CONDITIONING AN ADDITIVE FUNCTIONAL OF A MARKOV CHAIN TO STAY NON-NEGATIVE II: HITTING A HIGH LEVEL [1]

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Abstract

Let $(X_t)_{t\geq 0}$ be a continuous-time irreducible Markov chain on a finite statespace E, let $v : E \to \mathbb{R}\backslash\{0\}$ and let $(\varphi_t)_{t\geq 0}$ be defined by $\varphi_t = \int_0^t v(X_s)ds$. We consider the cases where the process $(\varphi_t)_{t\geq 0}$ is oscillating and where $(\varphi_t)_{t\geq 0}$ has a negative drift. In each of the cases we condition the process $(X_t, \varphi_t)_{t\geq 0}$ on the event that $(\varphi_t)_{t\geq 0}$ hits level y before hitting zero and prove weak convergence of the conditioned process as $y \to \infty$. In addition, we show the relation between conditioning the process $(\varphi_t)_{t\geq 0}$ with a negative drift to oscillate and conditioning it to stay non-negative until large time, and relation between conditioning $(\varphi_t)_{t\geq 0}$ with a negative drift to drift to drift to $+\infty$ and conditioning it to hit large levels before hitting zero.

1 Introduction

Let $(X_t)_{t\geq 0}$ be a continuous-time irreducible Markov chain on a finite statespace E, let v be a map $v : E \to \mathbb{R}\backslash\{0\}$, let $(\varphi_t)_{t\geq 0}$ be an additive functional defined by $\varphi_t = \int_0^t v(X_s)ds$ and let H_y, $y \in \mathbb{R}$, be the first hitting time of level y by the process $(\varphi_t)_{t\geq 0}$. In the previous paper Jacka, Najdanovic, Warren (2004) we discussed the problem of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative, that is the event $\{H_0 = +\infty\}$. In the oscillating case and in the case of the negative drift of the process $(\varphi_t)_{t\geq 0}$, when the event $\{H_0 = +\infty\}$ is of zero probability, the process $(X_t, \varphi_t)_{t\geq 0}$ can instead be conditioned on some approximation of the event $\{H_0 = +\infty\}$. In Jacka et al. (2004) we considered the approximation by the events $\{H_0 > T\}$, $T > 0$, and proved weak convergence as $T \to \infty$ of the process $(X_t, \varphi_t)_{t\geq 0}$ conditioned on this approximation.

In this paper we look at another approximation of the event $\{H_0 = +\infty\}$ which is the approximation by the events $\{H_0 > H_y\}$, $y \in \mathbb{R}$. Again, we are interested in weak convergence as $y \to \infty$ of the process $(X_t, \varphi_t)_{t\geq 0}$ conditioned on this approximation.

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Our motivation comes from a work by Bertoin and Doney. In Bertoin, Doney (1994) the authors considered a real-valued random walk $\{S_n, n \geq 0\}$ that does not drift to $+\infty$ and conditioned it to stay non-negative. They discussed two interpretations of this conditioning, one was conditioning S to exceed level n before hitting zero, and another was conditioning S to stay non-negative up to time n. As it will be seen, results for our process $(X_t, \varphi_t)_{t\geq 0}$ conditioned on the event $\{H_0 = +\infty\}$ appear to be analogues of the results for a random walk.

Furthermore, similarly to the results obtained in Bertoin, Doney (1994) for a real-valued random walk $\{S_n, n \geq 0\}$ that does not drift to $+\infty$, we show that in the negative drift case

(i) taking the limit as $y \to \infty$ of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on $\{H_y < +\infty\}$ and then further conditioning on the event $\{H_0 = +\infty\}$ yields the same result as the limit as $y \to \infty$ of conditioning $(X_t, \varphi_t)_{t\geq 0}$ on the event $\{H_0 > H_y\}$;

(ii) conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event that the process $(\varphi_t)_{t\geq 0}$ oscillates and then further conditioning on $\{H_0 = +\infty\}$ yields the same result as the limit as $T \to \infty$ of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on $\{H_0 > T\}$.

The organisation of the paper is as follows: in Section 2 we state the main theorems in the oscillating and in the negative drift case; in Section 3 we prove the main theorem in the oscillating case; in Section 4 we prove the main theorem in the negative drift case. Sections 5 and 6 deal with the negative drift case of the process $(\varphi_t)_{t\geq 0}$ and commuting diagrams in conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on $\{H_y < H_0\}$ and $\{H_0 > T\}$, respectively, listed in (i) and (ii) above. Finally, Section 7 is concerned with the Green's function of the process $(X_t, \varphi_t)_{t\geq 0}$ and some auxiliary results needed for the proofs in previous sections.

All the notation in the present paper is taken from Jacka et al. (2004).

2 Main theorems

For fixed $y > 0$, let $P^y_{(e,\varphi)}$ denote the law of the process $(X_t, \varphi_t)_{t\geq 0}$, starting at $(e, \varphi) \in E_0^+$, conditioned on the event $\{H_y < H_0\}$, and let $P^y_{(e,\varphi)}|_{\mathcal{F}_t}$, $t \geq 0$, be the restriction of $P^y_{(e,\varphi)}$ to $\mathcal{F}_t$. We are interested in weak convergence of $(P^y_{(e,\varphi)}|_{\mathcal{F}_t})_{T\geq 0}$ as $y \to +\infty$.

Theorem 2.1 *Suppose that the process $(\varphi_t)_{t\geq 0}$ oscillate. Then, for fixed $(e, \varphi) \in E_0^+$ and $t \geq 0$, the measures $(P^y_{(e,\varphi)}|_{\mathcal{F}_t})_{y\geq 0}$ converge weakly to the probability measure $P^r_{(e,\varphi)}|_{\mathcal{F}_t}$ as $y \to \infty$. The measure $P^r_{(e,\varphi)}$ is defined by*

$$P^r_{(e,\varphi)}(A) = \frac{E_{(e,\varphi)}\Big(I(A)h_r(X_t, \varphi_t)I\{t < H_0\}\Big)}{h_r(e, \varphi)}, \qquad t \geq 0,\ A \in \mathcal{F}_t,$$

where the function h_r *is given by*

$$h_r(e,y) = e^{-yV^{-1}Q} J_1 \Gamma_2 r(e), \qquad (e,y) \in E \times \mathbb{R},$$

and $V^{-1}Qr = 1$.

By comparing Theorem 2.1 and Theorem 2.1 in Jacka et al. (2004) we see that the measures $(P^y_{(e,\varphi)})_{y \geq 0}$ and $(P^T_{(e,\varphi)})_{T \geq 0}$ converge weakly to the same limit. Therefore, in the oscillating case conditioning $(X_t, \varphi_t)_{t \geq 0}$ on $\{H_y < H_0\}$, $y > 0$, and conditioning $(X_t, \varphi_t)_{t \geq 0}$ on $\{H_0 > T\}$, $T > 0$, yield the same result.

Theorem 2.2 *Suppose that the process* $(\varphi_t)_{t \geq 0}$ *drifts to* $-\infty$. *Then, for fixed* $(e,\varphi) \in E_0^+$ *and* $t \geq 0$, *the measures* $(P^y_{(e,\varphi)}|_{\mathcal{F}_t})_{y \geq 0}$ *converge weakly to the probability measure* $P^{f_{max}}_{(e,\varphi)}|_{\mathcal{F}_t}$ *as* $y \to \infty$ *given by*

$$P^{f_{max}}_{(e,\varphi)}(A) = \frac{E_{(e,\varphi)}\Big(I(A) h_{f_{max}}(X_t, \varphi_t) I\{t < H_0\}\Big)}{h_{f_{max}}(e,\varphi)}, \qquad t \geq 0,\ A \in \mathcal{F}_t$$

where the function $h_{f_{max}}$ *is*

$$h_{f_{max}}(e,y) = e^{-yV^{-1}Q} J_1 \Gamma_2 f_{max}(e), \qquad (e,y) \in E \times \mathbb{R}.$$

