ON DECOMPOSING RISK IN A FINANCIAL-INTERMEDIATE MARKET AND RESERVING.

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Abstract. We consider the problem of decomposing monetary risk in the presence of a fully traded market in *some* risks. We show that a mark-to-market approach to pricing leads to such a decomposition if the risk measure is time-consistent in the sense of Delbaen.

1. Introduction.

In many contexts, financial products are priced and sold in the absence of a market (i.e a fully traded market) in these products. Typically these products have a dependence (either explicit or implicit) on one or more securities or contracts in a traded (financial) market. An obvious example is insurance (and, in particular, life insurance), but other examples include (the benefits provided by) pension funds and stock and options in non-quoted companies. This paper is concerned with the questions of valuing the liabilities of an (intermediate) market maker in such products and of how to make and invest financial reserves for them.

We take the view that such a market maker is a price-taker in the traded financial market (hereafter referred to simply as the market) and is a price-maker in its own products (hereafter referred to as contracts)—the ultimate value of which are contingent on risks not present in the market. From this point of view we may add in other non-market risk such as, for example, interruption of business, fraud, litigation, insurable risks and economic factors (such as the behavior of price indices and salaries) which impinge upon the eventual settlement value (or payoff) of these contracts.

We adopt the view that in such a setup, the intermediate market maker (hereafter referred to as the intermediate) will adopt a coherent risk measure (on discounted final values) as their valuation method and show how this implies certain constraints on the form of this risk measure and finally, how if these constraints are met, the risk measure implies a reserving method and an investment strategy in the market.

2. Contracts contingent on lives

As we have already seen in the introduction, the issues we address are by no means limited to life assurance and related products, indeed they have relevance to monetary risk management in any conceivable context; nevertheless, historically, life assurance and

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annuities (the two main products of life insurance companies) are a major source of such issues. Consequently we shall briefly discuss the traditional approach to such problems.

Insurance as an institution gives its customers the ability to share the risk they may face in the future by buying a suitable contract. The law of averages or Strong Law of Large Numbers is used to reduce risk by sharing a part of it between a large group of customers. Given that N individuals are willing to buy N contracts of the same type that pay a fixed amount $X_0 = 1$ if the defined risk, death occurs during a time interval $[0, T]$ and by ignoring fees and taxes, the premium p should be a function of N, T, X_0 and q— the probability that the risk will happen during that interval. The SLLN says then that if we have independence between different individuals, then

$$p = \mathbb{E}\left(e^{-\delta T}\,\frac{Y_N}{N}\right),$$

where δ is the discount factor and Y_N is the number of customers who die, then $p = e^{-\delta T}\,q$ with

$$e^{-\delta T}\,\frac{Y_N}{N} \to p \;\; a.s.,$$

when N goes to infinity, so that p is a fair net premium to charge for the insurance.

In case the size of the loss is uncertain then the premium is given by $(1+\theta)\,p$, where p is the premium for the average losses and θ is a (safety) loading factor to cover possible fluctuations.

In practice also, customers are of different ages so that p varies and it is assumed that the type of contract influences mortality risk so that different values of p are used for different types of contracts.

In the presence of a financial risk (e.g equity-linked insurance contracts), the direct application of the SLLN principle may not give a suitable result as it does not take into account the possibility of investing in the financial market and the restriction of such pricing to purely financial claims does not necessarily respect the no-arbitrage property.

As we can see, this procedure implies the use of a coherent risk measure for valuing discounted monetary risks.

Many papers have been devoted to this kind of problem and many techniques have been proposed to price such contracts. We recall the risk-minimizing technique which considers the biometric risk as a non-tradable risk in an incomplete financial market, see T. Møller [5] for more details.

In this paper, we propose to build a pricing that respects both SLLN and no-arbitrage principles. In order to do this, we recall in section 3 some results on one-period coherent risk measure and the well-known theorem giving its representation in terms of test probabilities. In section 4, we work in a multi-period case, we define a chain of coherent risk measures that define prices along the time axis and introduce some properties, namely lower, weak and strong time-consistency. While lower time-consistency is a natural property in this context, the weak one suggests that the pricing is derived from a single set of test probabilities and the strong one allows us to hedge a claim by a trade at each period. In section 5, we consider the financial market as an embedded entity in the global market and decompose a given pricing into its financial and intermediary, or prerisk, parts. We show that the pricing can be constructed from its two parts under the time-consistency property. Finally, in section 7, we fix a no-arbitrage pricing mechanism Π on the financial market and derive the family of time-consistent pricing mechanisms that coincide with Π on the purely financial claims.

3. One-period coherent risk measures.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with $\mathcal{F}_0 \subset \mathcal{F}$ a sub-σ-algebra. In this section we recall the main result on the characterization of a one-period coherent risk measure defined on the vector space $L^\infty(\mathcal{F})$ with values in $L^\infty(\mathcal{F}_0)$. The σ-algebra $\mathcal{F}_0$ is not necessarily trivial.

Definition 3.1. *(See Delbaen [1]) We say that the mapping $\rho : L^\infty(\mathcal{F}) \to L^\infty(\mathcal{F}_0)$ is a coherent risk measure if it satisfies the following axioms :*

(1) *Monotonicity: For every $X, Y \in L^\infty(\mathcal{F})$,*
$$X \leq Y \text{ a.s } \Rightarrow \rho(X) \leq \rho(Y) \text{ a.s.}$$
(2) *Subadditivity: For every $X, Y \in L^\infty(\mathcal{F})$,*
$$\rho(X+Y) \leq \rho(X) + \rho(Y) \text{ a.s.}$$
(3) *Translation invariance: For every $X \in L^\infty(\mathcal{F})$ and $y \in L^\infty(\mathcal{F}_0)$,*
$$\rho(X+y) = \rho(X) + y \text{ a.s.}$$
(4) *$\mathcal{F}_0$-Positive homogeneity: For every $X \in L^\infty(\mathcal{F})$ and $a \in L^\infty_+(\mathcal{F}_0)$, we have*
$$\rho(a\,X) = a\,\rho(X) \text{ a.s.}$$

Definition 3.2. *The coherent risk measure $\rho : L^\infty(\mathcal{F}) \to L^\infty(\mathcal{F}_0)$ is said to satisfy the Fatou property if a.s $\rho(X) \leq \liminf \rho(X_n)$, for any sequence $(X_n)_{n\geq 1}$ uniformly bounded by 1 and converging to X in probability.*

Definition 3.3. *The coherent risk measure $\rho : L^\infty(\mathcal{F}) \to L^\infty(\mathcal{F}_0)$ is called relevant if for each set $A \in \mathcal{F}$ with $\mathbb{P}[A|\,\mathcal{F}_0] > 0$ a.s, we have that $\rho(1_A) > 0$ a.s.*

Proposition 3.4. *(See Delbaen [1]) Let the mapping $\rho : L^\infty(\mathcal{F}) \to L^\infty(\mathcal{F}_0)$ be a relevant coherent risk measure satisfying the Fatou property. Then*

(1) *The acceptance set $\mathcal{A}_\rho := \{X \in L^\infty(\mathcal{F})\ ;\ \rho(X) \leq 0 \text{ a.s}\}$ is a weak*-closed convex cone, arbitrage-free, stable under multiplication by bounded positive $\mathcal{F}_0$-measurable random variables and contains $L^\infty_-(\mathcal{F})$.*
(2) *There exists a convex set of probability measures $\mathcal{Q}$, all of them being absolutely continuous with respect to $\mathbb{P}$, with their densities forming an $L^1(\mathbb{P})$-closed set, and such that for $X \in L^\infty(\mathcal{F})$:*

$$\rho(X) = \textit{ess-sup}\,\{\mathbb{E}_\mathbb{Q}(X|\,\mathcal{F}_0)\ ;\ \mathbb{Q} \in \mathcal{Q}\}\,. \tag{3.1}$$

Proof. We sketch the proof. Since ρ is a coherent risk measure, $\mathcal{A}_\rho$ is a convex cone, closed under multiplication by bounded positive $\mathcal{F}_0$-measurable random variables and contains L^∞_-. Its weak*-closeness follows from the Fatou property and it is arbitrage-free since ρ is relevant. Now for the second assertion, we remark that, by applying the Hahn-Banach separation theorem with exhaustion argument (as in Schachermayer [6]), we may deduce that there exists some $g \in \mathcal{A}_\rho^*$, where $\mathcal{A}_\rho^*$ is the dual cone of $\mathcal{A}_\rho$ in L^1, such that $g > 0$ a.s, then we define

$$\mathcal{Q}^e = \left\{ \mathbb{Q} \ll \mathbb{P} \; ; \; \frac{d\mathbb{Q}}{d\mathbb{P}} \in \mathcal{A}_\rho^*, \frac{d\mathbb{Q}}{d\mathbb{P}} > 0 \text{ a.s} \right\},$$

and $\mathcal{Q} = \overline{\mathcal{Q}^e}$. Now let $X \in L^\infty(\mathcal{F})$ and $f^+ \in L^\infty_+(\mathcal{F}_0)$, then by the translation invariance property, we get that $f^+ (X - \rho(X)) \in \mathcal{A}_\rho$ and for every $\varepsilon > 0$, $f^+ (X - \rho(X)) + \varepsilon \notin \mathcal{A}_\rho$. Consequently, we deduce (3.3). □

Definition 3.5. *Given a coherent risk measure ρ, we define $\mathcal{Q}^\rho$ as follows:*

$$\mathcal{Q}^\rho = \left\{ \mathbb{Q} \ll \mathbb{P} \; ; \; \frac{d\mathbb{Q}}{d\mathbb{P}} \in \mathcal{A}_\rho^* \right\}. \tag{3.2}$$

Conversely, given $\mathcal{Q}$ a collection (not necessarily closed, or convex) of probability measures absolutely continuous with respect to $\mathbb{P}$, we define

$$\rho^{\mathcal{Q}}(X) = \textit{ess-sup} \left\{ \mathbb{E}_\mathbb{Q}(X | \mathcal{F}_0) \; ; \; \mathbb{Q} \in \mathcal{Q} \right\}. \tag{3.3}$$

The set $\mathcal{Q}^\rho$ is the largest subset $\mathcal{Q}$ for which $\rho = \rho^{\mathcal{Q}}$.

