Stability of (sub)critical spatial branching processes with and without immigration

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18 June 2026

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The original story

- That is, defining $Z_0 = z \in \mathbb{N}$, we have,

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_{n,i}, \quad n \geq 0,$$

where $\xi_{n,i} \sim_{iid} \xi$.

- The quantity $m = E[\xi]$ determines the average growth of the population:
  - $m > 1 \Leftrightarrow$ *supercritical*,
  - $m < 1 \Leftrightarrow$ *subcritical*,
  - $m = 1 \Leftrightarrow$ *critical*.

The original story

Critical Galton Watson asymptotics

For critical Galton–Watson processes with finite variance, σ^2, as $n \to \infty$:

$$\mathbb{P}(Z_n > 0) \sim \frac{2}{\sigma^2 n}$$

and

$$\left(\frac{Z_n}{n} \,\middle|\, Z_n > 0 \right) \Rightarrow \mathrm{Exp}(2/\sigma^2).$$

The original story

- Let $(Z_n)_{n\geq 1}$ be a Galton Watson process with offspring distribution ξ, and let $\tilde{\xi}$ be an independent random variable.

- The Galton Watson process with immigration is defined via

$$\tilde{Z}_{n+1} = \sum_{i=1}^{\tilde{Z}_n} \xi_{n,i} + \tilde{\xi}_{n+1}, \quad n \geq 0,$$

initiated from $\tilde{Z}_0 = z \in \mathbb{N}$.

The original story

Galton Watson with immigration asymptotics

- If $m < 1$, as $n \to \infty$,

$$\tilde{Z}_n \Rightarrow \tilde{Z}_\infty \quad \Leftrightarrow \quad \int_0^1 \frac{1 - g(s)}{f(s) - s} ds < \infty,$$

where $f(s) = E[s^\xi]$ and $g(s) = \tilde{E}[s^{\tilde{\xi}}]$.

- If $m = 1$, $\sigma^2 < \infty$ and $\lambda := \tilde{E}[\tilde{\xi}] \in (0, \infty)$ then

$$\left(\frac{Z_n}{n}\right) \Rightarrow \mathrm{Gamma}\left(\frac{2\lambda}{\sigma^2}, \frac{2}{\sigma^2}\right).$$

Question

How much of this remains true for general spatial branching Markov processes?

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Branching Markov process

- Particles evolve in a Polish space E according to a càdlàg (killed) Markov process $(\xi, \mathbf{P}_x)$.
- When at $x \in E$, at rate $\beta(x)$, the parent is replaced by new particles that are created according to the point process $(\mathcal{Z}, \mathcal{P}_x)$, where

$$\mathcal{Z} = \sum_{i=1}^{N} \delta_{x_i}.$$

- The branching mechanism is given by

$$G[g](x) = \beta(x)\mathcal{E}_x\left[\prod_{i=1}^{N} g(x_i) - g(x)\right].$$

Branching Markov process

- Let $\{x_i(t) : i = 1, \ldots, N_t\}$ denote the configuration of the system at time $t \geq 0$.

