov processes

- Let $\{x_i(t) : i = 1, \dots, N_t\}$ denote the configuration of the system at time $t \geq 0$.
- The branching process is given by

$$X_t := \sum_{i=1}^{N_t} \delta_{x_i(t)}.$$

- The law of $(X_t)_{t\geq 0}$ is characterised via the non-linear semigroup

$$v_t[g](x) := \mathbb{E}_{\delta_x}\left[\mathrm{e}^{-X_t[g]}\right],$$

where

$$X_t[g] = \sum_{i=1}^{N_t} g(x_i(t)).$$

Branching Markov processes

- Markov semigroup:

$$\mathtt{P}_t[g](x) := \mathbf{E}_x[g(\xi_t)\mathbf{1}_{t<\tau}], \quad t \geq 0, x \in E, g \in B^+(E).$$

- Modified Markov semigroup:

$$\hat{\mathtt{P}}_t[g](x) = \begin{cases} \mathtt{P}_t[g](x), & t < \tau \\ 1, & t \geq \tau. \end{cases}$$

- Branching mechanism:

$$\mathtt{G}[g](x) = \beta(x)\mathcal{E}_x\left[\prod_{i=1}^{N} g(y_i) - g(x)\right], \quad x \in E, g \in B_1^+(E).$$

Branching Markov processes

Proposition

We have that $(v_t, t \geq 0)$ is the unique solution to

$$v_t[g](x) = \hat{\mathtt{P}}_t[\mathrm{e}^{-g}](x) + \int_0^t \mathtt{P}_s[\mathtt{G}[v_{t-s}]](x)\mathrm{d}s.$$

Branching Markov processes

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Differentiating with respect to t, we obtain the generator equation

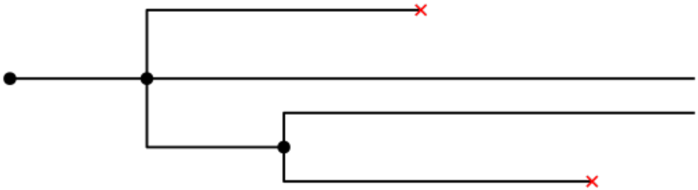
$$\frac{\partial}{\partial t} v_t[g](x) = \mathcal{L} v_t[g](x) + G[v_t[g]](x),$$

where $\mathcal{L}$ is the infinitesimal generator of the Markov process $(\xi, \mathbf{P})$ and $g \in \mathcal{D}(\mathcal{L})$.

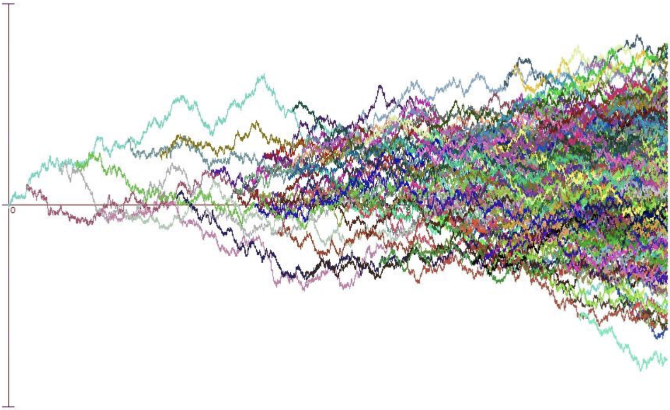
Example: continuous time Galton Watson process

- Non-linear semigroup: $v_t[\theta] = \mathbb{E}[\mathrm{e}^{-\theta N_t}]$
- Branching mechanism:

$$\mathtt{G}[\theta] = \mathcal{E}[\theta^N] - \theta = \sum_{k\geq 0} p_k \theta^k - \theta.$$



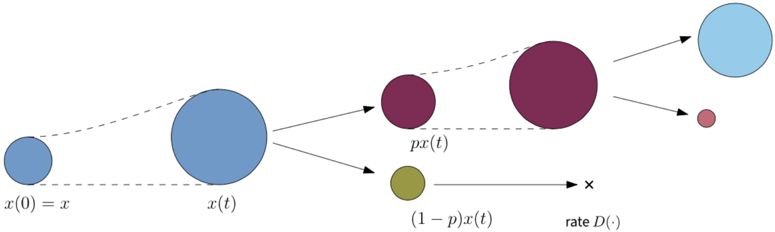
Example: BBM



$$\mathrm{P}_t[g](x) = \mathbf{E}_x[g(B_t)]$$

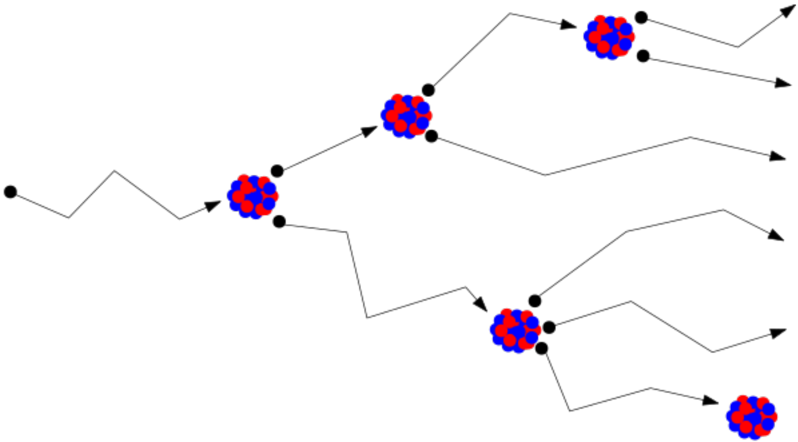
$$\frac{\partial}{\partial t} v_t[g](x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} v_t[g](x) + \beta v_t[g](x)(v_t[g](x) - 1)$$

Example: growth-fragmentation



$$\mathtt{P}_t[f](x) = \mathbf{E}_x[f(x(t))]$$
$$\mathtt{G}[f](x) = B(x)\left(\mathcal{E}_x\left[f(x(1-p))f(xp)\right] - f(x)\right)$$

Example: neutron transport



Branching Markov processes

We are also interested in the linear semigroup

$$\psi_t[g](x) := \mathbb{E}_{\delta_x}\left[X_t[g]\right],$$

which is the unique solution to

$$\psi_t[g](x) = \mathtt{P}_t[g](x) + \int_0^t \mathtt{P}_s[\mathtt{F}[\psi_{t-s}]](x)\mathrm{d}s,$$

where

$$\mathtt{F}[g](x) = \beta(x)\mathcal{E}_x\left[\sum_{i=1}^N g(y_i) - g(x)\right], \quad x \in E, g \in B^+(E).$$

For convenience, we will define

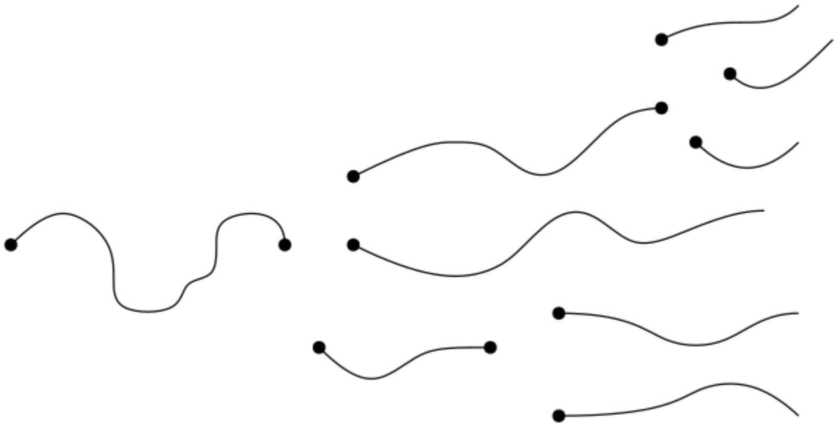
$$m[g](x) := \mathcal{E}_x[\mathcal{Z}[g]] = \mathcal{E}_x\left[\sum_{i=1}^N g(y_i)\right]$$

Many-to-one

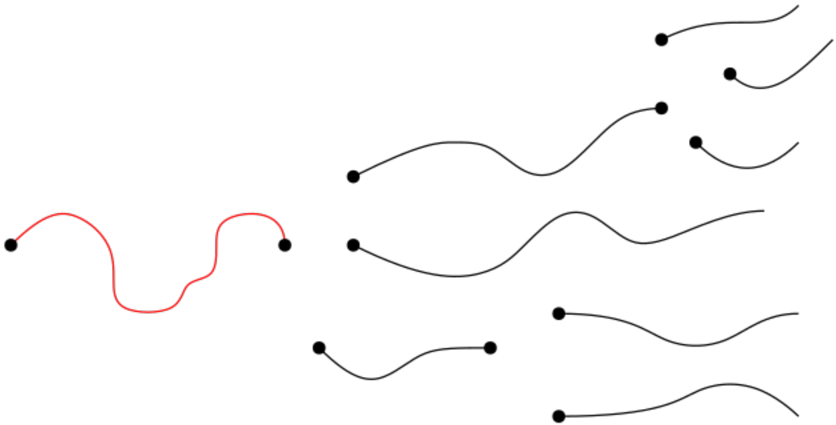
Let us consider the process $\hat{\xi}$, described as follows.

- From an initial position $x \in E$, $\hat{\xi}$ evolves as ξ.
- When at $y \in E$, at rate $\beta(y)m[1](y)$ the process is sent to a new position in E.
- The new position lies in $A \subset E$ with probability $m[\mathbf{1}_A](y)/m[1](y)$.

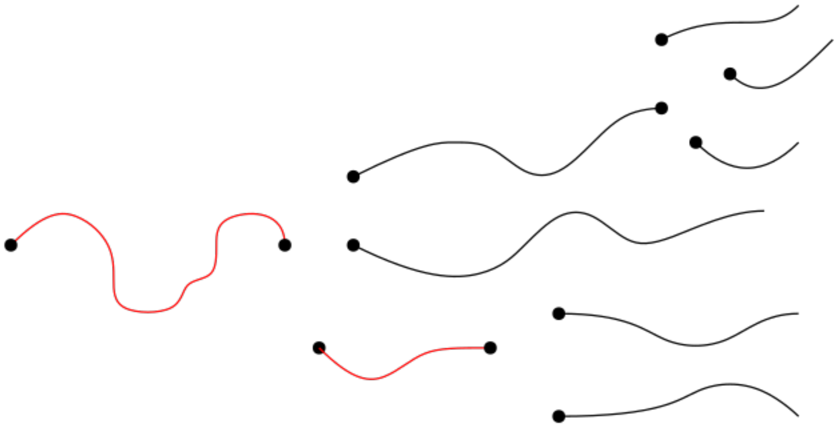
Many-to-one



Many-to-one



Many-to-one



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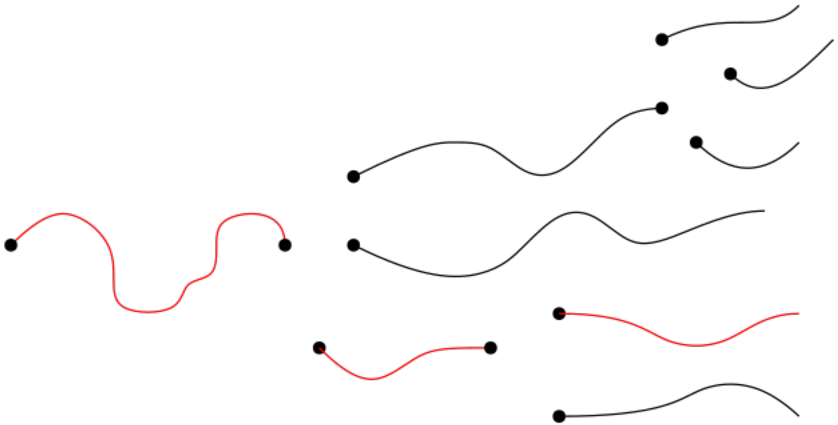

Many-to-one

Many-to-one lemma

Suppose that $m[1] \in B^+(E)$. Then for $x \in E$, $t \geq 0$, $g \in B^+(E)$, we have

$$\psi_t[g](x) = \hat{\mathbf{E}}_x\left[\mathrm{e}^{\int_0^t \gamma(\hat{\xi}_s)\mathrm{d}s} g(\hat{\xi}_t)\right],$$

where $\gamma(\hat{\xi}_s) = \beta(\hat{\xi}_s)(m[1](\hat{\xi}_s) - 1)$.

Many-to-one

- Continuous time GW process: $\mathbb{E}[N_t] = \mathrm{e}^{(m-1)t}$.
- BBM: $\mathbb{E}_{\delta_x}[X_t[f]] = \mathrm{e}^{\beta t}\mathbf{E}_x[f(B_t)]$.
- Growth-fragmentation: $\mathbb{E}_{\delta_x}[X_t[f]] = \mathbf{E}_x\left[\mathrm{e}^{\int_0^t B(Y_s)\mathrm{d}s} f(Y_t)\right]$.
- Neutron transport: $\mathbb{E}_{\delta_x}[X_t[f]] = \mathbf{E}_x\left[\mathrm{e}^{\int_0^t \sigma_f(R_s,\Upsilon_s)(m[1](R_s,\Upsilon_s)-1)\mathrm{d}s} f(R_t,\Upsilon_t)\right]$.