4. Risk measure versus Market.

Returning to our problem: we suppose that the intermediary is equipped with the probability space $(\Omega, \mathcal{G}, \mathbb{P})$, with a filtration $\mathbb{G} = (\mathcal{G}_t)_{t=0}^T$, with $\mathcal{G} = \mathcal{G}_T$, modelling the flow of information on the discrete time axis $\mathbb{T}^+ = \mathbb{T} \cup \{T\}$ with $\mathbb{T} = \{0, ..., T-1\}$.

We further suppose that the intermediary's pricing mechanism is $\underline{\rho} = (\rho_0, ..., \rho_{T-1})$ where each ρ_t denotes the price at time t of future (discounted) payoffs. Note that by choosing to price the discounted payoffs rather than the payoffs themselves, it's not necessary to introduce the discount rate in the property of translation invariance. Define the acceptance set of positions

$$\mathcal{A}^t = \{ X \in L^\infty(\mathcal{G}) \; ; \; \rho_t(X) \leq 0 \; \mathbb{P} \text{ a.s. } \},$$

the set of liabilities which the intermediary is willing to accept for no nett charge or no nett reserve at time t.

Definition 4.1. *We say that the vector $\underline{\rho} = (\rho_0, ..., \rho_{T-1})$ is a chain of coherent risk measures if for each $t \in \mathbb{T}$, the mapping $\rho_t : L^\infty(\mathcal{G}_T) \to L^\infty(\mathcal{G}_t)$ fulfills all the properties of a relevant coherent risk measure with the Fatou property (taking $\mathcal{F} = \mathcal{G}_T, \mathcal{F}_0 = \mathcal{G}_t$ in Definition 3.1).*

It follows from Proposition 3.4 that for all $t \in \mathbb{T}$, there exists an L^1-closed convex set of probabilities $\mathcal{Q}^t = \mathcal{Q}^{\rho_t}$, absolutely continuous w.r.t $\mathbb{P}$ such that for every $X \in L^\infty(\mathcal{G})$,

$$\rho_t(X) = \text{ess-sup}_{\mathbb{Q} \in \mathcal{Q}^t} \mathbb{E}_\mathbb{Q}(X | \mathcal{G}_t).$$

To determine the natural relationship between the subsets of probability measures

$$(\mathcal{Q}^0, ..., \mathcal{Q}^{T-1}) = (\mathcal{Q}^{\rho_0}, ..., \mathcal{Q}^{\rho_{T-1}}),$$

let us consider a contract $C_X^{t,T}$ issued at time t and paying the t-discounted amount $X \in L^\infty(\mathcal{G})$ (i.e discounted to time t) to the holder at time T. Its price at time t is given by $\rho_t(X)$. The buyer may choose, instead to buy another contract $C^{t,t+s}_{\rho_{t+s}(X)}$ paying

$\rho_{t+s}(X)$ at time $t+s$. Its price is given by $\rho_t \circ \rho_{t+s}(X)$. This contract can be seen also as a contract which gives the buyer, the right to choose at time $t+s$ between cash $\rho_{t+s}(X)$ or a new contract $C_X^{t+s,T}$. We conclude then that for every $t, t+s \in \mathbb{T}^+$ and X we should have

$$\rho_t(X) \leq \rho_t \circ \rho_{t+s}(X). \tag{4.1}$$

We say that $\underline{\rho}$ is lower time-consistent if $\underline{\rho}$ satisfies (4.1) which is equivalent to saying that the acceptance sets satisfy $\mathcal{A}^{t+s} \subset \mathcal{A}^t$ or by a duality argument that $\mathcal{Q}^t \subset \mathcal{Q}^{t+s}$. In the case where the inequality in (4.1) becomes equality we say that $\underline{\rho}$ is time-consistent w.r.t the filtration $(\mathcal{G}_t)_{t=0}^T$ or simply $\mathbb{G}$-time-consistent.

Definition 4.2. *Let $t \geq s$ with $t, s \in \mathbb{T}$, $\mathcal{H}$ and $\mathcal{H}'$ be two subsets of probability measures on $(\Omega, \mathcal{G})$. We say that $\mathcal{H} \subset_{s,t} \mathcal{H}'$ if for every $\mathbb{Q} \in \mathcal{H}$, there exists some $\mathbb{Q}' \in \mathcal{H}'$ such that for every $X \in L^\infty(\mathcal{G}_t)$, we have*

$$\mathbb{E}_{\mathbb{Q}}(X|\,\mathcal{G}_s) = \mathbb{E}_{\mathbb{Q}'}(X|\,\mathcal{G}_s).$$

We define the relation $\equiv_{s,t}$ in an analogous fashion and

$$[\mathcal{H}]_{s,t} = \{\mathbb{Q} \text{ a probability measure}: \ \{\mathbb{Q}\} \subset_{s,t} \mathcal{H}\}.$$

For a $\mathbb{P}$-absolutely continuous probability measure $\mathbb{R}$, we denote by $\Lambda^{\mathbb{R}}$ or $\Lambda(\mathbb{R})$ its density (with respect to $\mathbb{P}$) and define $\Lambda_t^{\mathbb{R}} = \mathbb{E}(\Lambda^{\mathbb{R}}|\mathcal{G}_t)$ for every $t \in \mathbb{T}^+$, so that $\Lambda_t^{\mathbb{R}}$ is the density of the restriction of $\mathbb{R}$ to $\mathcal{G}_t$.

Remark 4.3. *The set $[\mathcal{H}]_{s,t}$ defined in the previous definition, is not necessarily closed in L^1 even when $\mathcal{H}$ is.*

Definition 4.4. *Given a set of probability measures $\mathcal{Q}$,*

(1) *We define the associated chain of coherent risk measures $\underline{\rho}^{\mathcal{Q}} = (\rho_0^{\mathcal{Q}}, \ldots, \rho_{T-1}^{\mathcal{Q}})$ as follows: for all $t \in \mathbb{T}$ we define for $X \in L^\infty(\mathcal{G})$,*

$$\rho_t^{\mathcal{Q}}(X) = \text{ess-sup}\{\mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}_t);\ \mathbb{Q} \in \mathcal{Q}\}.$$

(2) *Let $\mathcal{A}$ be the dual cone of $\mathcal{Q}$; we define for $t \in \mathbb{T}$:*

$$\mathcal{A}_t = \{X;\ \alpha X \in \mathcal{A} \text{ for all } \alpha \in L_+^\infty(\mathcal{G}_t)\}.$$

Remark that $\mathcal{A} = \mathcal{A}_0$ since $\mathcal{G}_0$ is trivial.

Lemma 4.5. *Given a set of probability measures $\mathcal{Q}$ with the dual cone $\mathcal{A}$, then for all $t, t+s \in \mathbb{T}^+$, the dual cone of $[\mathcal{Q}]_{t,t+s}$ is given by $\mathcal{A}_t \cap L^\infty(\mathcal{G}_{t+s}) + L_-^\infty(\mathcal{G}_T)$.*