3 The oscillating case: Proof of Theorem 2.1

Let $t \geq 0$ be fixed and let $A \in \mathcal{F}_t$. We start by looking at the limit of $P^y_{(e,\varphi)}(A)$ as $y \to +\infty$. For $(e,\varphi) \in E_0^+$ and $y > \varphi$, by (viii) in Jacka et al. (2004), the event $P_{(e,\varphi)}(H_y < H_0) > 0$, $y > 0$. Hence, by the Markov property, for any $(e,\varphi) \in E_0^+$ and any $A \in \mathcal{F}_t$,

$$\begin{aligned} P^y_{(e,\varphi)}(A) &= P_{(e,\varphi)}(A \mid H_y < H_0) \\ &= \frac{1}{P_{(e,\varphi)}(H_y < H_0)} E_{(e,\varphi)}\Big(I(A)(I\{t < H_0 \wedge H_y\} P_{(X_t,\varphi_t)}(H_y < H_0) \\ &\qquad + I\{H_y \leq t < H_0\} + I\{H_y < H_0 \leq t\})\Big). \end{aligned} \tag{1}$$

Lemma 3.1 *Let* r *be a vector such that* $V^{-1}Qr = 1$. *Then*

$$(i) \qquad h_r(e,\varphi) \equiv e^{-\varphi V^{-1}Q} J_1 \Gamma_2 r(e) > 0, \quad (e,\varphi) \in E_0^+,$$

$$(ii) \quad \lim_{y \to +\infty} \frac{P_{(e',\varphi')}(H_y < H_0)}{P_{(e,\varphi)}(H_y < H_0)} = \frac{e^{-\varphi' V^{-1}Q} J_1 \Gamma_2 r(e')}{e^{-\varphi V^{-1}Q} J_1 \Gamma_2 r(e)}, \quad (e,\varphi), (e',\varphi') \in E_0^+.$$

Proof: (i) Let the matrices A_{-y} and C_{-y} be as given in (14). Then,

$$h_r(\cdot,\varphi) = e^{-\varphi V^{-1}Q} J_1 \Gamma_2 r = \begin{pmatrix} A_\varphi(r^+ - \Pi^- r^-) \\ C_\varphi(r^+ - \Pi^- r^-) \end{pmatrix}.$$

The outline of the proof is the following: first we show that the vector $A_\varphi(r^+ - \Pi^- r^-)$ is positive by showing that it is a Perron-Frobenius vector of some positive matrix. Then, because $C_\varphi(r^+ - \Pi^- r^-) = C_\varphi A_\varphi^{-1}\, A_\varphi(r^+ - \Pi^- r^-)$ and that the matrix $C_\varphi A_\varphi^{-1}$ is, by Lemma 7.2, Theorem 7.3 and by (viii) in Jacka et al. (2004), positive, we conclude that the vector $C_\varphi(r^+ - \Pi^- r^-)$ is also positive and that the function h_r is positive.

Therefore, all we have to prove is that the vector $A_\varphi(r^+ - \Pi^- r^-)$ is positive for any $\varphi \in \mathbb{R}$. Let r be fixed vector such that $V^{-1}Qr = 1$. Then

$$e^{yV^{-1}Q} r = r + y1 \quad \Leftrightarrow \quad \begin{aligned} A_{-y} r^+ + B_{-y} r^- &= r^+ + y1^+ \\ C_{-y} r^+ + D_{-y} r^- &= r^- + y1^-. \end{aligned}$$

By (17), the matrix A_φ is invertible. Thus, because $1^+ = \Pi^- 1^-$, $(A_{-y} - \Pi^- C_{-y}) = (A_y - \Pi^- C_y)^{-1}$ and $(B_{-y} - \Pi^- D_{-y}) = -(A_{-y} - \Pi^- C_{-y})\Pi^-$,

$$\left(A_\varphi (A_y - \Pi^- C_y)^{-1} A_\varphi^{-1}\right) A_\varphi(r^+ - \Pi^- r^-) = A_\varphi(r^+ - \Pi^- r^-).$$

By Theorem 7.3 the matrix $A_\varphi(A_y - \Pi^- C_y)^{-1}$ is positive. By Lemma 7.2, Theorem 7.3 and by (viii) in Jacka et al. (2004), the matrix A_φ^{-1} is also positive. Hence, the matrix $A_\varphi(A_y - \Pi^- C_y)^{-1} A_\varphi^{-1}$ is positive and it has the Perron-Frobenius eigenvector which is also positive.

Suppose that $A_\varphi(r^+ - \Pi^- r^-) = 0$. Then, because A_φ is invertible, $(r^+ - \Pi^- r^-) = 0$. If $r^+ = \Pi^- r^-$ then r is a linear combination of the vectors g_k, $k = 1, \ldots, m$ in the basis $\mathcal{B}$, but that is not possible because r is also in the basis $\mathcal{B}$ and therefore independent from g_k, $k = 1, \ldots, m$. Hence, the vector $A_\varphi(r^+ - \Pi^- r^-) \neq 0$ and by the last equation it is the eigenvector of the matrix $A_\varphi(A_{-y} - \Pi^- C_{-y}) A_\varphi^{-1}$ which corresponds to its eigenvalue 1.

It follows from

$$\left(A_\varphi (A_y - \Pi^- C_y)^{-1} A_\varphi^{-1}\right) A_\varphi(I - \Pi^- \Pi^+) = A_\varphi(I - \Pi^- \Pi^+)\, e^{yG^+} \tag{2}$$

that if α is a non-zero eigenvalue of the matrix G^+ with some algebraic multiplicity, then $e^{\alpha y}$ is an eigenvalue of the matrix $A_\varphi(A_y - \Pi^- C_y)^{-1} A_\varphi^{-1}$ with the same algebraic multiplicity. Since all $n-1$ non-zero eigenvalues of G^+ are with negative real parts, all eigenvalues $e^{\alpha_j y}$, $\alpha_j \neq 0$, $j = 1, \ldots, n$, of $A_\varphi(A_y - \Pi^- C_y)^{-1} A_\varphi^{-1}$ have real parts strictly less than 1. Thus, 1 is the Perron-Frobenius eigenvalue of the matrix $A_\varphi(A_y - \Pi^- C_y)^{-1} A_\varphi^{-1}$ and the vector $A_\varphi(r^+ - \Pi^- r^-)$ is its Perron-Frobenius eigenvector, and therefore positive.

(ii) The statement follows directly from the equality

$$\lim_{y \to +\infty} \frac{P_{(e',\varphi')}(H_y < H_0)}{P_{(e,\varphi)}(H_y < H_0)} = \lim_{y \to +\infty} \frac{G_0(\varphi', y)1(e')}{G_0(\varphi, y)1(e)},$$

where $G_0(\varphi, y)$ is the Green's function for the killed process defined in Appendix, and from the representation of $G_0(\varphi, y)$ given by

$$G_0(\varphi, y)1 = \sum_{j,\alpha_j \neq 0} a_j \ e^{-\varphi V^{-1}Q} J_1 \Gamma_2 \ e^{yV^{-1}Q} f_j + \ c \ e^{-\varphi V^{-1}Q} J_1 \Gamma_2 r,$$

for some constants a_j, $j = 1, \dots, n$ and $c \neq 0$. For the details of the proof see Najdanovic (2003). □

Proof of Theorem 2.1: For fixed $(e, \varphi) \in E_0^+$, $t \in [0, +\infty)$ and $y \geq 0$, let $h_y(e, \varphi, t)$ be a random variable defined on the probability space $(\Omega, \mathcal{F}, P_{(e,\varphi)})$ by

$$\begin{aligned} h_y(e, \varphi, t) \quad = \quad & \frac{1}{P_{(e,\varphi)}(H_y < H_0)} \Big(I\{t < H_0 \wedge H_y\} P_{(X_t, \varphi_t)}(H_y < H_0) \\ & + I\{H_y \leq t < H_0\} + I\{H_y < H_0 \leq t\} \Big). \end{aligned}$$

By Lemma 3.1 (ii) and by Lemmas 3.2, 3.3 and 3.4 in Jacka et al. (2004) the random variables $h_y(e, \varphi, t)$ converge to $\frac{h_r(X_t, \varphi_t)}{h_r(e,\varphi)} I\{t < H_0\}$ in $L^1(\Omega, \mathcal{F}, P_{(e,\varphi)})$ as $y \to +\infty$. Therefore, by (1), for fixed $t \geq 0$ and $A \in \mathcal{F}_t$,

$$\lim_{y \to +\infty} P^y_{(e,\varphi)}(A) = \lim_{y \to +\infty} E_{(e,\varphi)} \Big(I(A) h_y(e, \varphi, t) \Big) = P^r_{(e,\varphi)}(A),$$

which, by Lemma 3.3 (ii) in Jacka et al. (2004), implies that the measures $(P^y_{(e,\varphi)}|_{\mathcal{F}_t})_{y \geq 0}$ converge weakly to $P^r_{(e,\varphi)}|_{\mathcal{F}_t}$ as $y \to \infty$. □

4 The negative drift case: Proof of Theorem 2.2

Again, as in the oscillating case, we start with the limit of $P^y_{(e,\varphi)}(A)$ as $y \to +\infty$ by looking at $\lim_{y \to +\infty} \frac{P_{(e',\varphi')}(H_y < H_0)}{P_{(e,\varphi)}(H_y < H_0)}$. First we prove an auxiliary lemma.