Contents

A Perron Frobenius decomposition

- Consider the case where $(\xi_t, t \geq 0)$ is a continuous time Markov chain on $E = \{1, \dots, n\}$.

- Let $p_t(i,j) = \mathbf{P}_i(\xi_t = j) \quad \Rightarrow \quad \mathtt{P}_t[g](i) = \mathbf{E}_i[g(\xi_t)] = \sum_{j=1}^{n} p_t(i,j)g(j)$

- Assuming the chain is irreducible, Perron Frobenius theory tells us that there exist $\lambda_c \leq 0$ and vectors φ, η such that

$$\mathtt{P}_t[\varphi] = \mathrm{e}^{\lambda_c t}\varphi, \quad \eta[P_t[g]] = \mathrm{e}^{\lambda_c t}\eta[g],$$

and

$$p_t(i,j) \sim \mathrm{e}^{\lambda_c t}\varphi(i)\eta(j) + o(\mathrm{e}^{\lambda_c t}), \quad t \to \infty.$$

A Perron-Frobenius decomposition

Similarly, we would like to find

- $\lambda_* \in \mathbb{R}$,
- a positive function $\varphi \in B^+(E)$,
- a probability measure η on E

such that

$$\psi_t[\varphi] = \mathrm{e}^{\lambda_* t}\varphi, \quad \eta[\psi_t[g]] = \mathrm{e}^{\lambda_* t}\eta[g],$$

and

$$\psi_t[g](x) \sim \mathrm{e}^{\lambda_* t}\varphi(x)\eta[g], \qquad \text{as } t \to \infty.$$

A Perron Frobenius decomposition

Returning to the Markov chain example:

- If the chain is conservative, then $\lambda_c = 0$. Thus $\eta[P_t[g]] = \eta[g]$ and hence η is the stationary distribution.
- If the chain is non-conservative, then $\lambda_c < 0$. In this case, η is called the quasi-stationary distribution (QSD).

QSDs

- Let $(Y_t)_{t\geq 0}$ be a time-homogeneous Markov process on $E \cup \{\partial\}$ with probabilities $(\mathbf{P}^{\dagger}_x, x \in E)$ and semigroup $(\mathtt{P}^{\dagger}_t)_{t\geq 0}$.

- Assume that $\tau_\partial := \inf\{t > 0 : X_t = \partial\} < \infty$, $\mathbf{P}^{\dagger}_x$-almost surely for all $x \in E$.

- Assume further that for all $x \in E$, $\mathbf{P}^{\dagger}_x(t < \tau_\partial) > 0$.

QSDs

Definition

A **quasi-stationary distribution** (QSD) is a probability measure η on E such that

$$\eta = \lim_{t \to \infty} \mathbf{P}^{\dagger}_{\mu}(X_t \in \cdot | t < \tau_{\partial})$$

for some initial probability measure μ on E.

QSDs

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A **quasi-stationary distribution** (QSD) is a probability measure η on E such that

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Proposition

A probability measure η is a QSD if and only if, for any $t \geq 0$,

$$\eta = \mathbf{P}^{\dagger}_{\eta}(Y_t \in \cdot | t < \tau_{\partial}).$$

QSDs

- Méléard, S., & Villemonais, D. (2012). Quasi-stationary distributions and population processes.
- van Doorn, E. A., & Pollett, P. K. (2011). Quasi-stationary distributions. Memorandum 1945.
- Collet, P., Martínez, S., & San Martín, J. (2013). Quasi-stationary distributions: Markov chains, diffusions and dynamical systems (Vol. 1). Berlin: Springer.
- Works of Champagnat & Villemonais.

QSDs

Assumption A

There exists a probability measure ν on E such that

(A1) there exists $t_0, c_1 > 0$ such that for all $x \in E$,

$$\mathbf{P}^{\dagger}_x(Y_{t_0} \in \cdot | t_0 < \tau_\partial) \geq c_1 \nu(\cdot);$$

(A2) there exists $c_2 > 0$ such that for all $x \in E$ and $t \geq 0$,

$$\mathbf{P}^{\dagger}_\nu(t < \tau_\partial) \geq c_2 \mathbf{P}^{\dagger}_x(t < \tau_\partial).$$

QSDs

Theorem (Champagnat, Villemonais)

Under Assumption A, there exists a probability measure η on E and two constants $C, \epsilon > 0$ such that, for all $x \in E$,

$$\|\mathbf{P}^{\dagger}_x(Y_t \in \cdot | t < \tau_\partial) - \eta(\cdot)\|_{TV} \leq C\mathrm{e}^{-\epsilon t}, t \geq 0.$$

In this case, η is the unique QSD for the process.

Proposition

If η is a QSD then there exists $\lambda_c < 0$ such that, for all $t \geq 0$,

$$\mathbf{P}^{\dagger}_\eta(t < \tau_\partial) = \mathrm{e}^{\lambda_c t}, \quad \eta[\mathrm{P}^{\dagger}_t[g]] = \mathrm{e}^{\lambda_c t}\eta[g].$$

QSDs

Proposition (Champagnat, Villemonais)

There exists a non-negative function φ on $E \cup \{\partial\}$, positive on E and vanishing on ∂, defined by

$$\varphi(x) = \lim_{t\to\infty} \mathrm{e}^{-\lambda_c t}\mathbf{P}^{\dagger}_x(t < \tau_\partial),$$

where the convergence holds for the uniform norm on $E \cup \{\partial\}$ and $\eta[\varphi] = 1$. Moreover, φ is bounded and

$$\mathrm{P}^{\dagger}_t[\varphi] = \mathrm{e}^{\lambda_c t}\varphi.$$

A Perron Frobenius decomposition

- Define

$$\bar{\gamma} := \sup_{x\in E} \gamma(x) = \sup_{x\in E} \beta(x)(m[1](x) - 1).$$

- Let us introduce the semigroup $\psi^\dagger$ via

$$\begin{aligned}
\psi^\dagger_t[g](x) &:= \mathrm{e}^{-\bar{\gamma}t}\psi_t[g](x) \\
&= \hat{\mathbf{E}}_x\left[\mathrm{e}^{\int_0^t \gamma(\hat{\xi}_s)-\bar{\gamma}\mathrm{d}s} g(\hat{\xi}_t)\right] \\
&= \hat{\mathbf{E}}_x\left[g(\hat{\xi}_t)\mathbf{1}_{t<\kappa}\right] \\
&=: \mathbf{E}^\dagger_x[g(\hat{\xi}_t)],
\end{aligned}$$

where

$$\kappa := \inf\{t > 0 : \int_0^t \bar{\gamma} - \gamma(\hat{\xi}_s)\mathrm{d}s > \mathbf{e}\}.$$

QSDs

- Then, under Assumption A, we have

$$\psi_t^\dagger[\varphi](x) = \mathrm{e}^{\lambda_c t}\varphi(x), \quad \eta[\psi_t^\dagger[g]] = \mathrm{e}^{\lambda_c t}\eta[g]$$

and, for any $t \geq 0$,

$$\|\mathbf{P}_x^\dagger(\hat{\xi}_t \in \cdot | t < \tau_\partial) - \eta(\cdot)\| \leq C\mathrm{e}^{-\epsilon t}.$$

- Since $\varphi(x) = \lim_{t\to\infty} \mathrm{e}^{-\lambda_c t}\mathbf{P}_x^\dagger(t < \tau_\partial)$, it follows that

$$\sup_{x\in E, g\in B_1^+(E)} |\mathrm{e}^{-\lambda_c t}\varphi(x)^{-1}\psi_t^\dagger[g] - \eta[g]| \leq C\mathrm{e}^{-\epsilon t}.$$

- Since $\psi_t = \mathrm{e}^{\bar{\gamma} t}\psi_t^\dagger$, the same conclusion then holds for ψ_t with λ_c replaced by $\lambda_* = \lambda_c + \bar{\gamma}$.

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QSDs

Assumption A

There exists a probability measure ν on E such that

(A1) there exists $t_0, c_1 > 0$ such that for all $x \in E$,

$$\mathbf{P}^\dagger_x(Y_{t_0} \in \cdot | t_0 < \tau_\partial) \geq c_1 \nu(\cdot);$$

(A2) there exists $c_2 > 0$ such that for all $x \in E$ and $t \geq 0$,

$$\mathbf{P}^\dagger_\nu(t < \tau_D) \geq c_2 \mathbf{P}^\dagger_x(t < \tau_\partial).$$

- Under mild assumptions on the cross-sections and the domain, (A1) and (A2) are satisfied for the NTE.
- Birth-death processes.
- "Processes that come down from infinity".

QSDs

Assumption F

There exist $\gamma_1, \gamma_2, c_1, c_2, c_3, t_1 > 0$, a measurable function $\psi_1 : E \to [1, \infty)$, and a probability measure ν on a measurable subset $L \subset E$ such that

(F1) For all $x \in L$

$$\mathbf{P}^\dagger_x(X_{t_1} \in \cdot) \geq c_1 \nu(\cdot \cap L) \quad \text{and} \quad \sup_{t \in \mathbb{R}_+} \frac{\sup_{y \in L} \mathbf{P}^\dagger_y(t < \tau)}{\inf_{y \in L} \mathbf{P}^\dagger_y(t < \tau)} \leq c_2.$$

(F2) We have $\gamma_1 < \gamma_2$ and

$$\mathcal{L}\psi_1(x) \leq -\gamma_1 \psi_1(x) + c_3 \mathbf{1}_L(x), \quad x \in E$$
$$\gamma_2^{-t} \mathbf{P}^\dagger_x(X_t \in L) \to \infty \text{ as } t \to \infty, \text{ for all } x \in L.$$

QSDs

- Under Assumption F, Champagnat and Villemonais prove the existence of and convergence towards a QSD but the convergence is not uniform.
- In this case, there may be an infinite number of QSDs.
- The result captures the existence of the **minimal** QSD.

Recap

- Branching process $(X_t, t \geq 0)$.
- Non-linear semigroup: $\mathbb{E}_{\delta_x}[\mathrm{e}^{-X_t[f]}]$.
- Linear semigroup: $\mathbb{E}_{\delta_x}[X_t[f]] = \hat{\mathbf{E}}_x\left[\mathrm{e}^{\int_0^t \beta(\hat{\xi}_s)(m[1](\hat{\xi}_s)-1)\mathrm{d}s} f(\hat{\xi}_t)\right]$.
- The linear semigroup $(\psi_t, t \geq 0)$ is the unique solution to

$$\psi_t[g](x) = \mathtt{P}_t[g](x) + \int_0^t \mathtt{P}_s[\mathtt{F}[\psi_{t-s}]](x)\mathrm{d}s,$$

Recap

We will assume that $\beta, m[1] \in B^+(E)$ and that Assumption A holds.

Then we have

$$\psi_t[\varphi] = \mathrm{e}^{\lambda_* t}\varphi, \quad \eta[\psi_t[g]] = \mathrm{e}^{\lambda_* t}\eta[g]$$

and

$$\psi_t[g](x) \sim \mathrm{e}^{\lambda_* t}\varphi(x)\eta[g], \quad t \to \infty.$$

Spine decomposition

- The branching property and the fact that

$$\mathbb{E}_{\delta_x}[X_t[\varphi]] = \mathrm{e}^{\lambda_* t}\varphi(x),$$

imply that

$$W_t^1 := \mathrm{e}^{-\lambda_* t}\frac{X_t[\varphi]}{\varphi(x)}, \quad t \geq 0,$$

is a unit mean $\mathbb{P}_{\delta_x}$-martingale.

- Thus, we can define the change of measure

$$\left.\frac{\mathbb{P}^{\varphi}_{\delta_x}}{\mathbb{P}_{\delta_x}}\right|_{\mathcal{F}_t} := W_t^1, \quad t \geq 0, x \in E,$$

i.e. $\mathbb{P}^{\varphi}_{\delta_x}(A) = \mathbb{E}_{\delta_x}[\mathbf{1}_A W_t^1]$.

Spine decomposition

Under $\mathbb{P}^{\varphi}$, the branching process X can be constructed as follows.

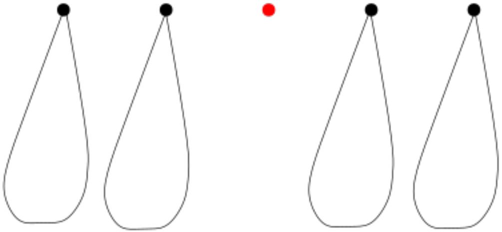
Spine decomposition

Under $\mathbb{P}^{\varphi}$, the branching process X can be constructed as follows.

1. From the initial configuration $\mu = \sum_{i=1}^{n} \delta_{x_i}$, the i^*-th individual is selected with probability $\varphi(x_{i^*})/\mu[\varphi]$ and marked the *spine*.

Spine decomposition

2. The individuals $j \neq i^*$ in the initial configuration each issue independent copies of $(X, \mathbb{P}_{\delta_{x_j}})$ respectively.