Proof. Let $X \in ([\mathcal{Q}]_{t,t+s})^*$ and define $Y = X - \rho'_{t+s}(X)$ where ρ' is the associated coherent risk measure to the set $[\mathcal{Q}]_{t,t+s}$, then $X = Y + \rho'_{t+s}(X)$. We want to show that $Y \in L_-^\infty(\mathcal{G}_T)$ and $\rho'_{t+s}(X) \in \mathcal{A}_t \cap L^\infty(\mathcal{G}_{t+s})$. Choose $g \in L_+^\infty(\mathcal{G}_T)$ and define the probability $\mathbb{Q}$ having the density

$$f = \frac{g}{\mathbb{E}(g|\mathcal{G}_{t+s})}\Lambda_{t+s},$$

where Λ is the density of a probability measure $\mathbb{R} \in \mathcal{Q}$. Remark that $\mathbb{Q} \equiv_{t,t+s} \mathbb{R}$, then $\mathbb{Q} \in [\mathcal{Q}]_{t,t+s}$ and

$$\mathbb{E}gY = \mathbb{E}_{\mathbb{Q}} \frac{\mathbb{E}(g|\mathcal{G}_{t+s})}{\Lambda_{t+s}} Y = \mathbb{E}_{\mathbb{Q}} \left(\frac{\mathbb{E}(g|\mathcal{G}_{t+s})}{\Lambda_{t+s}} \mathbb{E}_{\mathbb{Q}}(Y|\mathcal{G}_{t+s}) \right),$$

with $\mathbb{E}_{\mathbb{Q}}(Y|\mathcal{G}_{t+s}) = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}_{t+s}) - \rho'_{t+s}(X) \leq 0$. Hence $\mathbb{E}gY \leq 0$ for all $g \in L^\infty_+(\mathcal{G}_T)$, which leads to $Y \in L^\infty_-(\mathcal{G}_T)$. Now Choose $\mathbb{Q} \in \mathcal{Q}$ and $\alpha \in L^\infty_+(\mathcal{G}_t)$, we have

$$\mathbb{E}_{\mathbb{Q}}(\alpha \rho'_{t+s}(X)) = \mathbb{E}(\Lambda^{\mathbb{Q}}_{t+s} \alpha \rho'_{t+s}(X)) = a\mathbb{E}(\frac{\Lambda^{\mathbb{Q}}_{t+s}\alpha}{a} \rho'_{t+s}(X)) = a\mathbb{E}(f\, \rho'_{t+s}(X)),$$

with $a = \mathbb{E}(\Lambda^{\mathbb{Q}}_t \alpha)$ and

$$f = \frac{\Lambda^{\mathbb{Q}}_{t+s}\alpha}{a}.$$

Remark that there exists a sequence $\mathbb{Q}^n \in [\mathcal{Q}]_{t,t+s}$ such that the increasing sequence $\mathbb{E}_{\mathbb{Q}^n}(X|\mathcal{G}_{t+s})$ converges a.s to $\rho'_{t+s}(X)$. We denote by Λ^n, the density of $\mathbb{Q}^n$. We obtain

$$\mathbb{E}_{\mathbb{Q}}(\alpha \rho'_{t+s}(X)) = a \lim_{n\to\infty} \mathbb{E}(f \frac{\Lambda^n}{\Lambda^n_{t+s}} X).$$

Define

$$f^n = f\frac{\Lambda^n}{\Lambda^n_{t+s}},$$

and remark that $f^n_{t+s} = f^n_t = 1$, then the associated probability $\mathbb{Q}^n_1 \in [\mathcal{Q}]_{t,t+s}$. In consequence

$$\mathbb{E}_{\mathbb{Q}}(\alpha \rho'_{t+s}(X)) \leq a \lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}^n_1}(X) \leq 0.$$

Conversely let $\mathbb{Q} \in [\mathcal{Q}]_{t,t+s}$, there exists then some $\mathbb{Q}' \in \mathcal{Q}$ such that $\mathbb{Q} \equiv_{t,t+s} \mathbb{Q}'$. We obtain for all $X \in \mathcal{A}_t \cap L^\infty(\mathcal{G}_{t+s})$,

$$\mathbb{E}_{\mathbb{Q}}X = \mathbb{E}\Lambda_{t+s}X = \mathbb{E}_{\mathbb{Q}'}\frac{\Lambda_t}{\Lambda'_t}X,$$

with Λ and Λ' are respectively the densities of $\mathbb{Q}$ and $\mathbb{Q}'$. Define $Z = \dfrac{\Lambda_t}{\Lambda'_t}$ and for all n, define $Z^n = Z\mathbf{1}_{Z\leq n}$. Consequently

$$\mathbb{E}_{\mathbb{Q}}X = \mathbb{E}ZX = \lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}'}Z^nX \leq 0,$$

since $Z^nX \in \mathcal{A}$ for all n. $\square$

Lemma 4.6. *Let $\mathcal{Q}$ be a set of $\mathbb{P}$-absolutely continuous probability measures on $(\Omega, \mathcal{G})$ with $\mathcal{A}$ its dual cone. Then*

1. *for all $t, s \in \mathbb{T}$,*
$$(\mathcal{A}_t)_s = (\mathcal{A}_s)_t = \mathcal{A}_{t\vee s}.$$
2. *for all $t, s \in \mathbb{T}^+$,*
$$\overline{[\overline{[\mathcal{Q}]}_{s,T}]_{t,T}} = \overline{[\mathcal{Q}]}_{s\vee t,T},$$
with the closure taken in L^1.

Proof. Suppose $s \le t$, by definition $(\mathcal{A}_t)_s \subset \mathcal{A}_t$ and $(\mathcal{A}_s)_t \subset \mathcal{A}_t$, Now let $X \in \mathcal{A}_t$, then $\alpha_t X \in \mathcal{A}$ for all $\alpha_t \in L^\infty_+(\mathcal{G}_t)$ which means that $\beta_s \alpha_t X \in \mathcal{A}$ for all $\alpha_t \in L^\infty_+(\mathcal{G}_t)$ and $\beta_s \in L^\infty_+(\mathcal{G}_s)$, we deduce that $\alpha_t X \in \mathcal{A}_s$ (resp. $\beta_s X \in \mathcal{A}_t$) for all $\alpha_t \in L^\infty_+(\mathcal{G}_t)$ (resp. for all $\beta_s \in L^\infty_+(\mathcal{G}_s)$), therefore $X \in (\mathcal{A}_s)_t$ (resp. $X \in (\mathcal{A}_t)_s$). For the second assertion, we apply Lemma 4.5 and the assertion (1) and obtain

$$\left(\overline{[\overline{[\mathcal{Q}]}_{s,T}]_{t,T}}\right)^* = (\mathcal{A}_s)_t = \mathcal{A}_{s\vee t} = ([\mathcal{Q}]_{s\vee t,T})^* .$$

□

Remark 4.7. *Given a set of probability measures $\mathcal{Q}$, the associated chain of coherent risk measures $\underline{\rho}^{\mathcal{Q}}$ is lower time-consistent.*

Lemma 4.8. *Let $\underline{\rho} = (\rho_0, ..., \rho_{T-1})$ be a chain of coherent risk measures with the associated vector of test probabilities $(\mathcal{Q}^0, ..., \mathcal{Q}^{T-1})$. Then there exists a single $\mathcal{Q}$ such that for every $t \in \mathbb{T}$ we have $\rho_t = \rho_t^{\mathcal{Q}}$ on $L^\infty(\mathcal{G}_{t+1})$. Moreover, if for every $t \in \{0, \dots, T-2\}$ we have $\mathcal{Q}^{t+1} = \overline{[\mathcal{Q}^t]}_{t+1,T}$, then there exists a single $\mathcal{Q}$ (that we can take equal to $\mathcal{Q}^0$) such that for every $t \in \mathbb{T}$ we have $\rho_t = \rho_t^{\mathcal{Q}}$ on $L^\infty(\mathcal{G}_T)$.*

Proof. Let us define the subset $\mathcal{Q} = \bigcap_{t=0}^{T-1} [\mathcal{Q}^t]_{t,t+1}$. Remark that for all $t \in \mathbb{T}$, we have $\mathcal{Q} \subset [\mathcal{Q}^t]_{t,t+1}$, then $[\mathcal{Q}]_{t,t+1} \subset [\mathcal{Q}^t]_{t,t+1}$. Now let $\mathbb{Q} \in [\mathcal{Q}^t]_{t,t+1}$, then there exists some $\mathbb{Q}^t \in \mathcal{Q}^t$ such that $\mathcal{Q} \equiv_{t,t+1} \mathbb{Q}^t$. Let f^t denote the density of $\mathbb{Q}$ and define $\mathbb{Q}'$ as the probability measure associated to the density

$$f = \prod_{u\in\mathbb{T}} \frac{f^u_{u+1}}{f^u_u},$$

where each f^u is the density of a probability measure $\mathbb{Q}^u \in \mathcal{Q}^u$ for $u \neq t$. Then $\mathbb{Q} \equiv_{t,t+1} \mathbb{Q}'$ with $\mathbb{Q}' \equiv_{t,t+1} \mathbb{Q}^t \in \mathcal{Q}^t$ and for all $s \neq t$, we have $\mathbb{Q}' \equiv_{s,s+1} \mathbb{Q}^s \in \mathcal{Q}^s$. In consequence $\mathbb{Q} \in [\mathcal{Q}]_{t,t+1}$ and hence for all $t \in \mathbb{T}$ we have $\mathcal{Q} \equiv_{t,t+1} \mathcal{Q}^t$. We deduce then that $\rho_t^{\mathcal{Q}} = \rho_t$ on $L^\infty(\mathcal{G}_{t+1})$.