Lemma 4.1 *For any vector* g *on* E $\lim_{y \to +\infty} F(y) g = 0$.

In addition, for any non-negative vector g *on* E $\lim_{y \to +\infty} e^{-\alpha_{max} y} F(y) g = c \ J_1 f_{max}$ *for some positive constant* $c \in \mathbb{R}$.

Proof: Let

$$g = \begin{pmatrix} g^+ \\ g^- \end{pmatrix} \text{ and } g^+ = \sum_{j=1}^{n} a_j f_j^+,$$

for some coefficients a_j, $j = 1, \dots, n$, where vectors f_j^+, $j = 1, \dots, n$, form the basis $\mathcal{N}^+$ and are associated with the eigenvalues α_j, $j = 1, \dots, n$ (see Jacka *et.al* (2004)).

Then, the first equality in the lemma follows from

$$F(y)g = \begin{pmatrix} e^{yG^+} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g^+ \\ g^- \end{pmatrix} = \begin{pmatrix} e^{yG^+}g^+ \\ 0 \end{pmatrix} = \sum_{j=1}^{n} a_j \begin{pmatrix} e^{yG^+}f_j^+ \\ 0 \end{pmatrix}, \qquad y > 0, \tag{3}$$

since, for $Re(\alpha_j) < 0$, $j = 1, \dots, n$, $e^{yG^+}f_j^+ \to 0$ as $y \to +\infty$.

Moreover, by (iii) in Jacka et al. (2004), the matrix G^+ is an irreducible Q-matrix with the Perron-Frobenius eigenvalue α_{max} and Perron-Frobenius eigenvector f_{max}^+. Thus, for any non-negative vector g on E^+, by (VII) in Jacka et al. (2004),

$$\lim_{y\to+\infty} e^{-\alpha_{max}y}\ e^{yG^+}g(e) = c\ f_{max}^+(e), \tag{4}$$

for some positive constant $c \in \mathbb{R}$. Therefore, from (3) and (4)

$$\lim_{y\to+\infty} e^{-\alpha_{max}y}\ F(y)g = \lim_{y\to+\infty} \begin{pmatrix} e^{-\alpha_{max}y}\ e^{yG^+}g^+ \\ 0 \end{pmatrix} = c\ \begin{pmatrix} f_{max}^+ \\ 0 \end{pmatrix} = c\ J_1 f_{max}.$$

□

Now we find the limit $\lim_{y\to+\infty} \frac{P_{(e',\varphi')}(H_y<H_0)}{P_{(e,\varphi)}(H_y<H_0)}$.

Lemma 4.2

$$(i) \qquad h_{f_{max}}(e,\varphi) \equiv e^{-\varphi V^{-1}Q} J_1\Gamma_2 f_{max}(e) > 0, \quad (e,\varphi) \in E_0^+,$$

$$(ii) \quad \lim_{y\to+\infty} \frac{P_{(e',\varphi')}(H_y < H_0)}{P_{(e,\varphi)}(H_y < H_0)} = \frac{e^{-\varphi' V^{-1}Q} J_1\Gamma_2 f_{max}(e')}{e^{-\varphi V^{-1}Q} J_1\Gamma_2 f_{max}(e)}, \quad (e,\varphi), (e',\varphi') \in E_0^+.$$

Proof: (i) The function $h_{f_{max}}$ can be rewritten as

$$h_{f_{max}}(\cdot,\varphi) = e^{-\varphi V^{-1}Q} J_1\Gamma_2 f_{\max} = \begin{pmatrix} A_\varphi(I - \Pi^-\Pi^+)f_{max}^+ \\ C_\varphi(I - \Pi^-\Pi^+)f_{max}^+ \end{pmatrix}$$

where A_φ and C_φ are given by (14).

First we show that the vector $A_\varphi(I - \Pi^-\Pi^+)f_{max}^+$ is positive. By (16), (iv) and (ii) in Jacka et al. (2004) the matrix $(I - \Pi^-\Pi^+)$ is invertible and by (17) the matrix A_φ is invertible. Therefore,

$$A_\varphi(A_{-y} - \Pi^- C_{-y})A_\varphi^{-1} = A_\varphi(I - \Pi^-\Pi^+)e^{yG^+}(I - \Pi^-\Pi^+)^{-1}A_\varphi^{-1}.$$

By Theorem 7.3 the matrix $A_\varphi(A_y - \Pi^- C_y)^{-1}$ is positive and by Lemma 7.2, Theorem 7.3 and by (viii) in Jacka et al.(2004), the matrix A_φ^{-1} is also positive. Hence, the matrix $A_\varphi(A_{-y} - \Pi^- C_{-y})A_\varphi^{-1}$ is positive and is similar to e^{yG^+}. Thus, $A_\varphi(A_{-y} - \Pi^- C_{-y})A_\varphi^{-1}$ and e^{yG^+} have the same Perron-Frobenius eigenvalue and because the Perron-Frobenius eigenvector of e^{yG^+} is f^+_{max}, it follows that $A_\varphi(I-\Pi^-\Pi^+)f^+_{max}$ is the Perron-Frobenius eigenvector of $A_\varphi(A_{-y}-\Pi^- C_{-y})A_\varphi^{-1}$ and therefore positive. In addition,

$$C_\varphi(I-\Pi^-\Pi^+)f^+_{max} = C_\varphi A_\varphi^{-1}\ A_\varphi(I-\Pi^-\Pi^+)f^+_{max},$$

and by Lemma 7.2, Theorem 7.3 and by (viii) in Jacka et al. (2004), the matrix $C_\varphi A_\varphi^{-1}$ is positive. Therefore, the function $h_{f_{max}}$ is positive.

(ii) By Lemmas 7.2, 4.1 and Theorem 7.3

$$\lim_{y\to+\infty}\frac{P_{(e',\varphi')}(H_y<H_0)}{P_{(e,\varphi)}(H_y<H_0)} == \lim_{y\to+\infty}\frac{e^{-\varphi' V^{-1}Q}\Gamma\Gamma_2 F(y)1(e')}{e^{-\varphi V^{-1}Q}\Gamma\Gamma_2 F(y)1(e)}.$$

Since the vector 1 is non-negative and because $\Gamma\Gamma_2 J_1 f_{max} = J_1\Gamma_2 f_{max}$, the statement in the lemma follows from Lemma 4.1. ☐

The function $h_{f_{max}}$ has the property that the process $\{h_{f_{max}}(X_t,\varphi_t)I\{t<H_0\}, t\geq 0\}$ is a martingale under $P_{(e,\varphi)}$. We prove this in the following lemma.

Lemma 4.3 *The process* $\{h_{f_{max}}(X_t,\varphi_t)I\{t<H_0\}, t\geq 0\}$ *is a martingale under* $P_{(e,\varphi)}$.

Proof: The function $h_{f_{max}}(e,\varphi)$ is continuously differentiable in φ and therefore by (15) in Jacka et al. (2004) it is in the domain of the infinitesimal generator $\mathcal{G}$ of the process $(X_t,\varphi_t)_{t\geq0}$ and $\mathcal{G}h_{f_{max}}=0$. The rest of the proof is analogous to the proof of Lemma 3.3 in Jacka et al. (2004). ☐

Proof of Theorem 2.2: The theorem is proved in the same way as Theorem 2.1, the only difference is that Lemma 4.2 is used instead of Lemma 3.1. ☐

5 The negative drift case: conditioning $(\varphi_t)_{t\geq0}$ to drift to $+\infty$

The process $(X_t,\varphi_t)_{t\geq0}$ can also be conditioned first on the event that $(\varphi_t)_{t\geq0}$ hits large levels y regardless of crossing zero (that is taking the limit as $y\to\infty$ of conditioning $(X_t,\varphi_t)_{t\geq0}$ on $\{H_y<+\infty\}$), and then the resulting process can be conditioned on the event that $(\varphi_t)_{t\geq0}$ stays non-negative. In this section we show that these two conditionings performed in the stated order yield the same result as the limit as $y\to+\infty$ of conditioning $(X_t,\varphi_t)_{t\geq0}$ on $\{H_y<H_0\}$.