- The branching process is defined as

$$X_t := \sum_{i=1}^{N_t} \delta_{x_i(t)}.$$

- For a function f and measure μ, we will use the notation

$$\langle f, \mu \rangle := \int_E f(y)\mu(\mathrm{d}y).$$

Branching Markov process

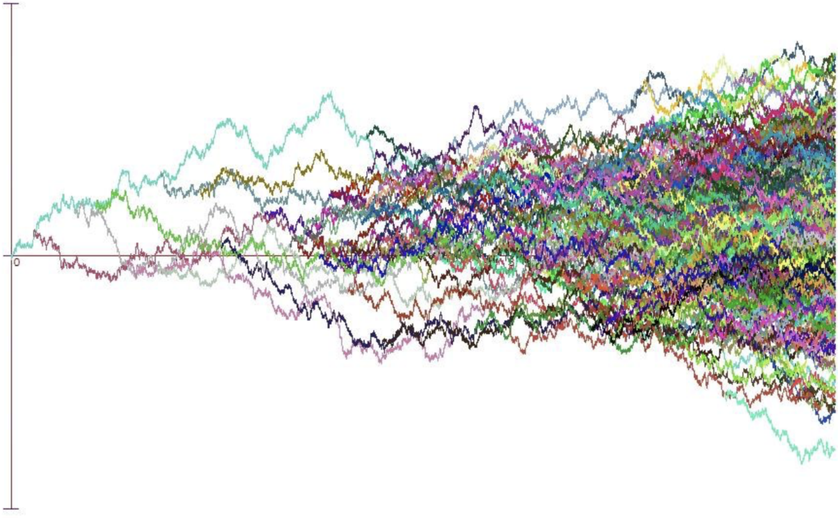
Define

$$\mathrm{e}^{-v_t[f](x)} = \mathbb{E}_{\delta_x}\left[\mathrm{e}^{-\langle f, X_t\rangle}\right].$$

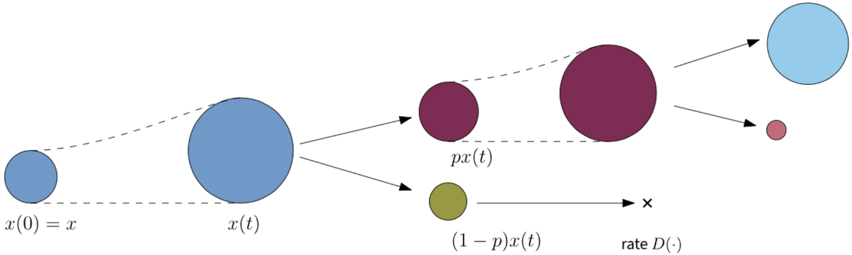
Then

$$\mathrm{e}^{-v_t[f](x)} = P_t[\mathrm{e}^{-f}](x) + \int_0^t P_s\left[G[\mathrm{e}^{-v_{t-s}[f]}]\right](x)\mathrm{d}s.$$

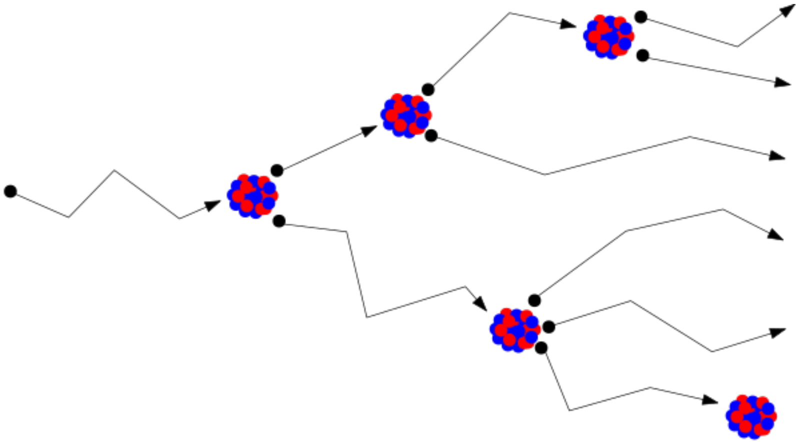
Example: BBM



Example: growth-fragmentation



Example: branching PDMP



With immigration

- Immigration events occur according to a homogeneous Poisson process of rate $\alpha > 0$.