Now suppose that for every $t \in \{0, \dots, T-2\}$ we have $\mathcal{Q}^{t+1} = \overline{[\mathcal{Q}^t]}_{t+1,T}$. We define $\mathcal{Q} = \mathcal{Q}^0$ and prove by induction on $t = 1, \dots, T-1$ that $\mathcal{Q}^t = \overline{[\mathcal{Q}^0]}_{t,T}$. By assumption $\mathcal{Q}^1 = \overline{[\mathcal{Q}^0]}_{1,T}$, we suppose that the induction hypothesis is true until t, then

$$\mathcal{Q}^{t+1} = \overline{[\mathcal{Q}^t]}_{t+1,T} = \overline{[\overline{[\mathcal{Q}^0]}_{t,T}]}_{t+1,T} = \overline{[\mathcal{Q}^0]}_{t+1,T},$$

where the last equality is due to Lemma 4.6. □

Definition 4.9. *We say that a chain $\underline{\rho} = (\rho_0, ..., \rho_{T-1})$ is weakly time-consistent if there exists a single $\mathcal{Q}$ such that $\underline{\rho} = \underline{\rho}^{\mathcal{Q}}$.*

Corollary 4.10. *Let $\underline{\rho} = (\rho_0, ..., \rho_{T-1})$ be a chain with the associated vector of test probabilities $(\mathcal{Q}^0, ..., \mathcal{Q}^{T-1})$. Then the chain is weakly time-consistent iff for every $t \in \mathbb{T}$ we have $\mathcal{Q}^t = \overline{[\mathcal{Q}^0]}_{t,T}$.*

Corollary 4.11. *Let $\underline{\rho} = (\rho_0, ..., \rho_{T-1})$ denote a chain of coherent risk measures with the associated vector of test probabilities $(\mathcal{Q}^0, ..., \mathcal{Q}^{T-1})$ and the family of dual cones $(\mathcal{A}^0, \dots, \mathcal{A}^{T-1})$ with $\mathcal{A} = \mathcal{A}^0$. Then $\underline{\rho}$ is weakly time-consistent iff for all $t \in \mathbb{T}$ we have $\mathcal{A}^t = \mathcal{A}_t$.*

Definition 4.12. *We say that a chain* $\underline{\rho} = (\rho_0, ..., \rho_{T-1})$ *is time-consistent if for every* $s, t \in \mathbb{T}$ *with* $s \le t$ *we have* $\rho_s = \rho_s \circ \rho_t$.

We note that every time-consistent chain is weakly time-consistent. For the converse to hold, the maximal associated set $\mathcal{Q}$ of probability measures has to satisfy the multiplicative stability property (see Delbaen [2]).

Definition 4.13. *We say that a set of* $\mathbb{P}$*-absolutely continuous probability measures* $\mathcal{Q}$*, is* $\mathbb{G}$*-m-stable (or just m-stable if there is no confusion as to the filtration) if for every* $\mathbb{Q} \in \mathcal{Q}$*,* $\mathbb{Q}' \in \mathcal{Q}^e$ *and* $t \in \mathbb{T}$*, the probability measure* $\tilde{\mathbb{Q}}$ *is contained in* $\mathcal{Q}$*, where*

$$\Lambda^{\tilde{\mathbb{Q}}} = \Lambda_t^{\mathbb{Q}} \frac{\Lambda^{\mathbb{Q}'}}{\Lambda_t^{\mathbb{Q}'}}.$$

Remark 4.14. *The property of m-stability was defined by Delbaen in* [2] *and the property was first introduced for EMMs by Jacka in* [4].

Lemma 4.15. *Let* $\mathcal{Q}$ *be a set of* $\mathbb{P}$*-absolutely continuous probability measures, then* $\bigcap_{t\in\mathbb{T}}[\mathcal{Q}]_{t,t+1}$ *is the smallest m-stable set of probability measures containing* $\mathcal{Q}$ *and therefore* $\mathcal{Q}$ *is m-stable iff* $\mathcal{Q} = \bigcap_{t\in\mathbb{T}}[\mathcal{Q}]_{t,t+1}$.

Proof. Let us define $\mathcal{H} \stackrel{def}{=} \bigcap_{t\in\mathbb{T}}[\mathcal{Q}]_{t,t+1}$. We show first that $\mathcal{H}$ is m-stable. In order to do this, let $t \in \mathbb{T}$, $\mathbb{Q} \in \mathcal{H}$ and $\mathbb{Q}' \in \mathcal{H}^e$ with respective densities Λ and Λ'. Define the probability measure $\mathbb{R}$ by

$$\Lambda(\mathbb{R}) = \Lambda_t \frac{\Lambda'}{\Lambda'_t}.$$

We want to show that $\mathbb{R} \in \mathcal{H}$, so it remains to show that $\mathbb{R} \in [\mathcal{Q}]_{s,s+1}$ for all $s \in \mathbb{T}$. Remark that

$$\frac{\Lambda_{s+1}(\mathbb{R})}{\Lambda_s(\mathbb{R})} = \begin{cases} \dfrac{\Lambda'_{s+1}}{\Lambda'_s} & \text{for } s \ge t \\[2ex] \dfrac{\Lambda_{s+1}}{\Lambda_s} & \text{for } s \le t-1. \end{cases}$$

In consequence $\mathbb{R} \equiv_{s,s+1} \mathbb{Q}'$ for $s \ge t$ and $\mathbb{R} \equiv_{s,s+1} \mathbb{Q}$ for $s \le t-1$. Remark that since $\mathbb{Q}, \mathbb{Q}' \in \mathcal{H} \subset [\mathcal{Q}]_{s,s+1}$, then there exists $\mathbb{Q}_s, \mathbb{Q}'_s \in \mathcal{Q}$ such that $\mathbb{Q}' \equiv_{s,s+1} \mathbb{Q}'_s$ and $\mathbb{Q} \equiv_{s,s+1} \mathbb{Q}_s$, therefore $\mathbb{R} \equiv_{s,s+1} \mathbb{Q}'_s$ for $s \ge t$ and $\mathbb{R} \equiv_{s,s+1} \mathbb{Q}_s$ for $s \le t-1$ with $\mathbb{Q}_s, \mathbb{Q}'_s \in \mathcal{Q}$.

Now let $\mathcal{H}'$ be an m-stable set of probability measures containing $\mathcal{Q}$ and let $\mathbb{Q} \in \mathcal{H}$, then there exists $\mathbb{Q}^0, \ldots, \mathbb{Q}^{T-1} \in \mathcal{Q}$ with their respective densities $\Lambda^0, \ldots, \Lambda^{T-1}$ such that

$$\Lambda^{\mathbb{Q}} = \prod_{u\in\mathbb{T}} \frac{\Lambda^u_{u+1}}{\Lambda^u_u}.$$

We define, for each $t \in \mathbb{T}$,

$$Z^t = \Lambda^t_{t+1} \prod_{u=t+1}^{T-1} \frac{\Lambda^u_{u+1}}{\Lambda^u_u}.$$

Remark that $\Lambda^{\mathbb{Q}} = Z^0$, then we prove by induction on $t = T-1, \ldots, 0$ that $Z^t \in \mathcal{H}'$. We have $Z^{T-1} = \Lambda^{T-1} \in \mathcal{Q} \subset \mathcal{H}'$, now suppose that $Z^{t+1} \in \mathcal{H}'$ and remark that

$$Z^t = \frac{Z^{t+1}}{Z^{t+1}_{t+1}} \Lambda^t_{t+1}.$$

Since $\mathcal{H}'$ is m-stable, we obtain $Z^t \in \mathcal{H}'$. The equivalence in Lemma 4.15 becomes straightforward. □

The following theorem is due to Delbaen ([2]).

Theorem 4.16. *Given a set of probability measures $\mathcal{Q}$ (not necessarily a closed convex set), the associated chain $\underline{\rho}^{\mathcal{Q}}$ is time-consistent iff $\overline{co(\mathcal{Q})}$ is m-stable. By a small abuse of language we say that $\mathcal{Q}$ is time-consistent when $\underline{\rho}^{\mathcal{Q}}$ is.*

Here we state some simple and interesting results on time-consistency.

Theorem 4.17. *Let $\underline{\rho} = (\rho_0, ..., \rho_{T-1})$ be a lower time-consistent chain and define for each $t \in \mathbb{T}$, the risk measure $\eta_t = \rho_t \circ ... \circ \rho_{T-1}$. Then $\underline{\eta} \stackrel{def}{=} (\eta_0, ..., \eta_{T-1})$ is the minimal time-consistent chain which dominates $\underline{\rho}$.*

Proof. By definition $\eta_t = \rho_t \circ \eta_{t+1}$ and $\eta_t = \rho_t$ on $L^\infty(\mathcal{G}_{t+1})$, so $\underline{\eta}$ is time-consistent. The fact that η dominates ρ follows by backwards induction.