Let $(e,\varphi)\in E_0^+$ and $y>\varphi$. Then, by (ix) in Jacka et al. (2004), the event $\{H_y<+\infty\}$ is of positive probability and the process $(X_t,\varphi_t)_{t\geq0}$ can be conditioned on $\{H_y<+\infty\}$ in the standard way.

For fixed $t\geq0$ and any $A\in\mathcal{F}_t$,

$$P_{(e,\varphi)}(A\mid H_y<+\infty) = \frac{E_{(e,\varphi)}\Big(I(A)P_{(X_t,\varphi_t)}(H_y<+\infty)I\{t<H_y\}+I(A)I\{H_y<t\}\Big)}{P_{(e,\varphi)}(H_y<+\infty)}. \tag{5}$$

Lemma 5.1 *For any* $(e,\varphi),(e',\varphi')\in E_0^+$,

$$\lim_{y\to+\infty}\frac{P_{(e',\varphi')}(H_y<+\infty)}{P_{(e,\varphi)}(H_y<+\infty)}=\frac{e^{-\alpha_{max}\varphi'}f_{max}(e')}{e^{-\alpha_{max}\varphi}f_{max}(e)}.$$

Proof: By Lemma 5.5 in Jacka et al. (2004), for $0\le\varphi<y$,

$$P_{(e,\varphi)}(H_y<+\infty)=P_{(e,\varphi-y)}(H_0<+\infty)=\Gamma F(y-\varphi)1.$$

The vector 1 is non-negative. Hence, by Lemma 4.1 and because $\Gamma J_1 f_{max}=f_{max}$,

$$\begin{aligned}\lim_{y\to+\infty}\frac{P_{(e',\varphi')}(H_y<+\infty)}{P_{(e,\varphi)}(H_y<+\infty)} &= \lim_{y\to+\infty}\frac{e^{-\alpha_{max}\varphi'}\Gamma e^{-\alpha_{max}(y-\varphi')}F(y-\varphi)1(e')}{e^{-\alpha_{max}\varphi}\Gamma e^{-\alpha_{max}(y-\varphi)}F(y-\varphi)1(e)}\\ &= \frac{e^{-\alpha_{max}\varphi'}f_{max}(e')}{e^{-\alpha_{max}\varphi}f_{max}(e)}.\end{aligned}$$

□

Let $h_{max}(e,\varphi)$ be a function on $E\times\mathbb{R}$ defined by

$$h_{max}(e,\varphi)=e^{-\alpha_{max}\varphi}f_{max}(e).$$

Lemma 5.2 *The process* $(h_{max}(X_t,\varphi_t))_{t\ge0}$ *is a martingale under* $P_{(e,\varphi)}$.

Proof: The function $h_{max}(e,\varphi)$ is continuously differentiable in φ which implies that it is in the domain of the infinitesimal generator $\mathcal{G}$ of the process $(X_t,\varphi_t)_{t\ge0}$. In addition, $\mathcal{G}h_{max}=0$. It follows that the process $(h_{max}(X_t,\varphi_t))_{t\ge0}$ is a local martingale under $P_{(e,\varphi)}$ and, because it is bounded on every finite interval, the process $(h_{max}(X_t,\varphi_t))_{t\ge0}$ is a martingale under $P_{(e,\varphi)}$. □

By Lemmas 5.1 and 5.2 we prove

Theorem 5.1 *For fixed* $(e,\varphi)\in E_0^+$, *let* $P^{h_{max}}_{(e,\varphi)}$ *be a measure defined by*

$$P^{h_{max}}_{(e,\varphi)}(A)=\frac{E_{(e,\varphi)}\Big(I(A)\ h_{max}(X_t,\varphi_t)\Big)}{h_{max}(e,\varphi)},\qquad t\ge0,A\in\mathcal{F}_t.$$

Then, $P^{h_{max}}_{(e,\varphi)}$ *is a probability measure and, for fixed* $t\ge0$,

$$\lim_{y\to+\infty}P_{(e,\varphi)}(A\mid H_y<+\infty)=P^{h_{max}}_{(e,\varphi)}(A),\quad A\in\mathcal{F}_t.$$

Proof: By the definition, the function h_{max} is positive. Hence $P^{h_{max}}_{(e,\varphi)}$ is a measure. In addition, by Lemma 5.2, the process $(h_{max}(X_t,\varphi_t))_{t\ge0}$ is a martingale under $P_{(e,\varphi)}$. Hence, $P^{h_{max}}_{(e,\varphi)}$ is a probability measure.

For fixed $(e,\varphi) \in E_0^+$ and $t, y \geq 0$, let $h_y(e,\varphi,t)$ be a random variable defined on the probability space $(\Omega, \mathcal{F}, P_{(e,\varphi)})$ by

$$h_y(e,\varphi,t) = \frac{P_{(X_t,\varphi_t)}(H_y < +\infty) I\{t < H_y\} + I(A) I\{H_y < t\}}{P_{(e,\varphi)}(H_y < +\infty)}.$$

The random variables $h_y(e,\varphi,t)$, $y \geq 0$, are non-negative and, by Lemma 5.1,

$$\lim_{y \to +\infty} h_y(e,\varphi,t) = \frac{h_{max}(X_t,\varphi_t)}{h_{max}(e,\varphi)}, \quad a.s..$$

The rest of the proof is analogous to the proof of Theorem 2.1. □

We now want to condition the process $(X_t,\varphi_t)_{t\geq 0}$ under $P^{h_{max}}_{(e,\varphi)}$ on the event $\{H_0 = +\infty\}$. By Theorem 7.4, $(X_t)_{t\geq 0}$ under $P^{h_{max}}_{(e,\varphi)}$ is Markov with the irreducible conservative Q-matrix $Q^{h_{max}}$ given by

$$Q^{h_{max}}(e,e') = \frac{f_{max}(e')}{f_{max}(e)} (Q - \alpha_{max} V)(e,e'), \qquad e, e' \in E,$$

and, by the same theorem, the process $(\varphi_t)_{t\geq 0}$ under $P^{h_{max}}_{(e,\varphi)}$ drifts to $+\infty$. We find the Wiener-Hopf factorization of the matrix $V^{-1}Q^{h_{max}}$.

Lemma 5.3 *The unique Wiener-Hopf factorization of the matrix* $V^{-1}Q^{h_{max}}$ *is given by* $V^{-1}Q^{h_{max}}\ \Gamma^{h_{max}} = \Gamma^{h_{max}}\ G^{h_{max}}$, *where, for any* $(e,e') \in E \times E$,

$$G^{h_{max}}(e,e') = \frac{f_{max}(e')}{f_{max}(e)}\ (G - \alpha_{max} I)(e,e') \quad \text{and} \quad \Gamma^{h_{max}}(e,e') = \frac{f_{max}(e')}{f_{max}(e)}\ \Gamma(e,e').$$

In addition, if

$$G^{h_{max}} = \begin{pmatrix} G^{h_{max},+} & 0 \\ 0 & -G^{h_{max},-} \end{pmatrix} \quad \text{and} \quad \Gamma^{h_{max}} = \begin{pmatrix} I & \Pi^{h_{max},-} \\ \Pi^{h_{max},+} & I \end{pmatrix},$$

then $G^{h_{max},+}$ *is a conservative* $Q-$*matrix and* $\Pi^{h_{max},+}$ *is stochastic, and* $G^{h_{max},-}$ *is not a conservative* $Q-$*matrix and* $\Pi^{h_{max},-}$ *is strictly substochastic.*

Proof: By the definition the matrices $G^{h_{max},+}$ and $G^{h_{max},-}$ are essentially non-negative. In addition, for any $e \in E^+$, $G^{h_{max},+}1(e) = 0$. Hence, $G^{h_{max},+}$ is a conservative Q-matrix. By Lemma 4.2 (i),

$$h^-_{f_{max}} = (\Pi^+ e^{-\varphi G^+} - e^{\varphi G^-}\Pi^+) f^+_{max} = e^{-\alpha_{max}\varphi}(I - e^{\varphi(G^- + \alpha_{max} I)}) f^-_{max} > 0.$$

Since

$$\lim_{\varphi \to 0} \frac{(I - e^{\varphi(G^- + \alpha_{max} I)})\ f^-_{max}}{\varphi} = -(G^- + \alpha_{max} I) f^-_{max},$$

and $(I - e^{\varphi(G^- + \alpha_{max} I)}) f^-_{max} > 0$, it follows that $(G^- + \alpha_{max} I) f^-_{max} \leq 0$. Thus, $G^{h_{max},-} 1^- \leq 0$ and so $G^{h_{max},-}$ is a Q-matrix. Moreover, if $(G^- + \alpha_{max} I) f^-_{max} = 0$ then $h_{f_{max}}(e, \varphi) = 0$ for $e \in E^-$ which is a contradiction to Lemma 4.2. Therefore, the matrix $G^{h_{max},-}$ is not conservative.