Spine decomposition

3. The marked individual, "spine", issues a single particle whose motion is determined by the semigroup

$$\mathtt{S}_t[f](x) := \mathbf{E}_x\left[\mathrm{e}^{\int_0^t \beta(\xi_s)\left(\frac{m[\varphi(\hat{\xi}_s)]}{\varphi(\hat{\xi}_s)} - 1\right)\mathrm{d}s} \frac{\varphi(\xi_t)}{\varphi(x)} f(\xi_t) \right] \qquad x \in E,\ f \in B^+(E).$$

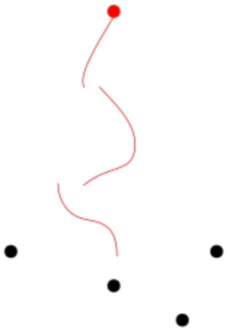
Spine decomposition

4. When at $x \in E$, the spine undergoes branching at rate

$$\rho(x) := \beta(x)\frac{m[\varphi](x)}{\varphi(x)}$$

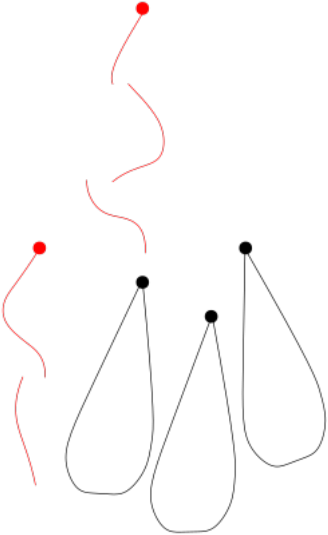
at which point, it produces particles according $(\mathcal{Z}, \mathcal{P}_x^{\varphi})$, where

$$\frac{\mathrm{d}\mathcal{P}_x^{\varphi}}{\mathrm{d}\mathcal{P}_x} = \frac{\mathcal{Z}[\varphi]}{m[\varphi](x)}.$$



Spine decomposition

5. Given $\mathcal{Z}$ from the previous step, μ is redefined as $\mu = \mathcal{Z}$ and Step 1 is repeated.



Spine decomposition

- From the many-to-one lemma,

$$\hat{\mathbf{E}}_x\left[\mathrm{e}^{\int_0^t \gamma(\hat{\xi}_s)\mathrm{d}s}\varphi(\hat{\xi}_t)\right] = \mathrm{e}^{\lambda_* t}\varphi(x).$$

- It follows that

$$W_t^2 := \mathrm{e}^{-\lambda_* t + \int_0^t \gamma(\hat{\xi}_s)\mathrm{d}s}\frac{\varphi(\hat{\xi}_t)}{\varphi(x)}, \quad t \geq 0.$$

  is a unit mean $\hat{\mathbf{P}}_x$-martingale.

- Thus, we can define a second change of measure

$$\left.\frac{\mathrm{d}\mathbf{P}_x^{\varphi}}{\mathrm{d}\hat{\mathbf{P}}_x}\right|_{\mathcal{G}_t} := W_t^2, \quad t \geq 0, x \in E.$$

Spine decomposition

Ergodicity of the spine

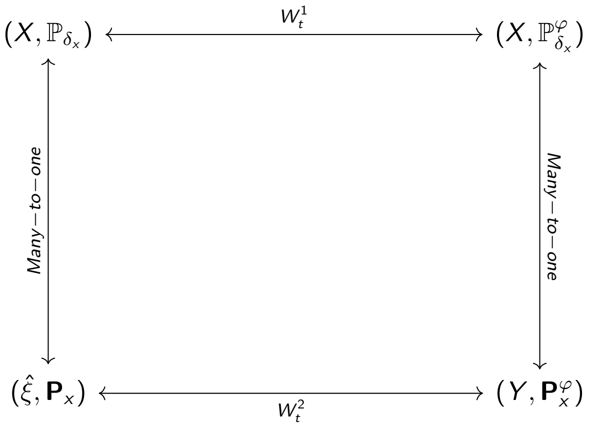
The spine process is equal in law to $(\hat{\xi}, \mathbf{P}^{\varphi})$. The semigroup $(\mathtt{P}^{\varphi}_t,\ t \geq 0)$ associated to $(\hat{\xi}, \mathbf{P}^{\varphi})$ is conservative, and satisfies

$$\mathtt{P}^{\varphi}_t[f](x) = \frac{\mathrm{e}^{-\lambda_* t}}{\varphi(x)} \psi_t[\varphi f], \qquad t \geq 0,\ f \in B^+(E),$$

with stationary distribution

$$\varphi(x)\eta(\mathrm{d}x), \qquad x \in E.$$

Spine decomposition



Q-process

Theorem (Champagnat, Villemonais)

Under Assumption A, the following three properties hold.

- There exists a family $(\mathbf{Q}_x)_{x\in E}$ of probability measures defined by

$$\lim_{t\to\infty} \mathbf{P}^\dagger_x(A|t<\tau) = \mathbf{Q}_x(A).$$

  The process $(\xi, \mathbf{Q})$ is an E-valued homogeneous Markov process. If, in addition, ξ is strong Markov under $\mathbf{P}^\dagger$ then it is also strong Markov under $\mathbf{Q}$.

- Letting $(\mathtt{Q}_t)_{t\geq 0}$ denote the semigroup of $(\xi, \mathbf{Q})$, we have

$$\mathtt{Q}_t[g](x) = \frac{\mathrm{e}^{-\lambda_c t}}{\varphi(x)}\mathtt{P}^\dagger_t[\varphi g](x).$$

- The probability measure on E given by $\varphi(x)\eta(\mathrm{d}x)$ is the unique invariant distribution of ξ under $\mathbf{Q}$.

Contents

Criticality

From the Perron Frobenius decomposition, we have

$$\psi_t[g](x) \sim \mathrm{e}^{\lambda_* t}\varphi(x)\eta[g], \quad t \to \infty.$$

- Subcritical: if $\lambda_* < 0$, the average mass decays at rate $-\lambda_*$.
- Critical: if $\lambda_* = 0$, the average mass remains constant.
- Supercritical: if $\lambda_* > 0$, the average mass in the system grows at rate λ_*.

Criticality

Define

$$\zeta := \inf\{t > 0 : N_t = 0\}.$$

- Subcritical: $\{\zeta < \infty\}$ almost surely.
- Critical: $\{\zeta < \infty\}$ almost surely.
- Supercritical: $\{\zeta = \infty\}$ with positive probability.

Criticality

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Yaglom limit for BGW processes

- Suppose $(Z_n)_{n\geq 0}$ is a BGW process,

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i, \qquad \xi_i \sim^{\text{iid}} \xi.$$

- Assume $\mathbb{E}[\xi] = 1$ so that the process is critical.
- Further assume that $\sigma^2 := \mathbb{E}[\xi^2] - \mathbb{E}[\xi] < \infty$.

- Kolmogorow limit (Kolmogorov '38):

$$\lim_{n\to\infty} n\mathbb{P}(Z_n > 0) = \frac{2}{\sigma^2}$$

- Yaglom limit (Yaglom '48):

$$\lim_{n\to\infty} \mathbb{E}\left[\exp\left(-\theta\frac{Z_n}{n}\right) \Big| Z_n > 0\right] = \frac{1}{1+\theta\sigma^2/2}.$$

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Yaglom limit for BBM on a compact domain

- Let $D \subset \mathbb{R}^d$ be compact.
- Let $(X_t)_{t\geq 0}$ denote a branching process where, given their point of creation, particles move independently according to a diffusion with generator L. Particles are killed on ∂D and at rate $\beta > 0$, they branch into a random number of particles with distribution A.

- Let λ denote the first eigenvalue of $-L$ on D.
- Assume $m := \mathbb{E}[A] > 1$, $\mathbb{E}[A^2] < \infty$ and $\lambda = \beta(m-1)$.

- Kolmogorov result (Powell '19):

$$\lim_{t\to\infty} t\mathbb{P}_x(N_t > 0) = C_1(x).$$

- Yaglom limit (Powell '19):

$$\lim_{t\to\infty} \mathbb{E}_x\left[\exp\left(-\frac{\theta}{t}\sum_{i=1}^{N_t} f(X_t^i)\right)\Big| N_t > 0\right] = \frac{1}{1+\theta C_2(f)}.$$

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General setting

Define

$$\mathcal{V}[g](x) := \mathcal{E}_x\left[\sum_{\substack{i,j=1\\ i\neq j}}^{N} g(y_i)g(y_j)\right], \quad x \in E, f \in B^+(E).$$

and

$$\Sigma = \eta[\beta\mathcal{V}[\varphi]].$$

Theorem

Under certain assumptions, we have

$$\lim_{t\to\infty}\sup_{x\in E}\left|\frac{t\mathbb{P}_{\delta_x}(N_t > 0)}{\varphi(x)} - \frac{2}{\Sigma}\right| = 0.$$

“Certain assumptions”

- For all t sufficiently large, $\sup_{x \in E} \mathbb{P}_{\delta_x}(N_t > 0) < 1$.
- There exists a constant $C > 0$ such that for all $g \in B^+(E)$,
$$\eta[\beta \mathcal{V}[g]] \geq C\eta[g]^2.$$
- The number of offspring produced at a branching event is bounded above by a constant n_{max}.

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General setting

Theorem (Yaglom limit)

Under the same assumptions, for each $f \in B^+(E)$,

$$\left(\frac{X_t[f]}{t} \,\middle|\, N_t > 0 \right) \to Y, \quad \text{as } t \to \infty,$$

in distribution, where Y is an exponential random variable with mean $\eta[f]\Sigma/2$.

Attempt 1



Method of moments

- Know that exponential distribution is characterised by its moments.
- Let us consider the moments of $t^{-1}X_t[f]$ under $\mathbb{P}_{\delta_x}(\cdot|N_t > 0)$:

$$\mathbb{E}_{\delta_x}\left[\left(\frac{X_t[f]}{t}\right)^k \middle| N_t > 0\right] = \frac{\frac{1}{t^{k-1}}\mathbb{E}_{\delta_x}[X_t[f]^k \mathbf{1}_{N_t>0}]}{t\mathbb{P}_{\delta_x}(N_t > 0)}$$

- If $f = \varphi$, we have

$$\mathbb{E}_{\delta_x}\left[\left(\frac{X_t[\varphi]}{t}\right)^k \middle| N_t > 0\right] = \frac{\frac{1}{t^{k-1}}\mathbb{E}_{\delta_x}[X_t[\varphi]^k \mathbf{1}_{N_t>0}]}{t\mathbb{P}_{\delta_x}(N_t > 0)} = \frac{\frac{\varphi(x)}{t^{k-1}}\mathbb{E}^{\varphi}_{\delta_x}[X_t[\varphi]^{k-1}]}{t\mathbb{P}_{\delta_x}(N_t > 0)}$$

Method of moments

The spine decomposition means that under the measure $\mathbb{E}^{\varphi}$, we may write

$$\frac{X_t[\varphi]}{t} = \frac{\varphi(\hat{\xi}_t)}{t} + \frac{1}{t}\sum_{i=1}^{n_t} \Xi_i(\hat{\xi}_{\mathbf{u}_t^i}, t - \mathbf{u}_t^i),$$

where the $\Xi_i(x, u)$ are independent and equal in law to

$$\sum_{\substack{j=1 \\ j\neq i^*}}^{N^i} X^j_{t-u}[\varphi] \quad \text{under} \quad \eta_x^{\varphi} := \mathcal{P}_x^{\varphi} \bigotimes_{\substack{j=1 \\ j\neq i^*}} \mathbb{P}_{\delta_{x_j}}.$$

Method of moments

Recall, for homogeneous Poisson processes

- The order of the arrivals is not important.
- Positions of events are uniformly distributed.

Method of moments

Recall, for homogeneous Poisson processes

- The order of the arrivals is not important.
- Positions of events are uniformly distributed.

Similarly, in this case, conditional on n_t and $\hat{\xi}$, the $\mathbf{u_t^i}$ are i.i.d with law

$$P_{(t,\hat{\xi})}(\mathbf{u}_t \in \mathrm{d}s) = \frac{\rho(\hat{\xi}_s)}{\int_0^t \rho(\hat{\xi}_s)\mathrm{d}s}.$$

Method of moments

Since the spine is ergodic, we have

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\mathbb{E}^{\varphi}_{\delta_x}[X_t[\varphi]] &= \lim_{t\to\infty}\mathbb{E}^{\varphi}_{\delta_x}\left[\frac{1}{t}\sum_{i=1}^{n_t}\eta^{\varphi}_{\hat{\xi}_{\mathbf{u}^i_t}}[\Xi_i(\hat{\xi}_{\mathbf{u}^i_t}, t-\mathbf{u}^i_t)]\right]\\ &= \lim_{t\to\infty}\mathbb{E}^{\varphi}_{\delta_x}\left[\frac{n_t}{t}\frac{\int_0^t\rho(\hat{\xi}_s)\eta^{\varphi}_{\hat{\xi}_s}[\Xi(\hat{\xi}_s, t-s)]\mathrm{d}s}{\int_0^t\rho(\hat{\xi}_s)\mathrm{d}s}\right]\\ &= \lim_{t\to\infty}\mathbf{E}^{\varphi}_{x}\left[\frac{1}{t}\int_0^t\rho(\hat{\xi}_s)\eta^{\varphi}_{\hat{\xi}_s}[\Xi(\hat{\xi}_s, t-s)]\mathrm{d}s\right]\\ &= \eta\left[\beta\mathcal{V}[\varphi]\right],\end{aligned}$$

where we recall that $\mathcal{V}[h](x) = \mathcal{E}_x\left[\sum_{\substack{i,j=1\\ i\neq j}}^{N} h(x_i)h(x_j)\right]$.