Now let $\underline{\xi} \stackrel{def}{=} (\xi_0, ..., \xi_{T-1})$ be a time-consistent chain of coherent risk measures which dominates $\underline{\rho}$. Therefore $\xi_{T-1} \geq \rho_{T-1} = \eta_{T-1}$ and by backwards induction on t we have

$$\xi_t = \xi_t \circ \xi_{t+1} \geq \xi_t \circ \eta_{t+1} \geq \rho_t \circ \eta_{t+1} = \eta_t.$$

□

Remark 4.18. *η corresponds to the smallest m-stable, closed convex set of probability measures containing $\mathcal{Q}^\rho$, i.e $\mathcal{Q}^\eta = \overline{\bigcap_{t\in\mathbb{T}} [\mathcal{Q}^\rho]_{t,t+1}}$.*

Lemma 4.19. *Suppose that $\mathcal{Q}$ is time-consistent and $s \in \mathbb{T}$. Let $X \in L^\infty(\mathcal{G}_T)$ be such that there exists a probability measure $\mathbb{Q} \in \mathcal{Q}^e$ satisfying $\rho_s(X) = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}_s)$. Then for every $t \geq s$ we have:*

$$\rho_t(X) = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}_t).$$

Proof. It suffices to remark that $\rho_t(X) \geq \mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}_t)$ and

$$\mathbb{E}_{\mathbb{Q}}\rho_t(X) = \mathbb{E}_{\mathbb{Q}}\mathbb{E}_{\mathbb{Q}}(\rho_t(X)|\mathcal{G}_s) \leq \mathbb{E}_{\mathbb{Q}}\rho_s \circ \rho_t(X) = \mathbb{E}_{\mathbb{Q}}\rho_s(X).$$

It's given that $\rho_s(X) = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}_s)$, then

$$\mathbb{E}_{\mathbb{Q}}\rho_t(X) \leq \mathbb{E}_{\mathbb{Q}}\mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}_s) = \mathbb{E}_{\mathbb{Q}} X = \mathbb{E}_{\mathbb{Q}}\mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}_t).$$

Then

$$\mathbb{E}_{\mathbb{Q}}\left(\rho_t(X) - \mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}_t)\right) = 0,$$

which means that a.s

$$\rho_t(X) = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}_t).$$

□

Theorem 4.20. *$\mathcal{Q}$ is time-consistent iff the process $(\rho^{\mathcal{Q}}_t(X))_{0\leq t\leq T}$ is a $\mathcal{Q}$-uniform-supermartingale for every $X \in L^\infty(\mathcal{G}_T)$.*

Proof. Suppose that $\mathcal{Q}$ is time-consistent. Then for $X \in L^\infty(\mathcal{G}_T)$ and $\mathbb{Q} \in \mathcal{Q}$ we have (suppressing the $\mathcal{Q}$-dependence of ρ):

$$\mathbb{E}_\mathbb{Q}(\rho_{t+s}(X)|\mathcal{G}_t) \leq \rho_t \circ \rho_{t+s}(X) = \rho_t(X).$$

Now suppose that the process $(\rho_t(X))_{t=0}^T$ is a $\mathcal{Q}$-uniform-supermartingale; which means that for every $\mathbb{Q} \in \mathcal{Q}$:

$$\mathbb{E}_\mathbb{Q}(\rho_{t+s}(X)|\mathcal{G}_t) \leq \rho_t(X).$$

It follows that $\rho_t \circ \rho_{t+s}(X) \leq \rho_t(X)$ and the result follows from lower time-consistency. □

In the next result we show the relationship between the time-consistency of the chain $\underline{\rho}$ and the decomposition of its acceptance set $\mathcal{A} = \{X \in L^\infty \ ; \ \rho_0(X) \leq 0\}$. Define for every $t \in \mathbb{T}$,

$$\mathcal{K}_t \stackrel{def}{=} \{X \in L^\infty(\mathcal{G}_{t+1}) \ ; \ \rho_t(X) \leq 0\} = \mathcal{A}_t \cap L^\infty(\mathcal{G}_{t+1}).$$

Theorem 4.21. *Suppose that the chain $\underline{\rho}$ is lower time-consistent, then it is time-consistent iff $\mathcal{A} = \mathcal{K}_0 + ... + \mathcal{K}_{T-1}$. In this case for all $t \in \mathbb{T}$, we have $\mathcal{A}_t = \mathcal{K}_t + ... + \mathcal{K}_{T-1}$.*

Proof. Suppose that $\underline{\rho}$ is time-consistent, then for every $X \in \mathcal{A}$ we get

$$X = \sum_{s=1}^{T-1} (\rho_{s+1}(X) - \rho_s(X)) + \rho_1(X),$$

with $\rho_T = id$. Let $u_s = \rho_{s+1}(X) - \rho_s(X) \in \mathcal{K}_s$ for $s \in \{1, ..., T-1\}$ and $u_0 = \rho_1(X) \in \mathcal{K}_0$. It follows that $\mathcal{A} \subset \mathcal{K}_0 + ... + \mathcal{K}_{T-1}$. Since $\mathcal{K}_0 + ... + \mathcal{K}_{T-1} \subset \mathcal{A}$ we have equality.

Now suppose that $\mathcal{A} = \mathcal{K}_0 + ... + \mathcal{K}_{T-1}$. Let $X \in L^\infty$ and $t \in \mathbb{T}$ be fixed. Since $\rho_t(X - \rho_t(X)) = 0$ and $\rho_0 \circ \rho_t \geq \rho_0$ it follows that $X - \rho_t(X) \in \mathcal{A}$, and so there exist $y_0 \in \mathcal{K}_0, ..., y_{T-1} \in \mathcal{K}_{T-1}$ such that

$$X - \rho_t(X) = y_0 + ... + y_{T-1}.$$

By applying ρ_t to both sides of this equality, we obtain

$$0 = y_0 + ... + y_{t-1} + \rho_t(y_t + ... + y_{T-1}),$$

and so by subadditivity

$$0 \leq y_0 + ... + y_{t-1} + \sum_{s=t}^{T-1} \rho_t(y_s)$$

and by lower time-consistency and the assumption that $y_s \in \mathcal{K}_s$, we get

$$0 \leq y_0 + ... + y_{t-1} + \sum_{s=t}^{T-1} \rho_t \circ \rho_s(y_s) \leq y_0 + ... + y_{t-1}.$$

But $y_0 + ... + y_{t-1} \in \mathcal{A}$, so $\mathbb{E}_\mathbb{Q}(y_0 + ... + y_{t-1}) \leq 0$ for some $\mathbb{Q} \in \mathcal{Q}^e$ and therefore $y_0 + ... + y_{t-1} = 0$.

Now it follows that $X - \rho_t(X) = y_t + ... + y_{T-1}$. By successively applying $\rho_{T-1}, \ldots, \rho_t$ on both sides and using the properties of ρ_s and $\mathcal{K}_s$, we obtain

$$\rho_t \circ \ldots \circ \rho_{T-1}(X) - \rho_t(X) = \eta_t(y_t + ... + y_{T-1}) \leq 0.$$

Finally, since $\rho_t \circ \ldots \circ \rho_{T-1}(X) \geq \rho_t(X)$ it follows that $\rho_t \circ \ldots \circ \rho_{T-1}(X) = \rho_t(X)$ and hence, by Lemma 4.17, it follows that $\underline{\rho}$ is time-consistent. □

Remark 4.22. *It is in this situation (where $\underline{\rho}$ is time-consistent) that we can replicate claims in $\mathcal{A}$ by a sequence of one-period trades. This explains the 'mark-to-market' requirement of section 1.*

5. The decomposition of the global market.

Example 5.1. *We consider a contract that provides one share of XYZ stock to the insured if he or she is still alive in one year's time, and nothing otherwise.*

(1) *What's the fair premium for this contract?*
(2) *What's the 'self financing ' strategy if it exists?*

To formulate this problem, let S denote the discounted price of the XYZ share in one year's time and let

$$Y = \begin{cases} 1 & \text{if the insured is alive then} \\ 0 & \text{otherwise} \end{cases}$$

We suppose that S and Y are defined respectively on two probability spaces $(\Omega^1, \mathcal{G}^1, \mathbb{P}^1)$ and $(\Omega^2, \mathcal{G}^2, \mathbb{P}^2)$, with $\mathcal{G}^1 = \sigma(S)$ and that the pricing of purely financial (resp. insurance) claims is given by p_F (resp. p_I). The payoff of the contract is $H = SY$. To price such a claim in one-period case, we remark first that $H = H(S)$ where $H(x) \stackrel{def}{=} xY$ for a scalar x. Remark also that for a fixed x, the claim xY is a purely insurance claim and it's priced by $p_I(xY)$ and that the claim $H^F \stackrel{def}{=} p_I(xY)|_{x=S}$ is a purely financial claim, priced by $p_F(H^F)$. We propose then the premium of H, $p(H) = p_F(H^F)$. The 'self financing ' strategy will be the one to hedge the claim H^F. We obtain then the decomposition of the claim H as follows: $H = p(H) + U^F + U^I$, where the claims $U^F \stackrel{def}{=} H^F - p_F(H^F)$ and $U^I \stackrel{def}{=} H - H^F$ are admissible.

Under the assumption that both P_F and P_I are coherent risk measures with $\mathcal{A}^F$ and $\mathcal{A}^I$ their respective acceptance sets, the risk measure p defined above is a coherent risk measure with acceptance set $\mathcal{A}$, satisfying $\mathcal{A} = \mathcal{A}^F + \mathcal{A}^I_s$, where

$$\mathcal{A}^I_s = \left\{ X : \ f(S)X \in \mathcal{A}^I, \text{ for all } f \in L^\infty_+(\mathbb{R}) \right\}.$$

In this section, we suppose that we're given a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and a coherent risk measure ρ and $\mathcal{F} \subset \mathcal{G}$ the financial sub-σ-algebra. Our aim is

(1) to construct, first in the one-period case, two coherent risk measures
$$\rho_F : L^\infty(\mathcal{F}) \to \mathbb{R} \text{ and } \rho_I : L^\infty(\mathcal{G}) \to L^\infty(\mathcal{F}),$$
such that $\rho = \rho_F$ on $L^\infty(\mathcal{F})$ and conditioning on $\mathcal{F}$, $\rho = \rho_I$.
(2) to establish necessary and sufficient conditions on ρ such that $\rho = \rho_F \circ \rho_I$.