- When immigration occurs, particles are distributed according to $(\widetilde{\mathcal{Z}}, \widetilde{\mathcal{P}})$, where

$$\widetilde{\mathcal{Z}} = \sum_{i=1}^{\widetilde{N}} \delta_{\tilde{x}_i}$$

- The immigration mechanism is given by

$$H[f] = \alpha \widetilde{\mathcal{E}} \left[1 - e^{-\langle f, \widetilde{Z} \rangle} \right].$$

With immigration

- Let $(D_t)_{t\geq 0}$ denote the Poisson process with rate α, $(\tau_j)_{j\geq 1}$ the arrival times and $(\widetilde{Z}_j)_{j\geq 1}$ be i.i.d. copies of $\widetilde{Z}$.
- We say that $(Y_t^{(\mu)})$ is a branching Markov process with immigration (BMPI) if

$$Y_t^{(\mu)} = X_t^{(\mu)} + \sum_{j=1}^{D_t} X_{t-\tau_j}^{(\widetilde{Z}_j)}.$$

- Moreover, given $(\widetilde{\mathcal{Z}}_j, j = 1, \cdots, D_t)$, the processes $X^{(\widetilde{\mathcal{Z}}_j)}$ are independent copies of $X^{(\mu)}$ issued from the respective measures $\mu = \widetilde{\mathcal{Z}}_j$.

With immigration

Define

$$\mathrm{e}^{-\mathtt{w}_t[f](x)} = \mathbb{E}_{\delta_x}\left[\mathrm{e}^{-\langle f, Y_t\rangle}\right].$$

Then,

$$\mathrm{e}^{-w_t[f](x)} = \exp\left(-v_t[f](x) - \int_0^t H[v_s[f]]\mathrm{d}s\right).$$

The mean semigroup and criticality

The expected mass is governed by the mean semigroup

$$T_t[f](x) = \mathbb{E}_{\delta_x}\left[\langle f, X_t\rangle\right].$$

(H1) Perron Frobenius assumption

There exist $\lambda \leq 0$, a right eigenfunction $\varphi > 0$, and a left eigenmeasure $\widetilde{\varphi}$ such that

$$T_t[\varphi] = e^{\lambda t}\varphi, \qquad \langle T_t[f], \widetilde{\varphi}\rangle = e^{\lambda t}\langle f, \widetilde{\varphi}\rangle.$$

Moreover, there exists $\varepsilon > 0$ such that

$$\sup_{x,f}\left|\varphi(x)^{-1}e^{-\lambda t}T_t[f](x) - \langle f, \widetilde{\varphi}\rangle\right| = O(e^{-\varepsilon t}), \quad t \to \infty.$$

Assumptions

(H2) $\sup_{x\in E} \beta(x) < \infty$ and $\sup_{x\in E} \mathcal{E}_x[N^2] < \infty$.

(H3) For each $x \in E$,

$$\mathbb{P}_{\delta_x}(\zeta < \infty) = 1,$$

where $\zeta = \inf\{t \geq 0 : N_t = 0\}$.

(H4) There exist constants $K > 0$ and $M > 0$ such that for all non-negative bounded f,

$$\langle \mathcal{V}_M[f], \tilde{\varphi} \rangle \geq K \langle f, \tilde{\varphi} \rangle^2,$$

where $\mathcal{V}_M$ is defined by

$$\mathcal{V}_M[f](x) = \beta(x)\mathcal{E}_x\left[\sum_{i,j=1;\, i\neq j}^{N} f(x_i)f(x_j)\mathbf{1}_{\{N\leq M\}}\right], \qquad x \in E,$$

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Results

Critical, no immigration

Assume (H1)-(H4) and $\lambda = 0$. Then, for all $x \in E$,

$$\lim_{t\to\infty} t\,\mathbb{P}_{\delta_x}(N_t > 0) = \frac{2\varphi(x)}{\langle \mathcal{V}[\varphi], \widetilde{\varphi}\rangle},$$

where $\mathcal{V}[f](x) = \lim_{M\to\infty} \mathcal{V}_M[f](x)$.