Method of moments

Proceed by induction:

$$\frac{1}{t^k}\mathbb{E}^{\varphi}_{\delta_x}\left[\left(\sum_{i=1}^{n_t}\Xi(\hat{\xi}_{\mathbf{u}^i_t}, t-\mathbf{u}^i_t)\right)^k\right]$$
$$=\frac{1}{t^k}\mathbb{E}^{\varphi}_{\delta_x}\left[\sum_{j=1}^{k}2^j\binom{n_t}{j}\mathbf{1}_{j\le n_t}\sum_{[k_1,\dots,k_j]_+}\binom{k}{k_1,\dots,k_j}\prod_{i=1}^{j}\eta^{\varphi}_{\hat{\xi}_{\mathbf{u}^i_t}}[\Xi(\hat{\xi}_{\mathbf{u}^i_t}, t-\mathbf{u}^i_t)^{k_i}]\right],$$

where $[k_1,\dots,k_j]_+$ is the set of all strictly positive $\{k_1,\dots,k_j\}$ that sum to k.

Method of moments

Theorem (Harris, H., Kyprianou, Wang)

Suppose that for some $k \geq 1$, $\sup_{x\in E} \mathcal{E}_x[\mathcal{Z}[1]^{k+1}] < \infty$. Then, for all $j \leq k$,

$$\lim_{t\to\infty} \sup_{x\in E} \left| \frac{1}{t^j} \mathbb{E}^{\varphi}_{\delta_x}[X_t[\varphi]^j] - (j+1)! \left(\frac{\Sigma}{2}\right)^j \right| = 0.$$

Yaglom limit

Then

$$
\begin{aligned}
\lim_{t\to\infty}\mathbb{E}_{\delta_x}\left[\left(\frac{X_t[\varphi]}{t}\right)^k \middle| N_t>0\right] &= \lim_{t\to\infty}\frac{\frac{1}{t^{k-1}}\mathbb{E}_{\delta_x}[X_t[\varphi]^k \mathbf{1}_{N_t>0}]}{t\mathbb{P}_{\delta_x}(N_t>0)}\\
&= \lim_{t\to\infty}\frac{\frac{\varphi(x)}{t^{k-1}}\mathbb{E}^{\varphi}_{\delta_x}[X_t[\varphi]^{k-1}]}{t\mathbb{P}_{\delta_x}(N_t>0)}\\
&= \frac{\varphi(x)k!(\Sigma/2)^{k-1}}{\varphi(x)2/\Sigma}\\
&= k!\left(\frac{\Sigma}{2}\right)^k.
\end{aligned}
$$

Yaglom limit

- For general f, write $f = f - \eta[f]\varphi + \eta[f]\varphi =: \tilde{f} + \eta[f]\varphi$.
- From the previous steps, if follows that replacing φ by $\eta[f]\varphi$ yields the correct result.
- To conclude, we show that $X_t[\tilde{f}]/t \to 0$ weakly under $\mathbb{P}_{\delta_x}(\cdot|N_t > 0)$.

Attempt 2




Attempt 2

- Proof of survival probability remains the same $\rightsquigarrow$ still need the n_{max} assumption ☹

- For the Yaglom result, recall that

$$\mathbb{E}_{\delta_x}\left[\left(\frac{X_t[f]}{t}\right)^k \middle| N_t > 0\right] = \frac{\frac{1}{t^{k-1}}\mathbb{E}_{\delta_x}[X_t[f]^k \mathbf{1}_{N_t>0}]}{t\mathbb{P}_{\delta_x}(N_t > 0)}$$

Method of moments #2

- Set $\psi_t^{(k)}[f](x) = \mathbb{E}_{\delta_x}[X_t[f]^k]$. Note that $\psi_t^{(1)} = \psi_t$.

- Our objective is to show that for $k \geq 2$, $f \in B^+(E)$ and $x \in E$,

$$\lim_{t\to\infty} g_k(t)\psi_t^{(k)}[f](x) = C_k(x, f),$$

  where $g_k(t)$ and $C_k(x, f)$ can be identified explicitly.

- The key is to notice that

$$\psi_t^{(k)}[f](x) = (-1)^k \frac{\partial^k}{\partial\theta^k}\mathbb{E}_{\delta_x}[\mathrm{e}^{-\theta X_t[f]}]\Big|_{\theta=0} = (-1)^k \frac{\partial^k}{\partial\theta^k} v_t[\theta f](x)\Big|_{\theta=0}.$$

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Method of moments #2

- Recall the evolution equation for $\mathtt{v}_t[f](x) = \mathbb{E}_{\delta_x}[\mathrm{e}^{-X_t[f]}]$:

$$\mathtt{v}_t[f](x) = \hat{\mathtt{P}}_t[\mathrm{e}^{-f}](x) + \int_0^t \mathtt{P}_s\left[\mathtt{G}[\mathtt{v}_{t-s}[f]]\right](x)\mathrm{d}s,$$

where $\hat{\mathtt{P}}_t[f](x) = \mathbb{E}_x[f(\xi_{t\wedge\tau_\partial})]$.

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where $\hat{\mathtt{P}}_t[f](x) = \mathbb{E}_x[f(\xi_{t\wedge\tau_\partial})]$.

- However, this is not the right evolution equation to work with.

Method of moments #2

Set

$$\mathtt{u}_t[f](x) = 1 - \mathtt{v}_t[f](x), \qquad t \geq 0,$$

and

$$\mathcal{A}[f](x) = \beta(x)\mathcal{E}_x\left[\prod_{i=1}^{N}(1 - f(x_i)) - 1 + \sum_{i=1}^{N} f(x_i)\right].$$

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Lemma

For all $x \in E$ and $t \geq 0$, $\mathrm{u}_t[g](x)$ satisfies

$$\mathrm{u}_t[g](x) = \psi_t[1 - \mathrm{e}^{-g}](x) - \int_0^t \psi_s\left[\mathcal{A}[\mathrm{u}_{t-s}[g]]\right](x)\mathrm{d}s.$$

Method of moments #2

Assume that $\sup_{x\in E}\mathcal{E}_x[\mathcal{Z}[1]^k] < \infty$. Then

$$\psi_t^{(k)}[f](x) = \psi_t[f^k](x) + \int_0^t \psi_s\left[\beta\eta_{t-s}^{(k-1)}[f]\right](x)\,\mathrm{d}s, \qquad t\geq 0, \tag{1}$$

where

$$\eta_{t-s}^{(k-1)}[f](x) = \mathcal{E}_x\left[\sum_{[k_1,\dots,k_N]_k^2}\binom{k}{k_1,\dots,k_N}\prod_{j:k_j>0}\psi_{t-s}^{(k_j)}[f](x_j)\right],$$

and $[k_1,\dots,k_N]_k^2$ is the set of all non-negative N-tuples $(k_1,\dots,k_N)$ such that $\sum_{i=1}^N k_i = k$ and at least two of the k_i are strictly positive.

Method of moments #2

Theorem (with Gonzalez Garcia & Kyprianou)

Assume that $\sup_{x\in E}\mathcal{E}_x[\mathcal{Z}[1]^k] < \infty$. Define

$$\Delta_t^{(\ell)} = \sup_{x\in E, f\in B^+(E)} \left| t^{-(\ell-1)}\varphi(x)^{-1}\psi_t^{(\ell)}[f](x) - \ell!\,\eta[f]^\ell(\Sigma/2)^{\ell-1}\right|,$$

Then, for all $\ell \le k$ and $\varepsilon > 0$

$$\sup_{t\ge\varepsilon}\Delta_t^{(\ell)} < \infty \text{ and } \lim_{t\to\infty}\Delta_t^{(\ell)} = 0.$$

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$$\sup_{t\ge\varepsilon} \Delta_t^{(\ell)} < \infty \text{ and } \lim_{t\to\infty} \Delta_t^{(\ell)} = 0.$$

i.e. $\psi_t^{(\ell)}[f](x) \sim t^{\ell-1} \ell! \, \varphi(x) \eta[f]^\ell (\Sigma/2)^{\ell-1}$

Recap

Theorem (Kolmogorov survival probability)

We have

$$\lim_{t\to\infty}\sup_{x\in E}\left|\frac{t\mathbb{P}_{\delta_x}(N_t>0)}{\varphi(x)}-\frac{2}{\Sigma}\right|=0.$$

Theorem (Yaglom limit)

For each $f\in B^+(E)$,

$$\left(\frac{X_t[f]}{t}\,\middle|\,N_t>0\right)\to Y,\qquad \text{as } t\to\infty,$$

in distribution, where Y is an exponential random variable with mean $\eta[f]\Sigma/2$.

Recap

- Method of moments: show that

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Attempt 3

New assumptions

- $\beta, m[1] \in B^+(E)$
- Assumption A holds and $\lambda_* = 0$.
- For t sufficiently large, $\sup_{x\in E} \mathbb{P}_{\delta_x}(N_t > 0) < 1$.
- There exist constants $C, M \in (0,\infty)$ such that $\forall g \in B^+(E)$,

$$\eta[\gamma \mathcal{V}_M[g]]\rangle \geq C\eta[g]^2,$$

where

$$\mathcal{V}_M[g](x) := \mathcal{E}_x[\sum_{i\neq j} g(x_i)g(x_j)\mathbf{1}_{\{N\leq M\}}].$$

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Attempt 3

- Idea: work directly with the Laplace transform

$$\mathbb{E}_{\delta_x}[\mathrm{e}^{-\theta X_t[\varphi]} | N_t > 0] = \frac{\mathbb{E}_{\delta_x}[\mathrm{e}^{-\theta X_t[\varphi]}\mathbf{1}_{N_t>0}]}{\mathbb{P}_{\delta_x}(N_t > 0)}$$

- $1 - \mathbb{E}_{\delta_x}[\mathrm{e}^{-\theta X_t[\varphi]}]$ and $\mathbb{P}_{\delta_x}(N_t > 0) = 1 - \mathbb{P}_{\delta_x}(N_t = 0)$ are both solutions to

$$\mathrm{u}_t[g](x) = \psi_t[1 - \mathrm{e}^{-g}](x) - \int_0^t \psi_s\left[\mathcal{A}[\mathrm{u}_{t-s}[g]]\right](x)\mathrm{d}s,$$

where

$$\mathcal{A}[f](x) = \beta(x)\mathcal{E}_x\left[\prod_{i=1}^{N}(1 - f(x_i)) - 1 + \sum_{i=1}^{N} f(x_i)\right].$$

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Magical explanation:

$$\eta_x^{\varphi}\Bigg[\sum_{\substack{j=1\\ j\neq i^*}}^{N} X_{t-u}^{j}[\varphi]\Bigg] = \mathcal{E}_x^{\varphi}\Bigg[\sum_{\substack{j=1\\ j\neq i^*}}^{N} \mathbb{E}_{\delta_{x_j}}[X_{t-u}^{j}[\varphi]]\Bigg]$$

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$$= \mathcal{E}_x\Bigg[\frac{\mathcal{Z}[\varphi]}{\mathcal{E}_x[\mathcal{Z}[\varphi]]}\sum_{k=1}^{N}\frac{\varphi(x_k)}{\mathcal{Z}[\varphi]}\sum_{\substack{j=1\\ j\neq k}}^{N} \mathbb{E}_{\delta_{x_j}}[X^j_{t-u}[\varphi]]\Bigg]$$

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$$= \frac{1}{\mathcal{E}_x[\mathcal{Z}[\varphi]]}\mathcal{E}_x\left[\sum_{k=1}^{N}\varphi(x_j)\sum_{\substack{j=1\\ j\neq k}}^{N}\varphi(x_j)\right].$$

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Analytic explanation:

$$t^{-k}\psi_t^{(k+1)}[f](x)$$
$$= t^{-k}\int_0^t \psi_s\left[\mathcal{E}_\cdot\left[\sum_{[k_1,\dots,k_N]^2_{k+1}}\binom{k+1}{k_1,\dots,k_N}\prod_{j:k_j>0}\psi_{t-s}^{(k_j)}[f](x_j)\right]\right](x)\mathrm{d}s$$

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&= \frac{1}{t}\int_0^t \psi_s\left[\mathcal{E}_\cdot\left[\sum_{[k_1,\dots,k_N]^2_{k+1}}\binom{k+1}{k_1,\dots,k_N}\frac{(t-s)^{k+1-\#\{j:k_j>0\}}}{t^{k-1}}\prod_{j:k_j>0}\frac{\psi_{t-s}^{(k_j)}[f](x_j)}{(t-s)^{k_j-1}}\right]\right](x)\mathrm{d}s
\end{aligned}
$$

Why the 2nd moments?