The matrices $G^{h_{max}}$ and $\Gamma^{h_{max}}$ satisfy the equality $V^{-1} Q^{h_{max}}\, \Gamma^{h_{max}} = \Gamma^{h_{max}}\, G^{h_{max}}$, which, by Lemma 5.4 in Jacka et al. (2004), gives the unique Wiener-Hopf factorization of the matrix $V^{-1} Q^{h_{max}}$. Finally, by (iv) in Jacka et al. (2004), $\Pi^{h_{max},+}$ is a stochastic and $\Pi^{h_{max},-}$ is a strictly substochastic matrix. □

Finally, we prove the main result in this section

Theorem 5.2 *Let $P^{f_{\max}}_{(e,\varphi)}$ be as defined in Theorem 2.2. Then, for any $(e, \varphi) \in E^+_0$ and any $t \geq 0$,*

$$P^{h_{max}}_{(e,\varphi)}(A\ |H_0 = \infty) = P^{f_{\max}}_{(e,\varphi)}(A), \qquad A \in \mathcal{F}_t.$$

Proof: By Theorem 7.4 the process $(\varphi_t)_{t\geq 0}$ under $P^{h_{max}}_{(e,\varphi)}$ drifts to $+\infty$. Since in the positive drift case the event $\{H_0 = +\infty\}$ is of positive probability, for any $t \geq 0$ and any $A \in \mathcal{F}_t$,

$$P^{h_{\max}}_{(e,\varphi)}(A\ |H_0 = \infty) = \frac{E^{h_{\max}}_{(e,\varphi)}\Big(I(A)\ P^{h_{max}}_{(X_t,\varphi_t)}(H_0 {=} {+}\infty)\ I\{t < H_0\}\Big)}{P^{h_{max}}_{(e,\varphi)}(H_0 {=} {+}\infty)}. \tag{6}$$

By Lemma 5.5 in Jacka et al. (2004) and by Lemma 5.3, for $\varphi > 0$,

$$\begin{aligned} P^{h_{max}}_{(e,\varphi)}(H_0 = +\infty) &= 1 - \frac{e^{\alpha_{max}\varphi}}{f_{max}(e)} \sum_{e'' \in E} \Gamma e^{-\varphi G}(e, e'')\ J_2 1(e'')\ f_{max}(e'') \\ &= \frac{1}{h_{max}(e,\varphi)} \Big(e^{-\alpha_{max}\varphi}\ f_{max} - \Gamma F(-\varphi) f_{max}\Big)(e) \\ &= \frac{h_{f_{max}}(e,\varphi)}{h_{max}(e,\varphi)}, \end{aligned} \tag{7}$$

where $h_{f_{max}}$ is as defined in Lemma 4.2. Similarly, for $e \in E^+$,

$$P^{h_{max}}_{(e,0)}(H_0 = +\infty) = \frac{f^+_{max} - \Pi^- f^-_{max})(e)}{f^+_{max}(e)} = \frac{h_{f_{max}}(e,0)}{h_{max}(e,0)}.$$

Therefore, the statement in the theorem follows from Theorem 5.1, (6) and (7). □

We summarize the results from this section: in the negative drift case, making the h-transform of the process $(X_t, \varphi_t)_{t\geq 0}$ with the function $h_{max}(e, \varphi) = e^{-\alpha_{max}\varphi} f_{max}(e)$ yields the probability measure $P^{h_{max}}_{(e,\varphi)}$ such that $(X_t)_{t\geq 0}$ under $P^{h_{max}}_{(e,\varphi)}$ is Markov and that $(\varphi_t)_{t\geq 0}$ under $P^{h_{max}}_{(e,\varphi)}$ is with a positive drift. The process $(X_t, \varphi_t)_{t\geq 0}$ under $P^{h_{max}}_{(e,\varphi)}$ is also the limiting process as $y \to +\infty$ in conditioning $(X_t, \varphi_t)_{t\geq 0}$ under $P_{(e,\varphi)}$ on $\{H_y < +\infty\}$. Further conditioning $(X_t, \varphi_t)_{t\geq 0}$ under $P^{h_{max}}_{(e,\varphi)}$ on $\{H_0 = +\infty\}$ yields the same result as the limit as $y \to +\infty$ of conditioning $(X_t, \varphi_t)_{t\geq 0}$ on $\{H_y < H_0\}$. In other words, the diagram in Figure 1 commutes.

$$\begin{tikzcd}
\begin{array}{c}(X_t,\varphi_t)_{t\geq 0} \text{ under } P_{(e,\varphi)} \\ \text{(the negative drift case)}\end{array} \arrow[rr, "\{H_y < H_0\},\ y\to+\infty"] \arrow[dr, swap, "\{H_y < +\infty\},\ y\to+\infty"] & & (X_t,\varphi_t)_{t\geq 0} \text{ under } P^{f_{max}}_{(e,\varphi)} \\
& \begin{array}{c}(X_t,\varphi_t)_{t\geq 0} \text{ under } P^{h_{max}}_{(e,\varphi)} \\ \text{(the positive drift case)}\end{array} \arrow[ur, swap, "\{H_0 = +\infty\}"] &
\end{tikzcd}$$

Figure 1: The negative drift case of conditioning the process $(X_t,\varphi_t)_{t\geq 0}$ on the events $\{H_y < H_0\}$, $y \geq 0$.

6 The negative drift case: conditioning $(\varphi_t)_{t\geq 0}$ to oscillate

In this section we condition the process $(\varphi_t)_{t\geq 0}$ with a negative drift to oscillate, and then condition the resulting oscillating process to stay non-negative. Let $P^h_{(e,\varphi)}$ denote the h-transformed measure $P_{(e,\varphi)}$ with a function h. We want to find a function h such that the process $(X_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ is Markov and that the process $(\varphi_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ oscillates. By Theorem 7.4, there does not exist such function defined on $E\times\mathbb{R}$. But, by Theorem 7.5, there exists exactly one such function defined on $E\times\mathbb{R}\times[0,+\infty)$ which is given by

$$h_0(e,\varphi,t) = e^{-\alpha_0 t}e^{-\beta_0\varphi}g_0(e),$$

where $\alpha(\beta)$ is the Perron-Frobenius eigenvalue of the matrix $(Q-\beta V)$, β_0 is the argmin of $\alpha(\cdot)$, $\alpha_0 = \alpha(\beta_0)$ and g_0 is the Perron-Frobenius eigenvector of the matrix $(Q-\beta_0 V)$.

For fixed $(e,\varphi)\in E^+_0$, let a measure $P^{h_0}_{(e,\varphi)}$ be defined by

$$P^{h_0}_{(e,\varphi)}(A) = \frac{E_{(e,\varphi)}\Big(I(A)h_0(X_t,\varphi_t,t)\Big)}{h_0(e,\varphi,0)}, \quad A\in\mathcal{F}_t,\ t\geq 0. \tag{8}$$

Then, the process $(X_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ is Markov with the Q-matrix Q^0 given by

$$Q^0(e,e') = \frac{g_0(e')}{g_0(e)}(Q-\alpha_0 I-\beta_0 V)(e,e'), \qquad e,e'\in E. \tag{9}$$

and, by Theorem 7.5, the process $(\varphi_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ oscillates.

The aim now is to condition $(X_t,\varphi_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ on the event that $(\varphi_t)_{t\geq 0}$ stays non-negative. The following theorem determines the law of this new conditioned process.