Results

Critical, no immigration

Assume (H1)-(H4) and $\lambda = 0$. Then, for non-negative bounded f and $\theta \geq 0$,

$$\lim_{t\to\infty} \mathbb{E}_\mu \left[\exp\left(-\theta \frac{\langle f, X_t \rangle}{t} \right) \mid N_t > 0 \right] = \frac{1}{1 + \theta \, \frac{1}{2} \langle f, \widetilde{\varphi} \rangle \langle \mathcal{V}[\varphi], \widetilde{\varphi} \rangle}.$$

Results

Critical, with immigration

Assume (H1), (H2), and $\lambda = 0$. Then $\langle f, Y_t\rangle / t$ converges weakly iff

$$I[\varphi] := \alpha \widetilde{\mathcal{E}}[\langle \varphi, \widetilde{Z}\rangle] < \infty.$$

In that case,

$$\lim_{t\to\infty} \mathbb{E}_\mu\left[\exp\left(-\theta \frac{\langle f, Y_t\rangle}{t}\right)\right] = \left(1 + \theta \frac{1}{2}\langle f, \widetilde{\varphi}\rangle \langle \mathcal{V}[\varphi], \widetilde{\varphi}\rangle\right)^{-2I[\varphi]/\langle \mathcal{V}[\varphi], \widetilde{\varphi}\rangle}.$$

Results

Subcritical, with immigration

Assume (H1), (H2), and $\lambda < 0$. Then $Y_t \Rightarrow Y_\infty$ iff the integral test holds:

$$\int_0^{z_0} \frac{H[z\varphi]}{z}\, dz < \infty \qquad \text{for some } z_0 > 0.$$

The limiting law is characterised by

$$\mathbb{E}_\mu[e^{-\langle f, Y_\infty\rangle}] = \exp\left\{-\int_0^\infty H[v_s[f]]\, ds\right\}.$$

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The key evolution equation

- The non-linear semigroup can be compared to the linear mean semigroup.

- Define
$$u_t[g](x) = 1 - \mathbb{E}_{\delta_x}\left[\prod_{i=1}^{N_t} g(x_i(t))\right].$$

- Then
$$u_t[g] = T_t[1-g] - \int_0^t T_s\left[A[u_{t-s}[g]]\right] ds,$$
where
$$A[h](x) = \beta(x)\mathcal{E}_x\left[\prod_{i=1}^{N}(1-h(x_i)) - 1 + \sum_{i=1}^{N} h(x_i)\right].$$

Survival probability: step 1

- The survival probability, $u_t(x) := u_t[1](x) = \mathbb{P}_{\delta_x}(N_t > 0)$ solves

$$u_t = T_t[1] - \int_0^t T_s[A[u_{t-s}]]\, ds.$$

- Project onto the left eigenmeasure: $a_t := \langle u_t, \widetilde{\varphi} \rangle$.

- Using $\langle T_s h, \widetilde{\varphi} \rangle = \langle h, \widetilde{\varphi} \rangle$ at criticality gives

$$a_t = 1 - \int_0^t \langle A[u_s], \widetilde{\varphi} \rangle\, ds, \qquad a_t' = -\langle A[u_t], \widetilde{\varphi} \rangle.$$

- So the proof reduces to identifying the asymptotics of $A[u_t]$.

Survival probability: step 2

- The Perron Frobenius assumption (H1) implies that at large time the profile of u_t is almost proportional to φ:

$$u_t(x) \approx a_t \varphi(x).$$

- More precisely, the proof establishes

$$\sup_x \left| \frac{u_t(x)}{\varphi(x)} - a_t \right| = O(t^{-2}).$$

- Thus the spatial shape is asymptotically fixed, and only the scalar amplitude a_t remains.