Probabilistic explanation: asymptotically, two children of the MRCA, each with at least 1 descendant alive at time t.

Recall the operator

$$\begin{aligned}\mathcal{A}[h](x) &= \beta(x)\mathcal{E}_x\left[1 - \prod_{i=1}^{N}(1 - h(x_i)) - \sum_{i=1}^{N} h(x_i)\right] \\ &= \beta(x)\mathcal{E}_x\left[\sum_{i\neq j} h(x_i)h(x_j) - \dots\right] \\ &= V[h](x) + h.o.t\end{aligned}$$

Why the exponential distribution?

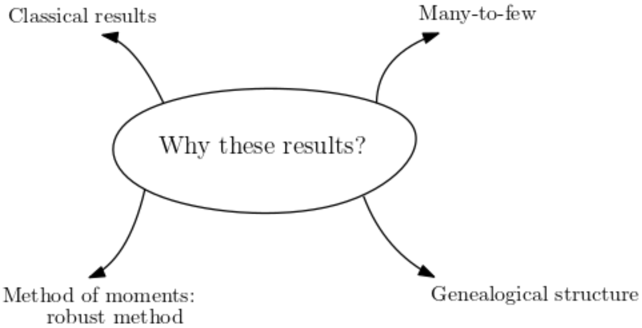
- There are asymptotically two children of the MRCA, each with at least 1 descendant alive at time t.
- Distribution of the time of the MRCA of the particles alive at time t is uniform.
- Therefore, under $\mathbb{P}_{\delta_x}(\cdot|N_t > 0)$,

$$\frac{X_t}{t} \approx U\left(\frac{X^{(1)}_{Ut}}{Ut} + \frac{X^{(2)}_{Ut}}{Ut}\right).$$

Literature

- Galton Watson processes: Kolmogorov '38, Yaglom '48, Kesten et. al. '66, Lyons et. al. '95, Geiger '99, Geiger '00, Vatutin et. al. '01, Ren et. al. '18.
- Spatial branching processes: Powell '19, Harris et. al. '22, Horton & Powell '24+.
- Superprocesses: Ren et. al. '19.
- Random/varying environment: Cardona-Tobòn & Palau '23.

Yaglom and Kolmogorov results



Robustness of the method of moments

- Define the occupation measure

$$\int_0^t X_s(\cdot)\mathrm{d}s, \qquad t \geq 0.$$

- Then, as $t \to \infty$,

$$\mathbb{E}_{\delta_x}\left[\left(\int_0^t X_s[g]\mathrm{d}s\right)^k\right] \sim t^{2k-1} C_k(x, g)$$

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Robustness of the method of moments

Consider now a Markov process $X := (X_t)_{t\geq 0}$ the space of *finite measures* on E, with probabilities $\mathbb{P} := (\mathbb{P}_\mu, \mu \in M(E))$ and transition semigroup

$$\mathbb{E}_\mu\left[\mathrm{e}^{-X_t[f]}\right] = \mathrm{e}^{-\mu[\mathtt{V}_t[f]]},$$

where

$$\mathtt{V}_t[f](x) = \mathtt{P}_t[f](x) - \int_0^t \mathtt{P}_s\left[\psi(\cdot, \mathtt{V}_{t-s}[f](\cdot)) + \phi(\cdot, \mathtt{V}_{t-s}[f])\right](x)\mathrm{d}s.$$

Method of moments

- E. Dumonteil and A. Mazzolo. Residence times of branching diffusion processes.
- S. Durham. Limit theorems for a general critical branching process.
- J. Fleischman. Limiting distributions for branching random fields.
- I. Iscoe. On the supports of measure-valued critical branching Brownian motion.
- A. Klenke. Multiple scale analysis of clusters in spatial branching models.

Many-to-few

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Recall the moment evolution equation:

$$\psi_t^{(k)}[f](x) = \psi_t[f^k](x) + \int_0^t \psi_s \left[\beta \eta_{t-s}^{(k-1)}[f]\right](x) \, \mathrm{d}s.$$

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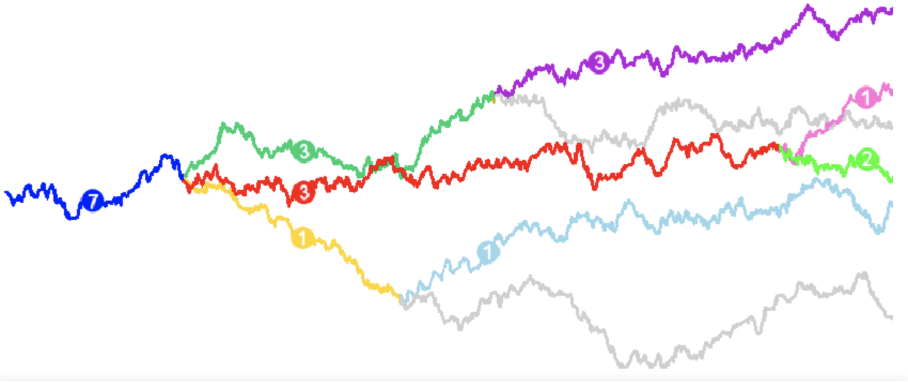
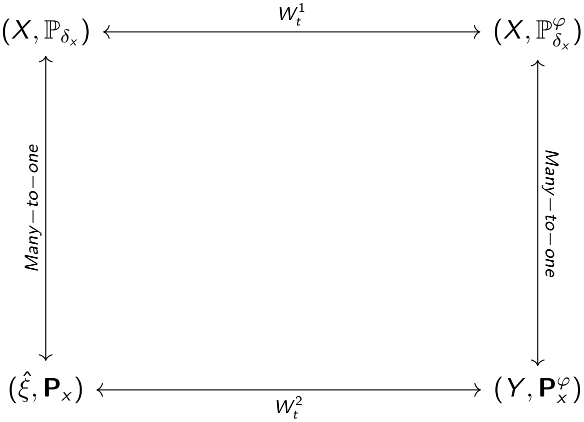


Many-to-few



$(X, \mathbb{P}_{\delta_x})$
W_t^1
$(X, \mathbb{P}^{\varphi}_{\delta_x})$
Many—to—one
Many—to—one
$(\hat{\xi}, \mathbf{P}_x)$
W_t^2
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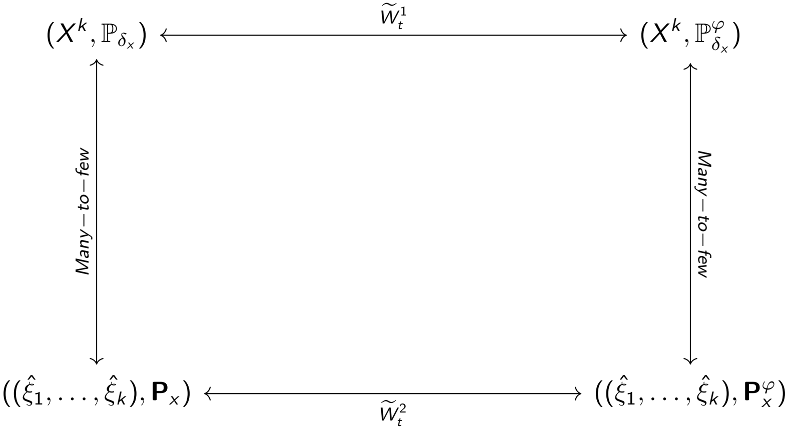
Many-to-few


$(X^k, \mathbb{P}_{\delta_x})$
$\widetilde{W}_t^1$
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Many − to − few
Many − to − few
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$((\hat{\xi}_1, \ldots, \hat{\xi}_k), \mathbf{P}^{\varphi}_x)$

Genealogical structure: ancestral trees

- Let $(X, \mathbb{P})$ denote a Markov branching process.
- Let $T > 0$. On the event $\{N_T \geq k\}$, choose k distinct particles $U_1, \ldots, U_k$ uniformly from those alive at time T.
- What does the ancestral tree formed from these k particles look like?

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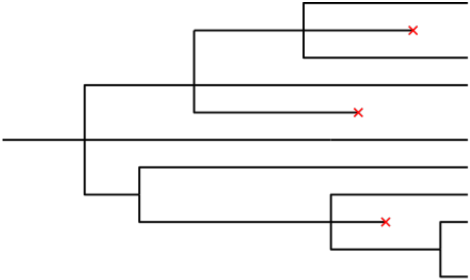
Ancestral trees



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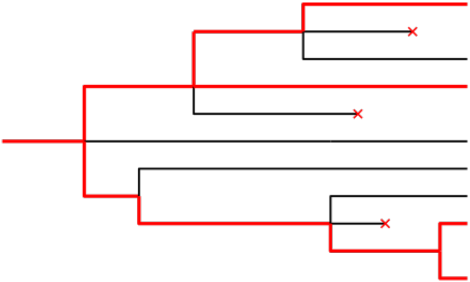

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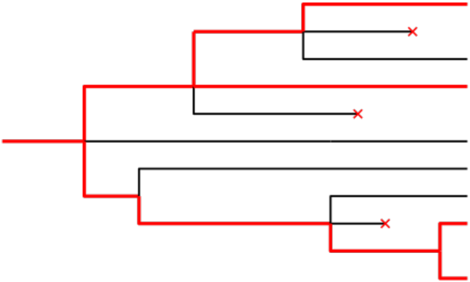

Equivalently, define the equivalence relation $\sim_t$ on $\{1, \ldots, k\}$ by

$$i \sim_t j \quad \Leftrightarrow \quad U_i \text{ and } U_j \text{ share a common ancestor alive at time } t.$$

Let $\pi_t^{k,T}$ denote the random partition of $\{1, \ldots, k\}$ corresponding to this equivalence relation.

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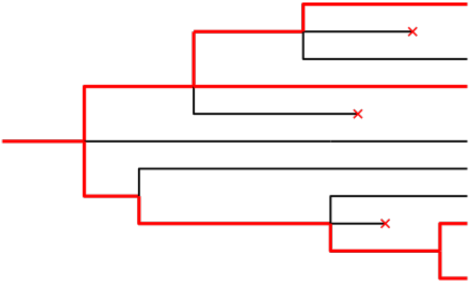

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What is the law of $(\pi_t^{k,T})_{t \geq 0}$ conditional on $N_T \geq k$?

Ancestral trees

Consider a continuous time Galton Watson with offspring distribution L.

Ancestral trees

Consider a continuous time Galton Watson with offspring distribution L.

Theorem (Lambert '03)

On $\{N_T \geq 2\}$, pick two distinct particles, uniformly from those alive at time T. Let τ denote the time of their most recent common ancestor (MRCA). Then

$$\mathbb{P}(\tau \in [t, T], N_T \geq 2) = \int_0^1 (1-s)\frac{F''_{T-t}(s)}{F'_{T-t}(s)} F''_T(s)\mathrm{d}s,$$

where $F_t(s) = \mathbb{E}[s^{N_t}]$.

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Theorem (Zubkov '76)

If $m = \mathbb{E}[L] = 1$, then conditioned on $N_T \geq 2$,

$$\frac{\tau}{T} \to \tau^C \in [0,1],$$

in distribution, as $T \to \infty$.

Ancestral trees

Theorem 3.1. *For any mesh* $(t_i)_{i\leq n}$*, and any chain of partitions* $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$ *of* $\{1, \ldots, k\}$*,*

$$\mathbb{P}(\pi_{t_1}^{k,L,T} = \gamma_1, \ldots, \pi_{t_n}^{k,L,T} = \gamma_n,\ N_T \geq k) = \int_0^1 \frac{(1-s)^{k-1}}{(k-1)!} \prod_{i=0}^{n} \prod_{\Gamma \in \gamma_i} F_{\Delta t_i}^{b_i(\Gamma)} \left(F_{T-t_{i+1}}(s)\right) ds, \quad (3.6)$$

where $\Delta t_i = t_{i+1} - t_i$.

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where $\Delta t_i = t_{i+1} - t_i$.

Theorem 3.6. *There exists a universal stochastic process* $(\bar{\pi}^{k,\mathrm{crit}}_t)_{t\in[0,1]}$ *such that for any tree with* $m = 1$ *and* $f''(1) < \infty$*, the process* $(\pi^{k,L,T}_{Tt})_{t\in[0,1]}$ *conditioned on* $\{N_T \ge k\}$ *converges in distribution to* $(\bar{\pi}^{k,\mathrm{crit}}_t)_{t\in[0,1]}$ *as* $T \to \infty$*. Moreover, the finite dimensional distributions of* $(\bar{\pi}^{k,\mathrm{crit}}_t)_{t\in[0,1]}$ *are given by*

$$\mathbb{P}(\bar{\pi}^{k,\mathrm{crit}}_{t_1} = \gamma_1, \ldots, \bar{\pi}^{k,\mathrm{crit}}_{t_n} = \gamma_n) \quad (3.14)$$

$$= \prod_{i=0}^{n} \prod_{\Gamma\in\gamma_i} b_i(\Gamma)! \int_0^\infty \frac{\theta^{k-1}}{(k-1)!} \prod_{i=0}^{n} (\Delta t_i)^{|\gamma_{i+1}|-|\gamma_i|} \left(\frac{1+(1-t_{i+1})\theta}{1+(1-t_i)\theta}\right)^{|\gamma_{i+1}|} d\theta. \quad (3.15)$$

Ancestral trees

This result was presented in a more general setting in Harris, Johnston, Roberts (2019).