Theorem 6.1 *For fixed $(e,\varphi) \in E_0^+$, let a measure $P^{h_0,h_r^0}_{(e,\varphi)}$ be defined by*

$$P^{h_0,h_r^0}_{(e,\varphi)}(A) = \frac{E^{h_0}_{(e,\varphi)}\Big(I(A) h_r^0(X_t,\varphi_t) I\{t < H_0\}\Big)}{h_r^0(e,\varphi)}, \quad A \in \mathcal{F}_t,\ t \geq 0,$$

where the function h_r^0 is given by $h_r^0(e,y) = e^{-yV^{-1}Q^0} J_1 \Gamma_2 r^0(e)$, $(e,y) \in E \times \mathbb{R}$, and $V^{-1}Q^0 r^0 = 1$. Then, $P^{h_0,h_r^0}_{(e,\varphi)}$ is a probability measure.

In addition, for $t \geq 0$ and $A \in \mathcal{F}_t$,

$$P^{h_0,h_r^0}_{(e,\varphi)}(A) = \lim_{y\to\infty} P^{h_0}_{(e,\varphi)}(A \mid H_y < H_0) = \lim_{T\to\infty} P^{h_0}_{(e,\varphi)}(A \mid H_0 > T),$$

and

$$P^{h_0,h_r^0}_{(e,\varphi)}(A) = P^{r^0}_{(e,\varphi)}(A),$$

where $P^{r^0}_{(e,\varphi)}$ is as defined in Theorem 2.2 in Jacka et al. (2004).

Proof: By Lemma 5.9 and (16) in Jacka et al. (2004), the Q-matrix Q^0 of the process $(X_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ is conservative and irreducible and the process $(\varphi_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ oscillates. Thus, if $P^{h_0,h_r^0}_{(e,\varphi)}$ denotes the law of $(X_t,\varphi_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ conditioned on $\{H_0 = +\infty\}$, then, by Theorem 2.1 in Jacka et al. (2004) and by Theorem 2.1, $P^{h_0,h_r^0}_{(e,\varphi)}$ is a probability measure and

$$P^{h_0,h_r^0}_{(e,\varphi)}(A) = \lim_{y\to\infty} P^{h_0}_{(e,\varphi)}(A|H_y < H_0) = \lim_{T\to\infty} P^{h_0}_{(e,\varphi)}(A|H_0 > T).$$

In addition, by definition (8) of the measure $P^{h_0}_{(e,\varphi)}$, for $t \geq 0$ and $A \in \mathcal{F}_t$,

$$P^{h_0,h_r^0}_{(e,\varphi)}(A) = \frac{E_{(e,\varphi)}\Big(I(A)\ h_0(X_t,\varphi_t,t)\ h_r^0(X_t,\varphi_t)\ I\{t < H_0\}\Big)}{h_0(e,\varphi,0)\ h_r^0(e,\varphi)} = P^{r^0}_{(e,\varphi)}(A),$$

since $h_0(e,\varphi,t)\ h_r^0(e,\varphi) = h_{r^0}(e,\varphi,t)$ where $h_{r^0}(e,\varphi,t)$ is as defined in Theorem 2.2 in Jacka et al. (2004). □

We summarize the results in this section: in the negative drift case, making the h-transform of the process $(X_t,\varphi_t,t)_{t\geq 0}$ with the function $h_0(e,\varphi) = e^{-\alpha_0\varphi} e^{-\beta_0\varphi} g_0(e)$ yields the probability measure $P^{h_0}_{(e,\varphi)}$ such that $(X_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ is Markov and that $(\varphi_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ oscillates. Then the law of $(X_t,\varphi_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ conditioned on the event $\{H_0 = +\infty\}$ is equal to $P^{h_0,h_r^0}_{(e,\varphi)} = P^{r^0}_{(e,\varphi)}$. On the other hand, by Theorem 2.2 in Jacka et al. (2004), under the condition that all non-zero eigenvalues of the matrix $V^{-1}Q^0$ are simple, $P^{r^0}_{(e,\varphi)}$ is the limiting law as $T \to +\infty$ of the process $(X_t,\varphi_t)_{t\geq 0}$ under $P^{r^0}_{(e,\varphi)}$ conditioned on $\{H_0 > T\}$. Hence, under the condition that all non-zero eigenvalues of the matrix $V^{-1}Q^0$ are simple, the diagram in Figure 2 commutes.

$$\begin{tikzcd}
\begin{array}{l}(X_t,\varphi_t)_{t\geq 0} \text{ under } P_{(e,\varphi)} \\ \text{(the negative drift case)}\end{array} \arrow[rr, "{\{H_0 > T\},\ T \to +\infty}"] \arrow[dr] & & (X_t,\varphi_t)_{t\geq 0} \text{ under } P^{r^0}_{(e,\varphi)} \\
& \begin{array}{l}(X_t,\varphi_t)_{t\geq 0} \text{ under } P^{h_0}_{(e,\varphi)} \\ \text{(the oscillating case)}\end{array} \arrow[ur, "{\begin{array}{l}\{T < H_0\},\ T \to +\infty \\ \{H_y < H_0\},\ y \to +\infty\end{array}}"'] &
\end{tikzcd}$$

Figure 2: The negative drift case of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the events $\{H_0 > T\}$, $T \geq 0$.

7 Appendix: The Green's function

The Green's function of the process $(X_t, \varphi_t)_{t\geq 0}$, denoted by $G((e,\varphi),(f,y))$, for any $(e,\varphi),(f,y) \in E \times \mathbb{R}$, is defined as

$$G((e,\varphi),(f,y)) = E_{(e,\varphi)}\Big(\sum_{0\leq s<\infty} I(X_s = f, \varphi_s = y)\Big),$$

noting that the process $(X_t, \varphi_t)_{t\geq 0}$ hits any fixed state at discrete times. For simplicity of notation, let $G(\varphi, y)$ denote the matrix $(G((\cdot,\varphi),(\cdot,y)))_{E\times E}$.

Theorem 7.1 *In the drift cases,*

$$G(0,0) = \Gamma_2^{-1} = \begin{pmatrix} (I - \Pi^-\Pi^+)^{-1} & \Pi^-(I - \Pi^+\Pi^-)^{-1} \\ \Pi^+(I - \Pi^-\Pi^+)^{-1} & (I - \Pi^+\Pi^-)^{-1} \end{pmatrix}.$$

In the oscillating case, $G(0,0) = +\infty$.

Proof: By the definition of $G(0,0)$ and the matrices Π^+, Π^- and Γ_2,

$$G(0,0) = \sum_{n=1}^{\infty} \begin{pmatrix} 0 & \Pi^- \\ \Pi^+ & 0 \end{pmatrix}^n = \sum_{n=1}^{\infty} (I - \Gamma_2)^n.$$

Suppose that the process $(\varphi_t)_{t\geq 0}$ drifts either to $+\infty$ of $-\infty$. Then by (16) and (IV) in Jacka et al. (2004) exactly one of the matrices Π^+ and Π^- is strictly substochastic.

In addition, the matrix $\Pi^-\Pi^+$ is positive and thus primitive. Therefore, the Perron-Frobenius eigenvalue λ of $\Pi^-\Pi^+$ satisfies $0 < \lambda < 1$ which, by the Perron-Frobenius theorem for primitive matrices, implies that

$$\lim_{n\to\infty} \frac{(\Pi^-\Pi^+)^n}{(1+\lambda)^n} = const. \neq 0.$$

Therefore, $(\Pi^-\Pi^+)^n \to 0$ elementwise as $n \to +\infty$, and similarly $(\Pi^+\Pi^-)^n \to 0$ elementwise as $n \to +\infty$. Hence, $(I - \Gamma_2)^n \to 0$, $n \to +\infty$. Since

$$I - (I - \Gamma_2)^{n+1} = \Gamma_2 \sum_{k=0}^{n} (I - \Gamma_2)^k,$$

and, by (II) in Jacka et al. (2004), Γ_2^{-1} exists, by letting $n \to +\infty$ we obtain

$$G(0,0) = \sum_{n=0}^{\infty} (I - \Gamma_2)^n = \Gamma_2^{-1}. \tag{10}$$

Suppose now that the process $(\varphi_t)_{t\geq 0}$ oscillates. Then again by (16) and (IV) in Jacka et al. (2004), the matries Π^+ and Π^- are stochastic. Thus, $(I - \Gamma_2)1 = 1$ and

$$G(0,0)1 = \sum_{n=0}^{\infty} (I - \Gamma_2)^n 1 = \sum_{n=0}^{\infty} 1 = +\infty. \tag{11}$$