Survival probability: step 3

- Once $u_t \approx a_t\varphi$, the second-order expansion gives

$$A[u_t] \sim \frac{1}{2}[u_t] \sim \frac{1}{2}a_t^2[\varphi].$$

- Therefore

$$a_t' = -\langle A[u_t], \widetilde{\varphi}\rangle \sim -\frac{1}{2}a_t^2\langle \mathcal{V}[\varphi], \widetilde{\varphi}\rangle.$$

- Solving the differential asymptotic

$$a_t' \sim -ca_t^2$$

gives

$$a_t \sim \frac{1}{ct} = \frac{2}{\langle \mathcal{V}[\varphi], \widetilde{\varphi}\rangle t}.$$

Yaglom limit: step 1

- Write

$$f = \langle f, \widetilde{\varphi} \rangle \varphi + \widetilde{f}, \qquad \langle \widetilde{f}, \widetilde{\varphi} \rangle = 0.$$

- Using Markov's inequality and second moment asymptotics shows that

$$\frac{\langle \widetilde{f}, X_t \rangle}{t} \longrightarrow 0 \qquad \text{under } \mathbb{P}_\mu(\,\cdot \mid N_t > 0).$$

- Hence it is enough to study

$$\frac{\langle \varphi, X_t \rangle}{t}.$$

Yaglom limit: step 2

- Hence, it is enough to show that

$$\lim_{t\to\infty} \mathbb{E}_\mu \left[\exp\left(-\frac{\theta}{t} \langle f, \widetilde{\varphi} \rangle \langle \varphi, X_t \rangle \right) \middle| N_t > 0 \right] = \frac{1}{1 + \frac{1}{2}\theta \langle f, \tilde{\varphi} \rangle \langle \mathcal{V}[\varphi], \tilde{\varphi} \rangle},$$

- Equivalently

$$\lim_{t\to\infty} \frac{\mathbb{E}_\mu \left[1 - \exp\left(-\theta \langle f, \widetilde{\varphi} \rangle \langle \varphi, X_t \rangle / t \right) \right]}{\mathbb{P}_\mu \left(\langle 1, X_t \rangle > 0 \right)} = \frac{\frac{1}{2}\theta \langle f, \tilde{\varphi} \rangle \langle \mathcal{V}[\varphi], \tilde{\varphi} \rangle}{1 + \frac{1}{2}\theta \langle f, \tilde{\varphi} \rangle \langle \mathcal{V}[\varphi], \tilde{\varphi} \rangle}.$$

- We now follow very similar steps as in the survival probability result.

Why the exponential distribution?

- There are asymptotically two children of the MRCA, each with at least 1 descendant alive at time t.
- Distribution of the time of the MRCA of the particles alive at time t is uniform.
- Therefore, under $\mathbb{P}_{\delta_x}(\cdot|N_t > 0)$,

$$\frac{X_t}{t} \approx U\left(\frac{X^{(1)}_{Ut}}{Ut} + \frac{X^{(2)}_{Ut}}{Ut}\right).$$

Now add immigration

- Recall
$$\mathbb{E}_\mu[e^{-\langle f, Y_t\rangle}] = \exp\left\{-\langle v_t[f], \mu\rangle - \int_0^t H[v_s[f]]\,ds\right\}.$$
- We know how to understand the asymptotics of v_t.
- Hence, the asymptotics for the process with immigration reduce to understanding the integral
$$\int_0^t H[v_s[f]]\,ds,$$
compared with the asymptotics for v.

Critical case with immigration

- In this case, the Laplace transform is controlled by

$$\int_0^t H\left[v_s\left[\theta f/t\right]\right] ds.$$

- The first order approximation of $H[f] = \alpha\tilde{\mathcal{E}}[1 - \mathrm{e}^{-\langle f, \tilde{\mathcal{Z}}\rangle}]$ is $\alpha\tilde{\mathcal{E}}[\langle f, \tilde{\mathcal{Z}}\rangle]$.