Remark that if we denote $\mathcal{G}_0 = \{\emptyset, \Omega\}$, $\mathcal{G}_{0^+} = \mathcal{F}$, $\mathcal{G}_1 = \mathcal{G}$ and suppose that ρ is time-consistent w.r.t the filtration $(\mathcal{G}_0, \mathcal{G}_{0^+}, \mathcal{G}_1)$, then the acceptance set $\mathcal{A}$ will be decomposed as follows: $\mathcal{A} = \mathcal{A}^F + \mathcal{A}^I$ where $\mathcal{A}^F = \mathcal{A} \cap L^\infty(\mathcal{F})$ is the financial part of the whole market, whilst the second component $\mathcal{A}^I = \mathcal{A}_{0^+}$, is the intermediary market, equivalent to the whole market in the absence of the financial market ($\mathcal{F} = \mathcal{G}_0$). Any claim then can be decomposed into its financial and intermediary parts. The corresponding pricing mechanisms are given respectively by $\mathcal{Q}^F = [\mathcal{Q}^\rho]_{0,0^+}$ and $\mathcal{Q}^I = [\mathcal{Q}^\rho]_{0^+,1}$. Here we adopted the notation of the last section.

To generalize this setting to a multi-period case, we introduce the following notation. Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space equipped with the filtration $(\mathcal{G}_t)_{t\in\mathbb{T}^+}$. Let $(\mathcal{F}_t)_{t\in\mathbb{T}^+}$ be the filtration modeling the information in the financial market such that for every $t \in \mathbb{T}^+$ we have $\mathcal{F}_t \subset \mathcal{G}_t$ with $\mathcal{F}_0$ and $\mathcal{G}_0$ trivial. We assume that the intermediary makes prices according to a pricing mechanism ρ, defined by a set of $\mathbb{P}$-absolutely continuous probabilities $\mathcal{Q}$ on Ω. We suppose w.l.o.g that $\mathbb{P} \in \mathcal{Q}$ and that the set $\mathcal{Q}$ is an $L^1(\mathbb{P})$-closed convex set. Define $\mathcal{Q}^e$ to be the subset of $\mathbb{P}$-equivalent probability measures in $\mathcal{Q}$, the intermediate σ-algebras $\mathcal{G}_{t^+} = \mathcal{G}_t \bigvee \mathcal{F}_{t+1}$ and the filtration

$$\mathbb{G}^* = (\mathcal{G}_0, \mathcal{G}_{0^+}, \mathcal{G}_1, ..., \mathcal{G}_T).$$

Define the subsets $\mathcal{Q}^F$ and $\mathcal{Q}^I$ as follows. For $t \in \mathbb{T}$, we define $\mathcal{Q}^{t,F} = [\mathcal{Q}]_{t,t^+}$ and $\mathcal{Q}^F = \bigcap_{t=0}^{T-1} \mathcal{Q}^{t,F}$. In the same way we define $\mathcal{Q}^{t,I} = [\mathcal{Q}]_{t^+,t+1}$ and $\mathcal{Q}^I = \bigcap_{t=0}^{T-1} \mathcal{Q}^{t,I}$. We denote respectively by ρ^F and ρ^I the coherent risk measures associated to the subsets $\mathcal{Q}^F$ and $\mathcal{Q}^I$. In the following lemmas we state some interesting properties of these two subsets of probabilities.

Definition 5.1. *Let $t \in \mathbb{T}$. We define the binary relation $\sim_{t,F}$, defined on the set of all $\mathbb{P}$-absolutely continuous probabilities, as follows:*

$$\mathbb{Q} \sim_{t,F} \mathbb{Q}' \textit{ iff } \{\mathbb{Q}\}^{t,F} = \{\mathbb{Q}'\}^{t,F}.$$

We define $\sim_{t,I}$ in the same fashion.

Lemma 5.2. *$\sim_{t,F}$ is an equivalence relation. Moreover $\mathcal{Q}^{t,F} = \bigcup_{\mathbb{Q}\in\mathcal{Q}} \{\mathbb{Q}\}^{t,F}$. The analogous results hold for $\sim_{t,I}$.*

Proof. The binary relation $\sim_{t,F}$ is obviously an equivalence relation. Take a probability measures $\mathbb{Q} \in \mathcal{Q}^{t,F}$, then there exists a probability measure $\mathbb{Q}' \in \mathcal{Q}$ such that $\mathbb{Q} \sim_{t,F} \mathbb{Q}'$, which means that $\mathbb{Q} \in \{\mathbb{Q}'\}^{t,F}$ and hence $\mathcal{Q}^{t,F} \subset \bigcap_{\mathbb{Q}\in\mathcal{Q}} \{\mathbb{Q}\}^{t,F}$. The reverse inclusion is obvious. □

Lemma 5.3. *For every $\mathbb{Q} \in \mathcal{Q}^F$ and $t \in \mathbb{T}$, there exists some $\mathbb{Q}^t \in \mathcal{Q}$ such that $\mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}^t}(X|\mathcal{G}_t)$ for every $X \in L^\infty(\mathcal{G}_{t^+})$. Analogously, for every $\mathbb{Q} \in \mathcal{Q}^I$ and $t \in \mathbb{T}$, there exists some $\mathbb{Q}^{t^+} \in \mathcal{Q}$ such that $\mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}_{t^+}) = \mathbb{E}_{\mathbb{Q}^{t^+}}(X|\mathcal{G}_{t^+})$ for every $X \in L^\infty(\mathcal{G}_{t+1})$.*

Proof. Immediate consequence of Definition 4.2. □

Theorem 5.4. *Let $\mathcal{Q}$ be a set of $\mathbb{P}$-absolutely continuous probability measures on Ω. Then*

(1) *$\mathcal{Q}$ is $\mathbb{G}^*$-m-stable if and only if $\mathcal{Q} = \mathcal{Q}^F \cap \mathcal{Q}^I$.*
(2) *The subsets $\mathcal{Q}^F$ and $\mathcal{Q}^I$ are $\mathbb{G}^*$-m-stable. Moreover*

$$\overline{(\mathcal{Q}^F)^I} = \overline{(\mathcal{Q}^I)^F} = \mathcal{P},$$

where $\mathcal{P}$ is the set of all $\mathbb{P}$-absolutely continuous probability measures and the closure is taken in $L^1(\Omega)$.

Proof. The first assertion is an immediate consequence of Lemma 4.15. The second assertion is an immediate consequence of assertion (1) since $(\mathcal{Q}^F)^F = \mathcal{Q}^F$ and $\mathcal{Q}^F \subset (\mathcal{Q}^F)^I$, so $\mathcal{Q}^F = (\mathcal{Q}^F)^F \cap (\mathcal{Q}^F)^I$. We make the same argument for the I-part. We remark also that $\mathcal{P}^e \subset (\mathcal{Q}^F)^I, (\mathcal{Q}^I)^F$ where $\mathcal{P}^e$ is the set of all $\mathbb{P}$-equivalent probability measures. Indeed let $\mathbb{Q} \in \mathcal{P}^e$ with $f = \Lambda^{\mathbb{Q}}$ and $t \in \mathbb{T}$ fixed. We define the probability $\mathbb{Q}^t$ by its density

$$\Lambda^t = \frac{f_{t+1}}{f_{t^+}}.$$

Then $\mathbb{Q}^t \in \mathcal{Q}^F$ since $\mathbb{Q}^t \sim_{s,F} \mathbb{P}$ for every $s \in \mathbb{T}$. Moreover $\mathbb{Q} \sim_{t,I} \mathbb{Q}^t$, therefore $\mathbb{Q} \in (\mathcal{Q}^F)^I$. We do the same for the inclusion $\mathcal{P}^e \subset (\mathcal{Q}^I)^F$. □

Let $\underline{\rho} = \underline{\rho}^{\mathcal{Q}}$ be the chain associated to the set of probabilities $\mathcal{Q}$ and $\rho = \rho_0$. Let us define the acceptance cone $\mathcal{A} = \mathcal{A}_\rho$ associated to the coherent risk measure ρ by

$$\mathcal{A} = \{X \in L^\infty(\mathcal{G}) \ ; \ \rho(X) \leq 0\}.$$

$\mathcal{A}$ is then a weak*-closed convex cone in L^∞. Our objective is to decompose this trading cone in the global market into the sum of two trading cones, one in the financial market and the other in the intermediary's market.