- The coalescent process obtained is topologically equivalent to Kingman's coalescent but with different coalescent rates.
- The $k-1$ split times of $(\bar{\pi}_t^{k,crit})_{t\in[0,1]}$ have joint pdf

$$P(u_1,\dots,u_{k-1}) = k\int_0^\infty \frac{\theta^{k-1}}{(1+\theta)^2}\prod_{i=1}^{k-1}\frac{1}{(1+\theta(1-u_i))^2}\mathrm{d}\theta,$$

  and are asymptotically independent of the Kingman tree topology.

Ancestral trees

Proposition (Harris, H., Kyprianou, Powell)

Let $a \in (0,1)$ and $x \in E$. Let T_t denote the time of the MRCA of two particles, one chosen uniformly from those alive at time t, and one chosen uniformly from those alive at time at. Then

$$\frac{T_t}{t} \to T,$$

in distribution as $t \to \infty$.

Literature

- O'Connell, The genealogy of branching processes and the age of our most recent common ancestor.
- Lambert, Coalescence times for the branching process.
- Harris & Roberts, The many-to-few lemma and multiple spines.
- Harris, Johnston & Roberts, The coalescent structure of continuous-time Galton-Watson trees.
- Harris, Horton, Kyprianou & Powell, Many-to-few for non-local branching Markov process.
- Johnston, The genealogy of Galton-Watson trees.
- Zubkov, Limiting distributions of the distance to the closest common ancestor.
- Athreya, Boenkost, Durrett, Foutel-Rodier, Le, Palau, Pardo, Schertzer, Schweinsberg, Tourniaire, . . .

Genealogical structure: convergence to the BCRT

- Aim is to look at the scaling limit of the continuous planar tree associated with a MBP.
- Ulam Harris notation:

$$\Omega = \bigcup_{n=0}^{\infty} \mathbb{N}^n.$$

- The label $\emptyset$ denotes the initial ancestor.
- Labels are of the form $u = \emptyset u_1 u_2 \dots u_n$, e.g. label $\emptyset 215$ means the particle is the 5th child of the 1st child of the 2nd child of the initial ancestor.

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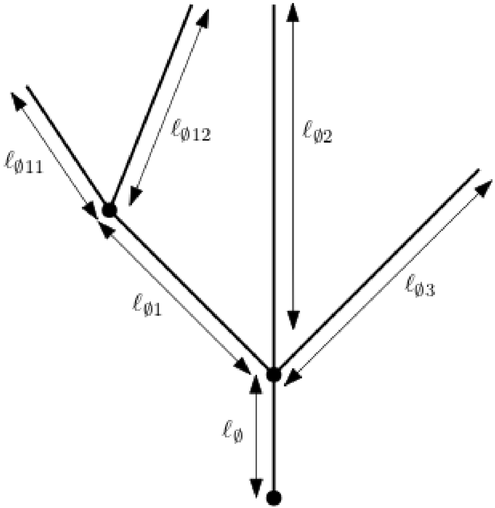
Convergence to the Brownian CRT



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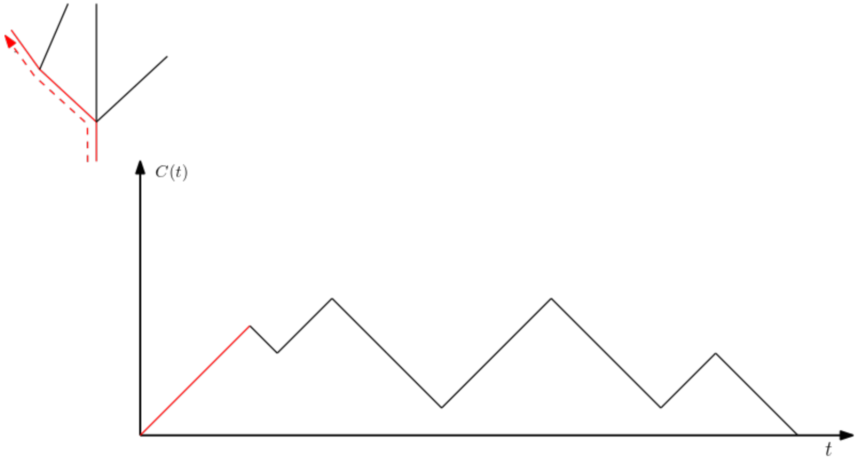
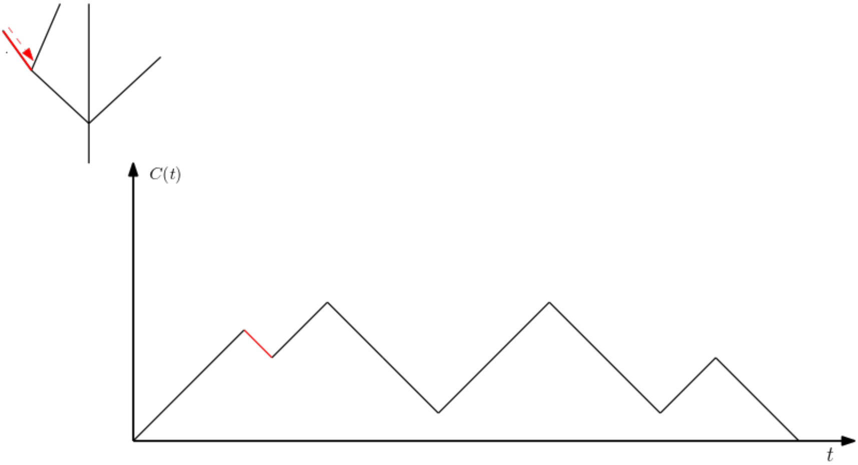


Convergence to the Brownian CRT



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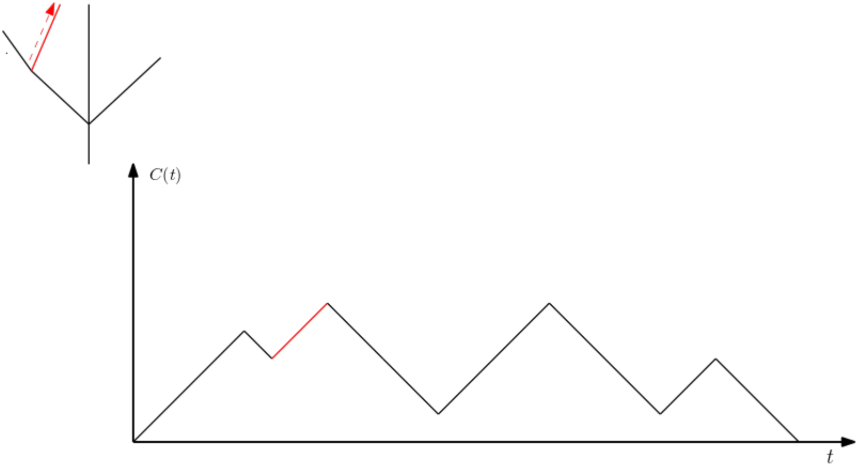

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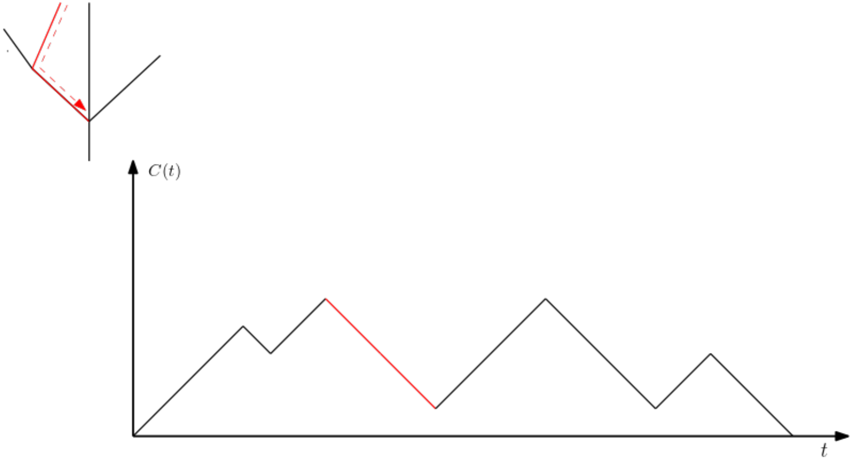

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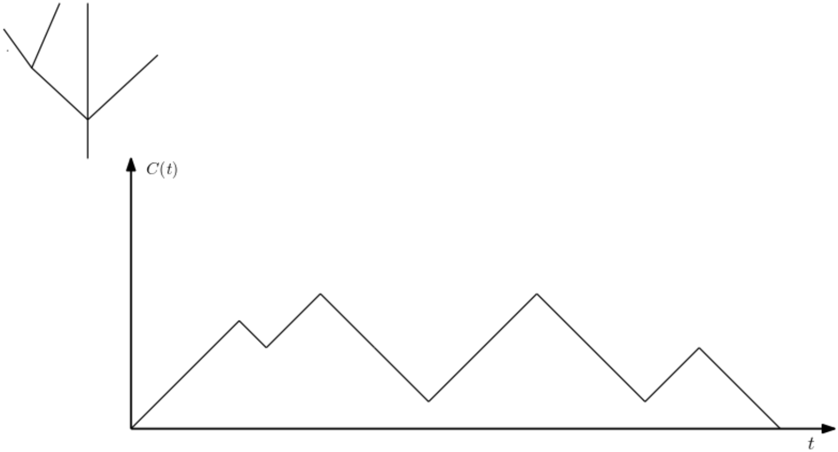

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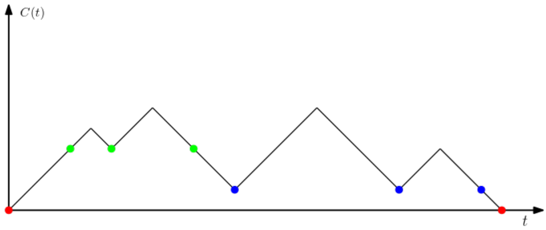

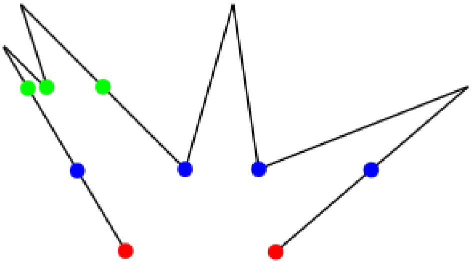

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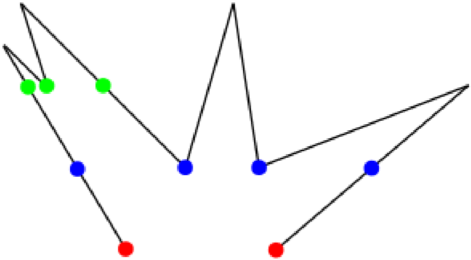

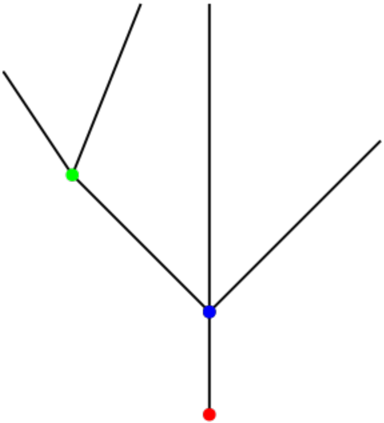

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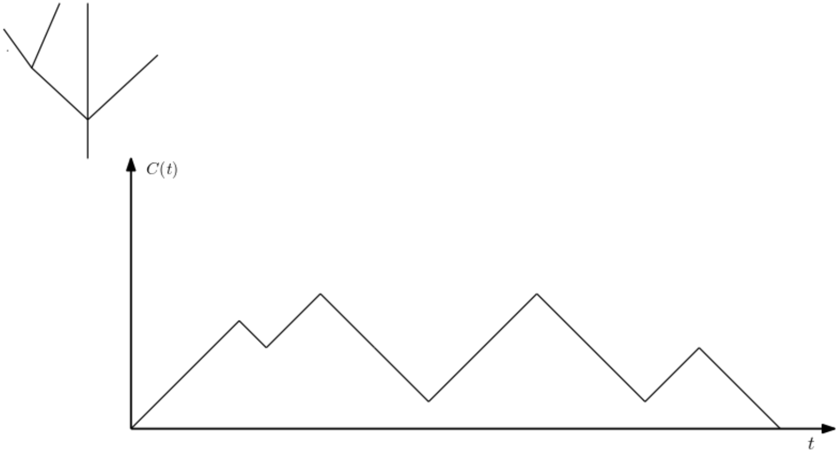

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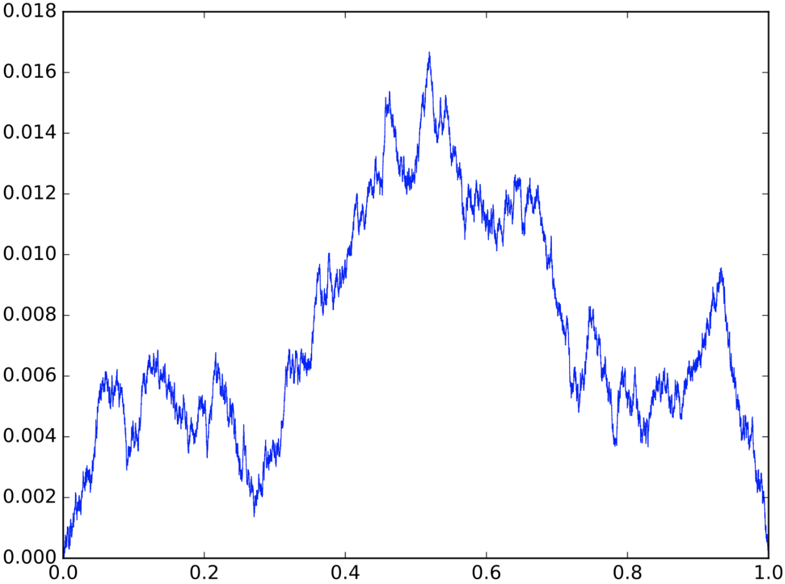

Convergence to the Brownian CRT

- Given a contour process $(C(t))_{t\geq 0}$, define

$$d(s,t) = C(s) + C(t) - 2\min_{r\in[s,t]} C(r), \quad 0 \leq s \leq t.$$

- Define

$$(\mathbf{T}_{t,x}, d_{t,x}) := (\mathbf{T}, \frac{1}{t}d) \text{ under } \mathbb{P}_{\delta_x}(\cdot|N_t > 0)$$

- Let $\mathbf{e}$ be a Brownian excursion conditioned to reach at least height 1.