- Similar steps to before give

$$H[v_s[\theta f/t]] \approx H[a_s(t)\varphi] \sim a_s(t)I[\varphi]$$

- The Yaglom result gives

$$a_s(t) \approx \frac{\theta\langle f, \widetilde{\varphi}\rangle}{t + \theta cs}, \qquad c = \frac{1}{2}\langle V[\varphi], \widetilde{\varphi}\rangle.$$

Critical case with immigration

- Putting the pieces together gives

$$\int_0^t H[v_s[\theta f/t]]\,ds \approx \frac{I[\varphi]\langle f, \widetilde{\varphi}\rangle}{c}\log(1+\theta c).$$

- Hence

$$\mathbb{E}_\mu\left[e^{-\theta\langle f, Y_t\rangle/t}\right] \to \exp\left\{-\frac{I[\varphi]\langle f, \widetilde{\varphi}\rangle}{c}\log(1+\theta c)\right\} = (1+\theta c)^{-I[\varphi]\langle f, \widetilde{\varphi}\rangle/c}.$$

Subcritical case with immigration

- When $\lambda < 0$, the nonlinear semigroup decays approximately as

$$v_s[f] \approx e^{\lambda s}\varphi \langle f, \widetilde{\varphi}\rangle.$$

- Thus the stationarity criterion is governed by

$$\int_0^\infty H[ce^{\lambda s}\varphi]\, ds.$$

- With the change of variables $z = ce^{\lambda s}$,

$$\int_0^\infty H[ce^{\lambda s}\varphi]\, ds < \infty \quad \Longleftrightarrow \quad \int_0^{z_0} \frac{H[z\varphi]}{z} dz < \infty.$$

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Example 1: branching Brownian motion in a bounded domain

- E is a bounded regular domain D with sufficiently smooth boundary.
- Particles move as Brownian motion between events.
- Local branching occurs at a constant rate β with offspring number N and mean $m = \mathcal{E}[N]$.
- Then, one can show that

$$T_t[f](x) = \mathbf{E}_x \left[e^{\beta(m-1)t} f(B_t);\ t < \tau_D \right].$$

Example 1: verifying the assumptions

- (H1) becomes the familiar condition $\mathcal{E}[N^2] < \infty$.

- (H2) follows from spectral theory for the Laplacian on D. In particular, we have
$$T_t[f](x) \sim e^{\lambda t}\varphi(x)\langle f, \widetilde{\varphi}\rangle,$$
where $\lambda = \lambda_1(D) + \beta(m-1)$.

- (H3) in the critical/subcritical case, extinction occurs almost surely without additional conditions.

Example 1: verifying the assumptions

- For local branching at rate β,

$$\mathcal{V}_M[g](x) = \beta g(x)^2 \mathcal{E}\left[N(N-1)\mathbf{1}_{\{N \le M\}}\right].$$

- Therefore

$$\langle \mathcal{V}_M[g], \widetilde{\varphi} \rangle = \beta \mathcal{E}\left[N(N-1)\mathbf{1}_{\{N \le M\}}\right] \langle g^2, \widetilde{\varphi} \rangle.$$

- By Jensen's inequality,

$$\langle g^2, \widetilde{\varphi} \rangle \ge \langle g, \widetilde{\varphi} \rangle^2.$$

- So (H4) follows once the truncated second factorial moment is positive.

Example 2: multi-type Galton Watson process

- Let $E = \{1, \dots, n\}$. When in state i, particles branch at rate β_i and produce offspring of all types.
- The process is a vector of integer masses, one coordinate per type.
- Let A be the $n \times n$ matrix whose entries are $m_{i,j} = \mathcal{E}_i[N_j] - \mathbf{1}_{i=j}$.
- The mean semigroup is an $n \times n$ positive matrix semigroup:

$$T_t = e^{At}.$$

Example 2: verifying the assumptions

- (H1) We assume second moments of the offspring distribution.
- (H2) follows from irreducibility/positivity of the mean matrix and the standard Perron Frobenius result for matrices.
- (H3) follows from (sub)criticality.
- (H4) also follows from a Jensen inequality argument.

Comments

- These results also hold for non-local superprocesses with and without immigration.
- Moreover, the proofs follow the same strategy and techniques.
- What about the case when immigration depends on the current population?

Thank you!