We define the following convex cones

$$\mathcal{K}_t^F = \{X \in L^\infty(\mathcal{G}_{t^+}) \ ; \ \rho_t(X) \leq 0\} = \mathcal{A}_t \cap L^\infty(\mathcal{G}_{t^+}),$$

$$\mathcal{K}_t^I = \{X \in L^\infty(\mathcal{G}_{t+1}) \ ; \ \rho_{t^+}(X) \leq 0\} = \mathcal{A}_{t^+} \cap L^\infty(\mathcal{G}_{t+1}),$$

for $t \in \mathbb{T}$, $\mathcal{K}_T^F = L^\infty_-(\mathcal{G})$,

$$\mathcal{A}^F = \mathcal{K}_0^F + ... + \mathcal{K}_T^F,$$

and

$$\mathcal{A}^I = \mathcal{K}_0^I + ... + \mathcal{K}_{T-1}^I.$$

Then

Lemma 5.5. *$\mathcal{A} = \mathcal{A}^F + \mathcal{A}^I$ iff $\mathcal{Q}$ is $\mathbb{G}^*$-time-consistent.*

Proof. Immediate consequence of Lemma 4.21. □

Remark 5.6. *Remark that this corresponds to mark-to-market approach valuation.*

The question now is to characterize the pricing mechanism in both trading cones. In the next lemma we prove that the cones $\mathcal{A}^F$ and $\mathcal{A}^I$ are respectively the acceptance sets of the risk measures ρ^F and ρ^I.

Lemma 5.7. *$\mathcal{A}^F = \mathcal{A}_{\rho^F}$ and $\mathcal{A}^I = \mathcal{A}_{\rho^I}$.*

Proof. Since $\mathcal{Q}^F$ is $\mathbb{G}^*$-time-consistent, then $\mathcal{A}_{\rho^F} = W + M$ where $W = W_0 + ... + W_T$ and $M = M_0 + ... + M_{T-1}$ with

$$W_t = \{X \in L^\infty(\mathcal{G}_{t^+}) \ ; \ \rho_t^F(X) \leq 0\},$$

and

$$M_t = \{X \in L^\infty(\mathcal{G}_{t+1}) \ ; \ \rho_{t^+}^F(X) \leq 0\},$$

for $t \in \mathbb{T}$ and $W_T = L^\infty_-(\mathcal{G})$. By definition $\mathcal{Q} \subset \mathcal{Q}^F$, and we deduce that each $W_t \subset \mathcal{K}_t^F$. Now let $X \in \mathcal{K}_t^F$ and $\mathbb{Q} \in \mathcal{Q}^F$. By applying Lemma 5.3, there exists some $\mathbb{Q}^t \in \mathcal{Q}$ such that

$$\mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}^t}(X|\mathcal{G}_t) \leq \rho_t(X) \leq 0.$$

In consequence $\rho_t^F(X) \leq 0$ and $X \in W_t$. We have that $\mathcal{K}_t^F = W_t$ and in consequence $\mathcal{A}^F = W$. It suffices therefore to prove that $M \subset L_-^\infty$ which follows if we can prove that each $M_t \subset L_-^\infty$ for $t \in \mathbb{T}$.

Let $X \in M_t$, which means that $X \in L^\infty(\mathcal{G}_{t+1})$ and $\rho_{t^+}^F(X) \leq 0$. Then for every $\mathbb{Q} \in \mathcal{Q}^F$ we have $\mathbb{E}_\mathbb{Q}(X|\mathcal{G}_{t^+}) \leq 0$. Now let $g \in L_+^\infty(\mathcal{G}_{t+1})$ with $g > 0$ a.s and $\mathbb{E}g = 1$. Define the probability measure $\mathbb{Q}_g$ by $d\mathbb{Q}_g = fd\mathbb{P}$ where

$$f = \frac{g}{\mathbb{E}(g|\mathcal{G}_{t^+})}.$$

Then $\mathbb{Q}_g \ll \mathbb{P}$ and for every $s \in \mathbb{T}$ and $B \in \mathcal{G}_{s^+}$ we have

$$\mathbb{Q}_g(B|\mathcal{G}_s) = \mathbb{E}\left(B, \frac{f_{s^+}}{f_s}|\mathcal{G}_s\right) = \mathbb{P}(B|\mathcal{G}_s),$$

since

$$\frac{f_{s^+}}{f_s} := \frac{\mathbb{E}(f|\mathcal{G}_{s^+})}{\mathbb{E}(f|\mathcal{G}_s)} = 1.$$

Consequently $\mathbb{Q}_g \in \mathcal{Q}^F$ and so

$$\mathbb{E}(g.X) = \mathbb{E}(\mathbb{E}(g|\mathcal{G}_{t^+})\, f\, X) = \mathbb{E}\left(\mathbb{E}(g|\mathcal{G}_{t^+})\, \mathbb{E}_{\mathbb{Q}_g}(X|\mathcal{G}_{t^+})\right) \leq 0,$$

for every $g \in L_+^1(\mathcal{G}_{t+1})$. Therefore $X \leq 0$. In the same way we prove that $\mathcal{A}^I = \mathcal{A}_{\rho^I}$. □

Corollary 5.8. *The convex cones $\mathcal{A}^F$ and $\mathcal{A}^I$ are weak*-closed in L^∞.*

Corollary 5.9. *Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two convex subsets in $\mathcal{P}$ with ρ^1 and ρ^2 their respective coherent risk measures. Then the following assertions are equivalent.*

1. *For all $t \in \mathbb{T}$, $\rho_t^2 \leq \rho_t^1$ on $L^\infty(\mathcal{G}_{t^+})$.*
2. *$\mathcal{A}^{\mathcal{Q}_1^F} \subset \mathcal{A}^{\mathcal{Q}_2^F}$.*
3. *$\overline{\mathcal{Q}_2^F} \subset \overline{\mathcal{Q}_1^F}$.*

Proof. The assertions (2) and (3) are equivalent by duality argument. Now let suppose that (2) is satisfied, then for all $t \in \mathbb{T}$, we have

$$K_t^1 \overset{def}{=} \mathcal{A}_t^{\mathcal{Q}_1} \cap L^\infty(\mathcal{G}_{t^+}) = \mathcal{A}_t^{\mathcal{Q}_1^F} \cap L^\infty(\mathcal{G}_{t^+}) \overset{def}{=} K_t^{1,F} \subset \mathcal{A}_t^{\mathcal{Q}_2^F} \cap L^\infty(\mathcal{G}_{t^+}) = K_t^{2,F} = K_t^2.$$

Take $X \in L^\infty(\mathcal{G}_{t^+})$, then $X - \rho_t^1(X) \in K_t^1 \subset K_t^2$. Therefore $\rho_t^2(X - \rho_t^1(X)) \leq 0$ which means that $\rho_t^2(X) \leq \rho_t^1(X)$. Conversely for all $t \in \mathbb{T}$,

$$K_t^{1,F} = K_t^1 \subset K_t^2 = K_t^{2,F}.$$

The assertion (2) is obtained. □

6. Example.

Consider the example, where sample spaces are $I = \{i, i'\}$, $F = \{f, f'\}$, $T = 1$ and the probabilities $\mathbb{I}$ and $\mathbb{F}$ are given by $\mathbb{I}(i) = \mathbb{F}(f) = 1/2$. The financial market can be seen then as associated to one risky asset taking only two values at time 1 and a constant interest rate. This market is complete and we suppose that $\mathbb{F}$ is the equivalent martingale measure. The sample space is given by $\Omega = I \times F = \{(i, f), (i, f'), (i', f), (i', f')\}$, the probability measure $\mathbb{P} = \mathbb{I} \otimes \mathbb{F}$ and $L^\infty(\Omega)$ is identified with the space of 2×2-matrices. We define the pricing set $\mathcal{Q}$ by:

$$\mathcal{Q} = \left\{\mathbb{Q} \ll \mathbb{P}\ ;\ \Lambda^{\mathbb{Q}} \leq 1+\varepsilon \text{ and } \Lambda^{\mathbb{Q}}_{0^+} = 1\right\}.$$

The subset $\mathcal{Q}$ can also be written as follows

$$\mathcal{Q} = \left\{(q_{ij})_{1\leq i,j \leq 2}\ ;\ \sum_{i,j} q_{ij} = 1,\ 0 \leq q_{ij} \leq 1/4(1+\varepsilon) \text{ and for each } j:\ q_{1j}+q_{2j} = 1/2\right\}.$$

Note that $\mathcal{Q}$ is chosen to have margin $\mathbb{F}$ on F, and to correspond to a TailVaR type construction on I.