- Let $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ denote the real tree encoded by $\mathbf{e}$.

Convergence to the Brownian CRT

Theorem

For any $x \in E$,

$$(\mathbf{T}_{t,x}, d_{t,x}) \to (\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}}) \text{ as } t \to \infty,$$

in distribution, with respect to the Gromov-Hausdorff topology.

Convergence to the Brownian CRT

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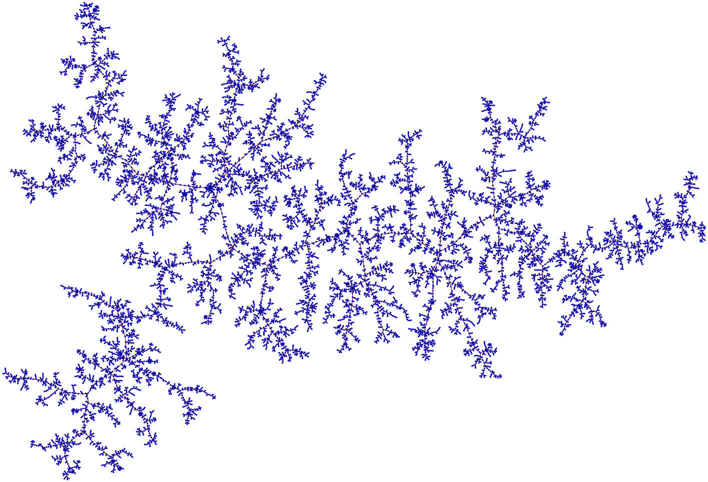
- GW trees: Aldous '93, Le Gall & Duquesne '02, Miermont '09.
- Branching diffusions: Powell '19.
- MBP: Horton & Powell '24+.

Convergence to the Brownian CRT



Comments

- All of this can be done in discrete time.
- Subcritical case: moment asymptotics, Yaglom limit, ...
- Supercritical case: law of large numbers, moment asymptotics, CLT, ...

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Contents

Monte Carlo methods: branching process

Recall the Perron Frobenius asymptotic,

$$\psi_t[g](x) \sim \mathrm{e}^{\lambda_* t}\eta[g]\varphi(x), \quad t \to \infty.$$

Manipulation of this allows us to estimate the eigen-elements, e.g.

$$\lambda_* = \lim_{t\to\infty} \frac{1}{t} \log \psi_t[\mathbf{1}](x) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_{\delta_x}[N_t]$$
$$\approx \frac{1}{T} \log \left(\frac{1}{N} \sum_{i=1}^{N} N_T^{(i)} \right).$$

Monte Carlo methods: many-to-one

Recall the many-to-one formula:

$$\mathbb{E}_{\delta_x}[X_t[g]] = \mathbf{E}_x \left[\mathrm{e}^{\int_0^t \gamma(\hat{\xi}_s)\mathrm{d}s} g(\hat{\xi}_t) \mathbf{1}_{t<\tau} \right].$$

We can replace the branching process by a single weighted trajectory, e.g.

$$\lambda_* = \lim_{t\to\infty} \frac{1}{t} \log \psi_t[\mathbf{1}](x) = \lim_{t\to\infty} \frac{1}{t} \log \mathbf{E}_x \left[\mathrm{e}^{\int_0^t \gamma(\hat{\xi}_s)\mathrm{d}s} \mathbf{1}_{t<\tau} \right]$$
$$= \frac{1}{T} \log \left(\frac{1}{N} \sum_{i=1}^{N} \mathrm{e}^{\int_0^T \gamma(\hat{\xi}_s^{(i)})\mathrm{d}s} \mathbf{1}_{T<\tau^{(i)}} \right).$$

Monte Carlo methods: Importance sampling

- If only we could find a single trajectory that survives forever...

- Recall that

$$\left.\frac{\mathrm{d}\mathbf{P}_x^{\varphi}}{\mathrm{d}\hat{\mathbf{P}}_x}\right|_{\mathcal{G}_t} := \mathrm{e}^{-\lambda_* t+\int_0^t \gamma(\hat{\xi}_s)\mathrm{d}s}\frac{\varphi(\hat{\xi}_t)}{\varphi(x)}, \quad t \geq 0, x \in E.$$

Then

$$\psi_t[g](x) = \mathbf{E}_x\left[\mathrm{e}^{\int_0^t \gamma(\hat{\xi}_s)\mathrm{d}s} g(\hat{\xi}_t)\mathbf{1}_{t<\tau}\right] = \mathbf{E}_x^{\varphi}\left[\mathrm{e}^{\lambda_* t}\frac{\varphi(x)}{\varphi(\hat{\xi}_t)} g(\hat{\xi}_t)\right].$$

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Monte Carlo methods: Importance sampling

- Instead, let's make a "guess" for φ, say h.
- Define the change of measure

$$\left.\frac{\mathrm{d}\mathbf{P}_x^h}{\mathrm{d}\hat{\mathbf{P}}_x}\right|_{\mathcal{G}_t} := \mathrm{e}^{-\int_0^t \frac{\mathcal{J}h(\hat{\xi}_s)}{h(\hat{\xi}_s)}\mathrm{d}s} \frac{h(\hat{\xi}_t)}{h(x)},$$

where $\mathcal{J}$ is the generator of $\hat{\xi}$.

- Then

$$\psi_t[g](x) = h(x)\mathbf{E}_x^h\left[\mathrm{e}^{\int_0^t \gamma(\hat{\xi}_s)+\frac{\mathcal{J}h(\hat{\xi}_s)}{h(\hat{\xi}_s)}\mathrm{d}s} \frac{g(\hat{\xi}_t)}{h(\hat{\xi}_t)}\right].$$

- Cox, A. M. G., Harris, S. C., Kyprianou, A. E., & Wang, M. (2022). Monte Carlo methods for the neutron transport equation. SIAM/ASA Journal on Uncertainty Quantification, 10(2), 775-825.

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- Then

$$\psi_t[g](x) = h(x)\mathbf{E}^h_x\left[\mathrm{e}^{\int_0^t \gamma(\hat{\xi}_s)+\frac{\mathcal{J} h(\hat{\xi}_s)}{h(\hat{\xi}_s)}\mathrm{d}s} \frac{g(\hat{\xi}_t)}{h(\hat{\xi}_t)}\right].$$

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Toy model

- $D = (-L, L),\ V = \{-1, +1\}$.
- We consider a system governed by the following NTE:

$$
\begin{aligned}
\frac{\partial}{\partial t}\psi_t(r,v) =& v \cdot \nabla \psi_t(r,v) - (\Sigma_{\mathtt{s}} + \Sigma_{\mathtt{f}})\psi_t(r,v) \\
&+ \frac{\Sigma_{\mathtt{s}}}{2}\big(\psi_t(r,v) + \psi_t(r,-v)\big) \\
&+ \frac{\Sigma_{\mathtt{f}}\nu}{2}\big(\psi_t(r,v) + \psi_t(r,-v)\big).
\end{aligned}
$$

- Boundary condition: $\psi_t(L,1) = 0 = \psi_t(-L,-1)$.

Toy model

- Standard ODE techniques allow one to solve the associated eigenvalue problem explicitly.
- Critical case:

$$L_c = \frac{\arctan(1/\sqrt{\bar{c}-1})}{(\Sigma_{\mathtt{s}} + \Sigma_{\mathtt{f}})\sqrt{\bar{c}-1}}.$$

- Eigenfunctions:

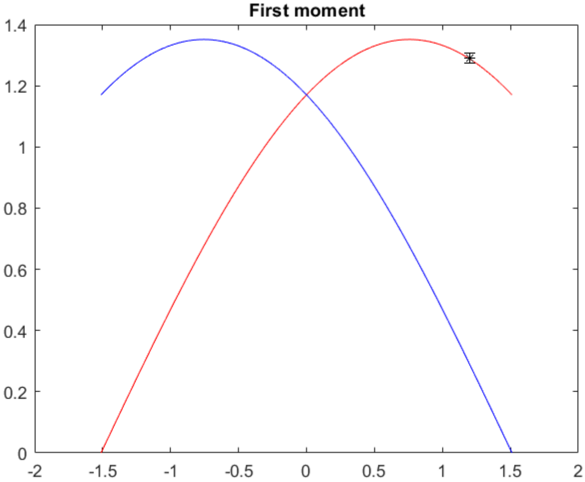
$$\varphi(r, v) \propto \phi(r)\mathbf{1}_{\{v=+1\}} + \phi(-r)\mathbf{1}_{\{v=-1\}}$$
$$\eta(r, v) \propto \phi(-r)\mathbf{1}_{\{v=+1\}} + \phi(r)\mathbf{1}_{\{v=-1\}},$$

where

$$\phi(r) = \cos(\alpha_1 r) - \sin(\alpha_1 r)\cot(\alpha_1 L).$$

Toy model

Joint work with Eric Dumonteil and Andrea Zoia, CEA.



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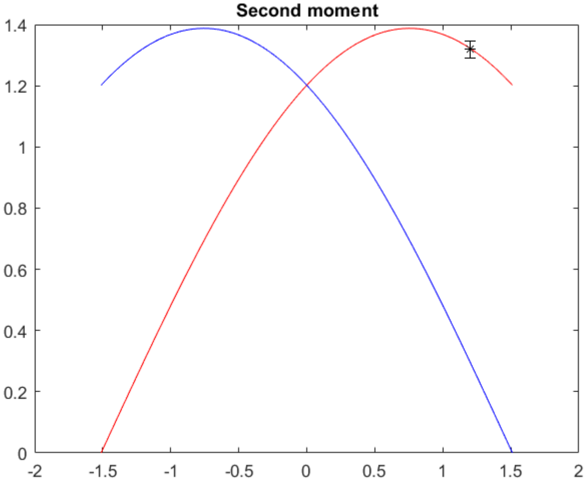
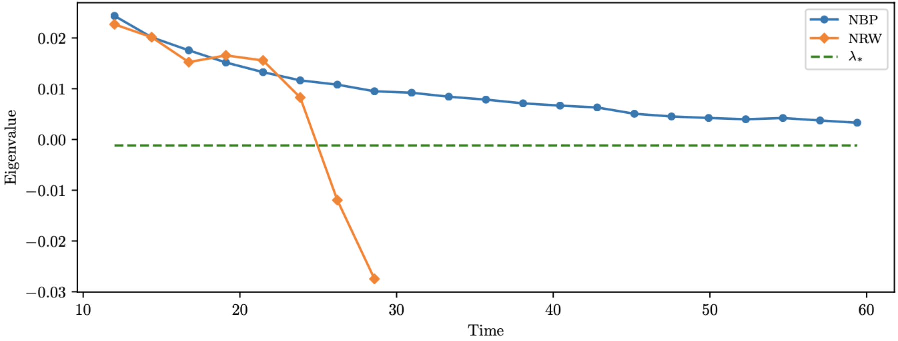


Toy model

Cox et. al., Monte Carlo methods for the neutron transport equation.