Since the matrix Q is irreducible, it follows that $G(0,0) = +\infty$. □

Theorem 7.2 *In the drift cases, the Green's function* $G((e,\varphi),(f,y))$ *of the process* $(X_t,\varphi_t)_{t\geq 0}$ *is given by the* $E \times E$ *matrix* $G(\varphi,y)$, *where*

$$G(\varphi,y) = \begin{cases} \Gamma\ F(y-\varphi)\ \Gamma_2^{-1}, & \varphi \neq y \\ \Gamma_2^{-1}, & \varphi = y. \end{cases}$$

Proof: By Theorem 7.1, $G(y,y) = G(0,0) = \Gamma_2^{-1}$. and by Lemma 5.5 in Jacka et al. (2004),

$$P_{(e,\varphi-y)}(X_{H_0} = e', H_0 < +\infty) = \Gamma\ F(y-\varphi)(e,e'), \qquad \varphi \neq y.$$

The theorem now follows from

$$G((e,\varphi),(f,y)) = \sum_{e'\in E} P_{(e,\varphi-y)}(X_{H_0} = e', H_0 < +\infty)\ G((e',0),(f,0)).$$

□

The Green's function $G_0((e,\varphi),(f,y))$, $(e,\varphi),(f,y) \in E \times \mathbb{R}$, (in matrix notation $G_0(\varphi,y)$) of the process $(X_t,\varphi_t)_{t\geq 0}$ killed when the process $(\varphi_t)_{t\geq 0}$ crosses zero is defined by

$$G_0((e,\varphi),(f,y)) = E_{(e,\varphi)}\Big(\sum_{0\leq s<H_0} I(X_s=f,\varphi_s=y)\Big).$$

It follows that $G_0(\varphi,y) = 0$ if $\varphi y < 0$, that $G_0(\varphi,0) = 0$ if $\varphi \neq 0$, and that $G_0(0,0) = I$. To calculate $G_0(\varphi,y)$ for $|\varphi| \leq |y|$, $\varphi y \geq 0$, $y \neq 0$, we use the following lemma:

Lemma 7.1 *Let $(f,y) \in E^+ \times (0,+\infty)$ be fixed and let the process $(X_t,\varphi_t)_{t\geq 0}$ start at $(e,\varphi) \in E \times (0,y)$. Let $(e,\varphi) \mapsto h((e,\varphi),(f,y))$ be a bounded function on $E \times (0,y)$ such that the process $(h((X_{t\wedge H_0\wedge H_y},\varphi_{t\wedge H_0\wedge H_y}),(f,y)))_{t\geq 0}$ is a uniformly integrable martingale and that*

$$h((e,0),(f,y)) = 0, \qquad e \in E^- \tag{12}$$

$$h((e,y),(f,y)) = G_0((e,y),(f,y)). \tag{13}$$

Then

$$h((e,\varphi),(f,y)) = G_0((e,\varphi),(f,y)), \quad (e,\varphi) \in E \times (0,y).$$

Proof: The proof of the lemma is based on the fact that a uniformly integrable martingale in a region which is zero on the boundary of that region is zero everywhere. Therefore we omit the proof. □

Let A_y, B_y, C_y and D_y be components of the matrix $e^{-yV^{-1}Q}$ such that, for any $y \in \mathbb{R}$,

$$e^{-yV^{-1}Q} = \begin{pmatrix} A_y & B_y \\ C_y & D_y \end{pmatrix}. \tag{14}$$

Theorem 7.3 *The Green's function $G_0((e,\varphi),(f,y))$, $|\varphi| \leq |y|$, $\varphi y \geq 0$, $y \neq 0$, $e,f \in E$, is given by the $E \times E$ matrix $G_0(\varphi,y)$ with the components*

$$G_0(\varphi,y) = \begin{cases} \begin{pmatrix} A_\varphi(A_y-\Pi^-C_y)^{-1} & A_\varphi(A_y-\Pi^-C_y)^{-1}\Pi^- \\ C_\varphi(A_y-\Pi^-C_y)^{-1} & C_\varphi(A_y-\Pi^-C_y)^{-1}\Pi^- \end{pmatrix}, & 0 \leq \varphi < y \\ \begin{pmatrix} B_\varphi(D_y-\Pi^+B_y)^{-1}\Pi^+ & B_\varphi(D_y-\Pi^+B_y)^{-1} \\ D_\varphi(D_y-\Pi^+B_y)^{-1}\Pi^+ & D_\varphi(D_y-\Pi^+B_y)^{-1} \end{pmatrix}, & y < \varphi \leq 0, \\ \begin{pmatrix} (I-\Pi^-C_yA_y^{-1})^{-1} & \Pi^-(I-C_yA_y^{-1}\Pi^-)^{-1} \\ C_yA_y^{-1}(I-\Pi^-C_yA_y^{-1})^{-1} & (I-C_yA_y^{-1}\Pi^-)^{-1} \end{pmatrix}, & \varphi = y > 0 \\ \begin{pmatrix} (I-B_yD_y^{-1}\Pi^+)^{-1} & ByD_y^{-1}(I-\Pi^+B_yD_y^{-1})^{-1} \\ \Pi^+(I-B_yD_y^{-1}\Pi^+)^{-1} & (I-\Pi^+B_yD_y^{-1})^{-1} \end{pmatrix}, & \varphi = y < 0, \end{cases}$$

In the drift cases, $G_0(\varphi,y)$ written in matrix notation is given by

$$G_0(\varphi,y) = \begin{cases} \Gamma\ e^{-\varphi G}\ \Gamma_2\ F(y)\ \Gamma_2^{-1}, & 0 \leq \varphi < y \quad \text{or} \quad y < \varphi \leq 0 \\ \Gamma\ F(-\varphi)\ \Gamma_2\ e^{yG}\ \Gamma_2^{-1}, & 0 < y < \varphi \quad \text{or} \quad \varphi < y < 0 \\ \Big(I - \Gamma F(-y)\Gamma F(y)\Big)\Gamma_2^{-1}, & \varphi = y \neq 0. \end{cases}$$

In addition, the Green's function $G_0(\varphi, y)$ is positive for all $\varphi, y \in \mathbb{R}$ except for $y = 0$ and for $\varphi y < 0$.

Proof: We prove the theorem for $y > 0$. The case $y < 0$ can be proved in the same way.

Let $y > 0$. First we calculate the Green's function $G_0(y, y)$. Let Y_y denote a matrix on $E^- \times E^+$ with entries

$$Y_y(e, e') = P_{(e,y)}(X_{H_y} = e', H_y < H_0).$$

Then

$$G_0(y, y) = \begin{pmatrix} I & \Pi^- \\ Y_y & I \end{pmatrix} \begin{pmatrix} \sum_{n=0}^{\infty}(\Pi^- Y_y)^n & 0 \\ 0 & \sum_{n=0}^{\infty}(Y_y \Pi^-)^n \end{pmatrix}.$$

By (viii) in Jacka et al. (2004), the matrix Y_y is positive and $0 < Y_y 1^+ < 1^-$. Hence, $\Pi^- Y_y$ is positive and therefore irreducible and its Perron-Frobenius eigenvalue λ satisfies $0 < \lambda < 1$. Thus,

$$\lim_{n \to \infty} \frac{(\Pi^- Y_y)^n}{(1 + \lambda)^n} = const. \neq 0,$$

which implies that $(\Pi^- Y_y)^n \to 0$ elementwise as $n \to +\infty$. Similarly, $(Y_y \Pi^-)^n \to 0$ elementwise as $n \to +\infty$.

Furthermore, the essentially non-negative matrices $(\Pi^- Y_y - I)$ and $(Y_y \Pi^- - I)$ are invertible because their Perron-Frobenius eigenvalues are negative and, by the same argument, the matrices $(I - \Pi^- Y_y)^{-1}$ and $(I - Y_y \Pi^-)^{-1}$ are positive. Since

$$\begin{array}{rcl} \sum_{k=0}^{n}(\Pi^- Y_y)^k & = & (I - \Pi^- Y_y)^{-1} \, (I - (\Pi^- Y_y)^{n+1}) \\ \sum_{k=0}^{n}(Y_y \Pi^-)^k & = & (I - Y_y \Pi^-)^{-1} \, (I - (Y_y \Pi^-)^{n+1}). \end{array}$$

by letting $n \to \infty$ we finally obtain

$$G_0(y, y) = \begin{pmatrix} (I - \Pi^- Y_y)^{-1} & \Pi^-(I - \Pi^- Y_y)^{-1} \\ Y_y(I - Y_y \Pi^-)^{-1} & (I - Y_y \Pi^-)^{-1} \end{pmatrix} = \begin{pmatrix} I & -\Pi^- \\ -Y_y^{-1} & I \end{pmatrix}^{-1}. \tag{15}$$

By (i) and (viii) in Jacka et al. (2004), the matrices Π^- and Y_y are positive. Since the matrices $(I - \Pi^- Y_y)^{-1}$ and $(I - Y_y \Pi^-)^{-1}$ are also positive, it follows that $G_0(y, y)$, $y > 0$ is positive.