To compute the corresponding quantities $\rho_{0^+}(X)$ for $X \in L^\infty(\Omega)$, we remark that the extreme points of the set $\mathcal{Q}$ are given by

$$\mathbb{Q}^{a,b} = \frac{1}{4}\begin{pmatrix} 1+a\varepsilon & 1+b\varepsilon \\ 1-a\varepsilon & 1-b\varepsilon \end{pmatrix}$$

with $(a,b) \in \{(1,1),(1,-1),(-1,1),(-1,-1)\}$. Therefore we may check easily that for $\omega \in F$

$$\rho_{0^+}(\mathbf{1}_{(i,\omega)}) = \frac{1}{2}(1+\varepsilon)\mathbf{1}_{(\omega)},$$

and

$$\rho_{0^+}(-\mathbf{1}_{(i,\omega)}) = -\frac{1}{2}(1-\varepsilon)\mathbf{1}_{(\omega)}.$$

That means that for a real x we have:

$$\rho_{0^+}(x\mathbf{1}_{(i,f)}) = \frac{1}{2}(x+\varepsilon|x|)\mathbf{1}_{(f)}.$$

Consequently, for every $X \in L^\infty(\mathcal{G}_1)$ we have:

$$\rho_{0^+}(X) = \alpha_X(f)\mathbf{1}_f + \alpha_X(f')\mathbf{1}_{f'},$$

with

$$\alpha_X(g) = \frac{1}{2}\left(X(i,g)+\varepsilon|X(i,g)| + X(i',g)+\varepsilon|X(i',g)|\right).$$

For every $X \in L^\infty(\mathcal{G}_{0^+})$ we have:

$$\rho_0(X) = \mathbb{E}(X).$$

We conclude then that

$$\mathcal{Q}^I = \left\{(q_{ij})_{1\leq i,j\leq 2}\ ;\ \sum_{i,j} q_{ij} = 1, q_{ij} \geq 0 \text{ and for each } j:\ \frac{1}{\delta_\varepsilon} \leq \frac{q_{1j}}{q_{2j}} \leq \delta_\varepsilon\right\}.$$

with $\delta_\varepsilon = \dfrac{1+\varepsilon}{1-\varepsilon}$. Moreover

$$\mathcal{Q}^F = \left\{\begin{pmatrix} \alpha & \beta \\ 1/2-\alpha & 1/2-\beta \end{pmatrix}\ ;\ 0 \leq \alpha,\beta \leq 1/2\right\}.$$

The set $\mathcal{Q}$ is $\mathbb{G}^*$-time-consistent since $\mathcal{Q} = \mathcal{Q}^F \cap \mathcal{Q}^I$.

7. Pricing.

In this section, we suppose we are in the same situation as in Example 5.1, where the financial market is equipped with a no-arbitrage pricing Π (namely a closed convex set of probability measures); defined on a probability space $(\Omega_F, \mathcal{F}, \mathbb{P}_F)$ with the filtration $(\mathcal{F}_t)_{t\in\mathbb{T}^+}$. Moreover we consider a probability space $(\Omega_I, \mathcal{I}, \mathbb{P}_I)$ with the filtration $(\mathcal{I}_t)_{t\in\mathbb{T}^+}$, to model the biometric risk.

Our aim is to build the class of pricing mechanism ρ (or $\mathcal{Q}^\rho$) that prices the purely financial claims as Π does.

Define the product probability space $(\Omega, \mathcal{G}, \mathbb{P})$ as follows: $\Omega = \Omega_F \times \Omega_I, \mathcal{G} = \mathcal{F} \otimes \mathcal{I}$ and $\mathbb{P} = \mathbb{P}_F \otimes \mathbb{P}_I$, equipped with the filtration $(\mathcal{G}_t)_{t\in\mathbb{T}^+}$ given by $\mathcal{G}_t = \mathcal{F}_t \otimes \mathcal{I}_t$ and $\mathcal{G}_{t^+} = \mathcal{F}_{t+1} \otimes \mathcal{I}_t$. Let $\hat{\Pi}$ denote the extension of Π to the product space, i.e

$$\hat{\Pi} = \{\mathbb{Q} \otimes \mathbb{P}_I : \ \mathbb{Q} \in \Pi\}.$$

We state first the following result and identify the probability measures with their densities.

Lemma 7.1. *Let $\mathcal{Q}_1, \mathcal{Q}_2 \subset \mathcal{P}$, then the set $\mathcal{Q}$ defined by $\mathcal{Q} = \mathcal{Q}_1^F \cap \mathcal{Q}_2^I$, satisfies the following: $\mathcal{Q}^F = \mathcal{Q}_1^F, \mathcal{Q}^I = \mathcal{Q}_2^I$ and then $\mathcal{Q}$ is $\mathbb{G}^*$-m-stable.*

Proof. Remark that $\mathcal{Q}^F \subset (\mathcal{Q}_1^F)^F = \mathcal{Q}_1^F$ since $\mathcal{Q} \subset \mathcal{Q}_1^F$. Now let $\mathbb{Q} \in \mathcal{Q}_1^F$, then for all $t \in \mathbb{T}$, there exists a probability measure $\mathbb{R}^t \in \mathcal{Q}_1$ such that $\mathbb{Q} \sim_{t,F} \mathbb{R}^t$. Define the probability measure $\mathbb{Q}^t$ by the density

$$f^t = \prod_{u\in\mathbb{T}} \left(\frac{\Lambda^u_{u^+}}{\Lambda^u_u} \frac{\Lambda^{u^+}_{u+1}}{\Lambda^{u^+}_{u^+}} \right),$$

where Λ^u and Λ^{u^+} are respectively the densities of probability measures $\mathbb{R}^u \in \mathcal{Q}_1$ and $\mathbb{R}^{u^+} \in \mathcal{Q}_2$ for all $u \in \mathbb{T}$. Remark that $\mathbb{Q} \sim_{t,F} \mathbb{Q}^t$ and for all $u \in \mathbb{T}$, we have $\mathbb{Q}^t \sim_{u,F} \mathbb{R}^u$ and $\mathbb{Q}^t \sim_{u,I} \mathbb{R}^{u^+}$. Then $\mathbb{Q}^t \in \mathcal{Q}_1^F \cap \mathcal{Q}_2^I = \mathcal{Q}$ and hence $\mathbb{Q} \in \mathcal{Q}^F$. We make the same argument for the I-part. ☐

Remark 7.2. *The set $\mathcal{Q}$ defined in Lemma 7.1 is given by:*

$$\mathcal{Q} = \left\{ \prod_{t\in\mathbb{T}} \left(\frac{Z^t_{t^+}}{Z^t_t} \times \frac{W^t_{t+1}}{W^t_{t^+}} \right); \ Z^t \in \mathcal{Q}_1, \, W^t \in \mathcal{Q}_2 \text{ for all } t \in \mathbb{T} \right\}.$$

Now we characterize the class, denoted by $\Psi(\Pi)$, of time-consistent coherent risk measures ρ that satisfy $\mathcal{Q}^F = \hat{\Pi}^F$ with $\mathcal{Q} = \mathcal{Q}^\rho$.

Theorem 7.3. *$\rho \in \Psi(\Pi)$ iff there exists some non empty set Φ of $\mathbb{P}$-absolutely continuous probabilities measures such that $\mathcal{Q} = \hat{\Pi}^F \cap \Phi^I$. In this case*

$$\rho_t = \rho_t^{\hat{\Pi}} \circ \rho_{t^+}^{\Phi} \circ \rho_{t+1},$$

for $t \in \mathbb{T}$. In particular if we suppose that Π is time-consistent w.r.t the filtration $(\mathcal{F}_t)_{t\in\mathbb{T}}$, then for all purely financial claims $X \in L^\infty(\mathcal{F})$, we have:

$$\rho_t(X) = \rho_t^{\Pi}(X),$$

for $t \in \mathbb{T}$.

Proof. Suppose that $\rho \in \Psi(\Pi)$ and define $\Phi = \mathcal{Q}$, we obtain then by Lemma 5.4, $\mathcal{Q} = \mathcal{Q}^F \cap \mathcal{Q}^I = \Pi^F \cap \Phi^I$. Conversely suppose that there exists some non empty set $\Phi \subset \mathcal{P}$ such that $\mathcal{Q} = \Pi^F \cap \Phi^I$. From Lemma 7.1 we have $\mathcal{Q}^F = \Pi^F$ and $\mathcal{Q}^I = \Phi^I$, we deduce that $\mathcal{Q}$ is time-consistent. To prove the last assertion remark that for all $t \in \mathbb{T}$ and $Y \in L^\infty(\mathcal{F}_{t+1})$, we have:

$$\rho^\Phi_{t^+}(Y) \overset{def}{=} \text{ess-sup}_{\mathbb{Q}\in\Phi}\mathbb{E}_{\mathbb{Q}}(Y|\mathcal{F}_{t+1} \otimes \mathcal{I}_t) = Y,$$

and

$$\rho^{\hat{\Pi}}_t(Y) = \rho^\Pi_t(Y).$$

We prove then by induction on $t = T-1, \ldots, 0$ that $\rho_t(X) = \rho^\Pi_t(X)$ for all $X \in L^\infty(\mathcal{F})$. For $t = T-1$ we obtain

$$\rho_{T-1}(X) = \rho^{\hat{\Pi}}_{T-1} \circ \rho^\Phi_{(T-1)^+}(X) = \rho^\Pi_{T-1}(X).$$

Suppose that the induction hypothesis is true until $t+1$, we shall prove it for t. We get

$$\rho_t(X) = \rho^{\hat{\Pi}}_t \circ \rho^\Phi_{t^+} \circ \rho_{t+1}(X) = \rho^{\hat{\Pi}}_t \circ \rho^\Phi_{t^+} \circ \rho^\Pi_{t+1}(X).$$

Remark that $\rho^\Pi_{t+1}(X) \in L^\infty(\mathcal{F}_{t+1})$, then

$$\rho_t(X) = \rho^\Pi_t \circ \rho^\Pi_{t+1}(X) = \rho^\Pi_t(X),$$

from the time-consistency of Π. □

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