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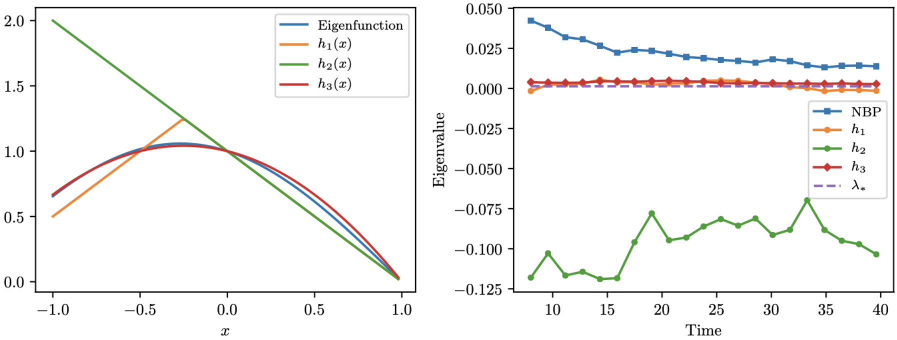
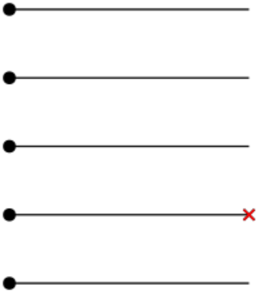


Monte Carlo methods: Fleming Viot

- Let $(Y_t)_{t\geq 0}$ be a time-homogeneous Markov process on $E \cup \{\partial\}$ with probabilities $(\mathbf{P}_x^{\dagger}, x \in E)$ and semigroup $(\mathtt{P}_t^{\dagger})_{t\geq 0}$.
- Assume that $\tau_\partial := \inf\{t > 0 : X_t = \partial\} < \infty$, $\mathbf{P}_x^{\dagger}$-almost surely for all $x \in E$.
- Assume further that for all $x \in E$, $\mathbf{P}_x^{\dagger}(t < \tau_\partial) > 0$.

Monte Carlo methods: Fleming Viot

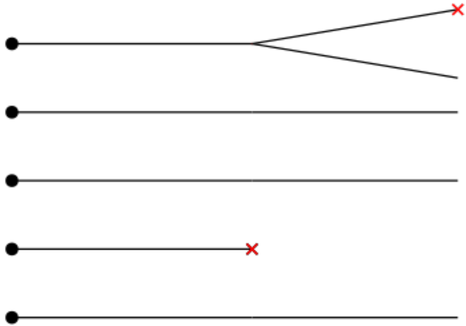
Simulate $N \geq 1$ independent copies of $(Y, \mathbf{P}^\dagger)$ until one of the particles is absorbed.



Fleming Viot particle system

Monte Carlo methods: Fleming Viot

When this happens, duplicate one of the remaining $N-1$ particles and return to the previous step.



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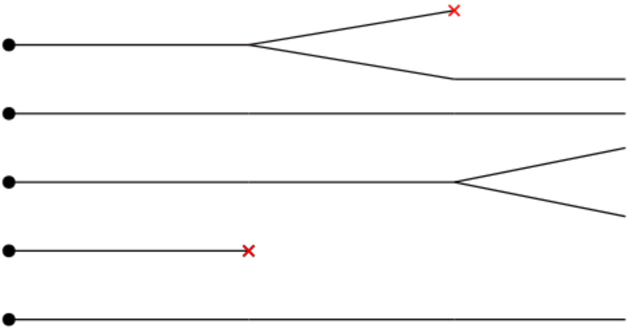
Monte Carlo methods: Fleming Viot



Fleming Viot particle system

Monte Carlo methods: Fleming Viot

- Let $\{Y_t^i, i = 1, \ldots, N\}$ denote the configuration of the Fleming Viot system at time $t \geq 0$.
- Let A_t denote the number of rebirths up to time t.

Monte Carlo methods: Fleming Viot

- Let $\{Y_t^i, i = 1, \dots, N\}$ denote the configuration of the Fleming Viot system at time $t \geq 0$.
- Let A_t denote the number of rebirths up to time t.

Theorem (Villemonais '14)

Assume that for any $x \in E$ and $t \geq 0$,

- $\mathbf{P}_x^\dagger(\tau_\partial = t) = 0$,
- $A_t < \infty$ almost surely.

For any continuous, bounded function $f : E \to [0, \infty)$, we have

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{Y_t^i} \to \mathbf{E}_{\mu_0}[f(Y_t)|t < \tau_\partial],$$

in law, as $N \to \infty$.

Monte Carlo methods: Fleming Viot

- Let $\{Y_t^i, i = 1, \dots, N\}$ denote the configuration of the Fleming Viot system at time $t \geq 0$.
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Theorem

Assume that for any $x \in E$ and $t \geq 0$,

- $\mathbf{P}_x^\dagger(\tau_\partial = t) = 0$,
- $A_t < \infty$ almost surely.

Then, for any $f \in B^+(E)$ and $T > 0$,

$$\mathbf{E}_x^\dagger[f(Y_T)] = \mathbb{E}_x\left[\left(\frac{N-1}{N}\right)^{A_T^N} \frac{1}{N}\sum_{i=1}^N f(Y_T^i)\right].$$

Monte Carlo methods: Fleming Viot

Idea of proof:

- Define $\nu_t^f = \left(\frac{N-1}{N}\right)^{A_t} \frac{1}{N}\sum_{i=1}^{N} \mathrm{P}^{\dagger}_{T-t}[f](Y_t^i)$.

- Martingale decomposition:

$$\nu_T^f - \nu_0^f = \int_0^T \left(\frac{N-1}{N}\right)^{A_{s-}^N} \mathrm{d}\mathbb{M}_s + \frac{N}{N-1}\int_0^T \left(\frac{N-1}{N}\right)^{A_{s-}^N} \mathrm{d}\mathcal{M}_s. \quad (2)$$

- Taking expectations yields the result.

Monte Carlo methods: Fleming Viot

Recall that we can create a subMarkov process from the branching process via

$$\mathrm{e}^{-\bar{\gamma}t}\psi_t[g](x) = \mathrm{e}^{-\bar{\gamma}t}\hat{\mathbf{E}}_x\left[\mathrm{e}^{\int_0^t \gamma(\hat{\xi}_s)\mathrm{d}s} g(\hat{\xi}_t)\mathbf{1}_{t<\tau}\right] = \mathbf{E}_x^{\dagger}[g(\hat{\xi}_t)].$$

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Then, playing the same game, we have

$$\mathbb{E}_{\delta_x}[X_t[g]] = \mathrm{e}^{\bar{\gamma}t}\mathbb{E}\left[\left(\frac{N-1}{N}\right)^{A_t}\frac{1}{N}\sum_{i=1}^{N} f(X_t^i)\right]$$

and

$$\lambda_* = \bar{\gamma} + \lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\frac{1}{N}\left(\frac{N-1}{N}\right)^{A_t}\right].$$

Sequential Monte Carlo

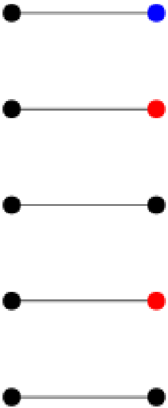
- Sequential Monte Carlo
- Particle filters
- Genetic algorithms
- Evolutionary population
- Diffusion Monte Carlo
- Quantum Monte Carlo
- Sampling Algorithms
- ...

Sequential Monte Carlo

- Sequential Monte Carlo $\rightsquigarrow$ Sampling + Resampling
- Particle filters $\rightsquigarrow$ Prediction + Updating
- Genetic algorithms $\rightsquigarrow$ Mutation + Selection
- Evolutionary population $\rightsquigarrow$ Exploration + Branching-selection
- Diffusion Monte Carlo $\rightsquigarrow$ Free evolution + Absorption
- Quantum Monte Carlo $\rightsquigarrow$ Walkers motion + Reconfiguration
- Sampling Algorithms $\rightsquigarrow$ Transition proposals + Accept-reject-recycle
- ...

Sequential Monte Carlo

Sequential Monte Carlo



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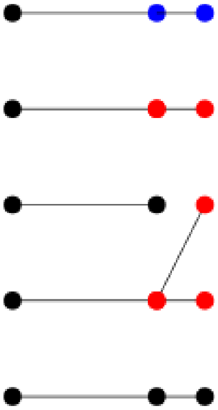

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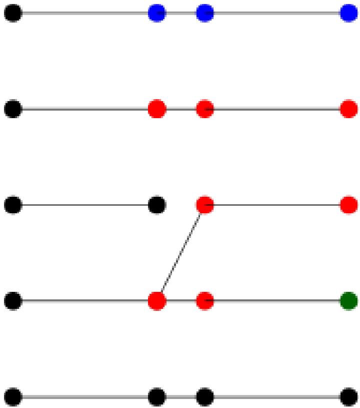

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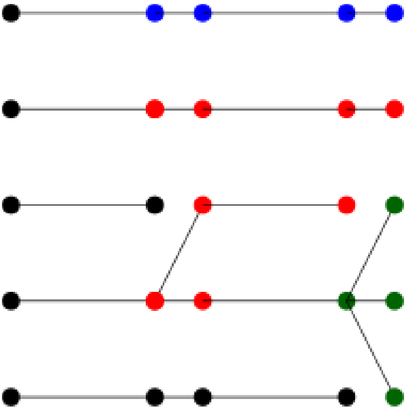

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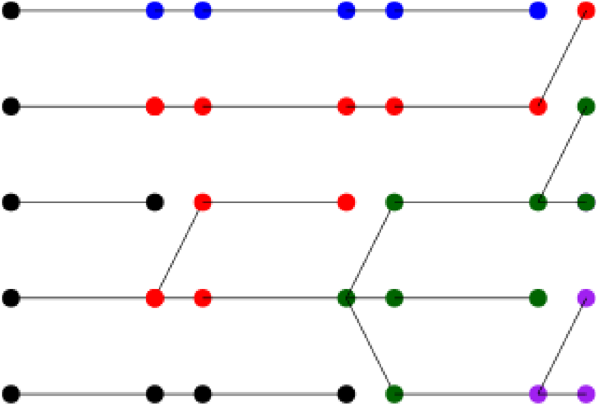

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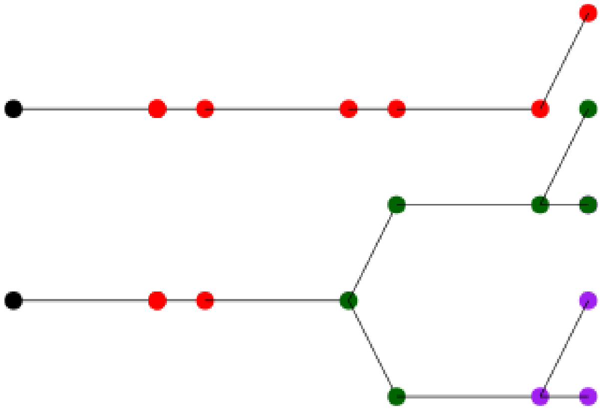


Sequential Monte Carlo

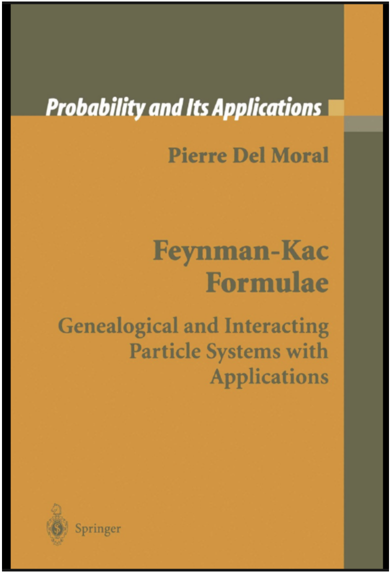
- Initiate a set of N particles, $\xi_0^i \sim \mu$.
- Evolve each particle independently according to a Markov semigroup M, until some time T.
- Compute weights $G_T(\xi_T^i)$ for each $i = 1, \ldots, N$.
- Select the new population according to:

$$G_T(\xi_T^i)\delta_{\xi_T^i} + (1 - G_T(\xi_T^i)) \sum_{j \neq i} \frac{G_T(\xi_T^j)}{Z_T^N} \delta_{\xi_T^j}.$$

Sequential Monte Carlo

- Fleming Viot:
  - Motion: $(Y, \mathbf{P}^{\dagger})$,
  - Time step: $T = \min_{i=1,\dots,N} \inf\{t > 0 : Y_t^i = \partial\}$,
  - Weight: $G(x) = \mathbf{1}_E(x)$.
- Confinements:
  - Motion: discrete time random walk, $(Y_n)_{n\geq 0}$, in $\mathbb{Z}^d$,
  - Time step: $T = 1$,
  - Weight: $G(x) = \mathbf{1}_{[-L,L]}(x)$.
- Self avoiding walks:
  - Motion: $\mathbf{Y}_n = (Y_0, \dots, Y_n)$,
  - Time step: $T = 1$,
  - Weight: $G_n(\mathbf{x}) = \mathbf{1}_{x_n \notin \{x_0,\dots,x_{n-1}\}}$.

Literature


Probability and Its Applications
Pierre Del Moral
Feynman-Kac Formulae
Genealogical and Interacting Particle Systems with Applications
Springer

Future work

- Systematic comparison of MC methods for simulating branching processes.
- Developing SMC methods that incorporate branching.
- Understanding the genealogy of SMC algorithms.
- Rare events.
- Time inhomogeneous systems.
- Machine learning.
- ...

¡Gracias!