Now we calculate the Green's function $G_0(\varphi, y)$ for $0 \leq \varphi < y$. Let $(f, y) \in E^+ \times (0, +\infty)$ be fixed and let the process $(X_t, \varphi_t)_{t \geq 0}$ start in $E \times (0, y)$. Let

$$h((e, \varphi), (f, y)) = e^{-\varphi V^{-1} Q} g_{f,y}(e), \tag{16}$$

for some vector $g_{f,y}$ on E. Since by (15) in Jacka et.al (2004) $\mathcal{A}h = 0$, the process $(h((X_t, \varphi_t), (f, y)))_{t \geq 0}$ is a local martingale, and because the function h is bounded on every finite interval, it is a martingale. In addition, $(h((X_{t \wedge H_0 \wedge H_y}, \varphi_{t \wedge H_0 \wedge H_y}), (f, y)))_{t \geq 0}$ is a bounded martingale and therefore a uniformly integrable martingale.

We want the function h to satisfy the boundary conditions in Lemma 7.1. Let $h_y(\varphi)$ be an $E \times E^+$ matrix with entries

$$h_y(\varphi)(e,f) = h((e,\varphi),(f,y)).$$

Then, from (16) and the boundary condition (12),

$$h_y(\varphi) = \begin{pmatrix} A_\varphi & B_\varphi \\ C_\varphi & D_\varphi \end{pmatrix} \begin{pmatrix} M_y \\ 0 \end{pmatrix} = \begin{pmatrix} A_\varphi M_y \\ C_\varphi M_y \end{pmatrix}, \quad 0 \le \varphi < y,$$

for some $E^+ \times E^+$ matrix M_y. From the boundary condition (13),

$$A_y M_y = (I - \Pi^- Y_y)^{-1} \qquad \text{and} \qquad C_y M_y = Y_y (I - \Pi^- Y_y)^{-1}, \tag{17}$$

which implies that $M_y = (A_y - \Pi^- C_y)^{-1}$ and $Y_y = C_y A_y^{-1}$. Hence,

$$h_y(\varphi) = \begin{pmatrix} A_\varphi (A_y - \Pi^- C_y)^{-1} \\ C_\varphi (A_y - \Pi^- C_y)^{-1} \end{pmatrix}, \qquad 0 \le \varphi < y,$$

and the function $h((e,\varphi),(f,y))$ satisfies the boundary conditions (12) and (13) in Lemma 7.1. Therefore, for $0 \le \varphi < y$, $G_0(\varphi, y) = h_y(\varphi)$ on $E \times E^+$, and because $G_0(\varphi, y) = h_y(\varphi)\Pi^-$ on $E \times E^-$,

$$G_0(\varphi, y) = \begin{pmatrix} A_\varphi (A_y - \Pi^- C_y)^{-1} & A_\varphi (A_y - \Pi^- C_y)^{-1}\Pi^- \\ C_\varphi (A_y - \Pi^- C_y)^{-1} & C_\varphi (A_y - \Pi^- C_y)^{-1}\Pi^- \end{pmatrix}, \qquad 0 \le \varphi < y.$$

Finally, since $G_0(y,y)$, $y > 0$, is positive, by irreducibility $G_0(\varphi, y)$ for $0 \le \varphi < y$ is also positive. □

Lemma 7.2 *For $y \neq 0$ and any $(e,f) \in E \times E$*

$$\begin{aligned} &P_{(e,\varphi)}(X_{H_y} = f, H_y < H_0) = G_0(\varphi, y)(G_0(y,y))^{-1}(e,f), \qquad 0 < |\varphi| < |y|, \\ &P_{(e,y)}(X_{H_y} = f, H_y < H_0) = \Big(I - (G_0(y,y))^{-1}\Big)(e,f). \end{aligned}$$

Proof: By Theorem 7.3, the matrix $G_0(y,y)$ is invertible. Therefore, the equalities

$$\begin{gathered} G_0((e,\varphi),(f,y)) = \textstyle\sum_{e' \in E} P_{(e,\varphi)}(X_{H_y} = e', H_y < H_0)\ G_0((e',y),(f,y)),\ \varphi \neq y \neq 0, \\ G_0((e,y),(f,y)) = I(e,f) + \textstyle\sum_{e' \in E} P_{(e,y)}(X_{H_y} = e', H_y < H_0) G_0((e',y),(f,y)), y \neq 0, \end{gathered}$$

prove the lemma. □

We close the section by stating two results which were proved in Najdanovic (2003) and which were used in the previous sections. Let $h(e, \varphi, t)$ be a positive function on $E \times \mathbb{R} \times [0, +\infty)$ such that the process $(h(X_t, \varphi_t, t))_{t \ge 0}$ is a martingale. For fixed $(e, \varphi) \in E \times \mathbb{R}$, define a probability measure $P^h_{(e,\varphi)}$ by

$$P^h_{(e,\varphi)}(A) = \frac{E_{(e,\varphi)}\Big(I(A)\ h(X_t, \varphi_t, t)\Big)}{h(e,\varphi,0)}, \qquad A \in \mathcal{F}_t. \tag{18}$$

Theorem 7.4 *There exist only two functions* $h(e, \varphi)$ *on* $E \times \mathbb{R}$ *continuously differentiable in* φ *such that the process* $(X_t)_{t\geq 0}$ *under* $P^h_{(e,\varphi)}$ *is Markov and they are*

$$h_{max}(e, \varphi) = e^{-\alpha_{max}\varphi} f_{max}(e) \qquad \text{and} \qquad h_{min}(e, \varphi) = e^{-\beta_{min}\varphi} g_{min}(e).$$

Moreover,
1) if the process $(\varphi_t)_{t\geq 0}$ *drifts to* $+\infty$ *then* $h_{max} = 1$ *and the process* $(\varphi_t)_{t\geq 0}$ *under* $P^{h_{min}}_{(e,\varphi)}$ *drifts to* $-\infty$*;*
2) if the process $(\varphi_t)_{t\geq 0}$ *drifts to* $-\infty$ *then* $h_{min} = 1$ *and the process* $(\varphi_t)_{t\geq 0}$ *under* $P^{h_{max}}_{(e,\varphi)}$ *drifts to* $+\infty$*;*
3) if the process $(\varphi_t)_{t\geq 0}$ *oscillates then* $h_{max} = h_{min} = 1$.

Theorem 7.5 *All functions* $h(e, \varphi, t)$ *on* $E \times \mathbb{R} \times [0, +\infty)$ *continuously differentiable in* φ *and* t *for which the process* $(X_t)_{t\geq 0}$ *under* $P^h_{(e,\varphi)}$ *is Markov are of the form*

$$h(e, \varphi, t) = e^{-\alpha t} e^{-\beta\varphi} g(e), \quad (e, \varphi, t) \in E \times \mathbb{R} \times [0, +\infty),$$

where, for fixed $\beta \in \mathbb{R}$, α *is the Perron-Frobenius eigenvalue and* g *is the right Perron-Frobenius eigenvector of the matrix* $(Q - \beta V)$.

Moreover, there exists unique $\beta_0 \in \mathbb{R}$ *such that*

$$\begin{array}{rcc} (\varphi_t)_{t\geq 0} \text{ under } P^h_{(e,\varphi)} \text{ drifts to } +\infty & \text{iff} & \beta < \beta_0 \\ (\varphi_t)_{t\geq 0} \text{ under } P^h_{(e,\varphi)} \text{ oscillates} & \text{iff} & \beta = \beta_0 \\ (\varphi_t)_{t\geq 0} \text{ under } P^h_{(e,\varphi)} \text{ drifts to } -\infty & \text{iff} & \beta > \beta_0, \end{array}$$

and β_0 *is determined by the equation* $\alpha'(\beta_0) = 0$, *where* $\alpha(\beta)$ *is the Perron-Frobenius eigenvalue of* $(Q - \beta V)$.

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