Multi-currency reserving for coherent risk measures

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A very partial history

Stochastic models of security prices: Bachelier (1900)

Ruin probability: Cramer Lundberg (1903), Sparre Anderson, Powers, Gerber Shiu (1995,98)

Portfolio Theory: Markovic (1952), Treynor, Sharpe, Lintner and Mossman (1962-65)

Immunisation: Redington (1952)

Dynamic Hedging (via stochastic calculus, complete markets): Black-Scholes (1973), Harrison and Pliska (1983)

Incomplete Markets: Dalang, Morton and Willinger (1990), Jacka (hedging 1991), Delbaen and Schachermayer (FTAP) (92, 94)

Optimal stopping, American options: McKean (1965), van Moerbeke(1976), Jacka, Kim(1990) and Superhedging: Kramkov (1996)

Proportional transaction costs: Davis and Norman (1990), Jouini, Kalal (1995), Cvitanic and Karatzas (1996), Kabanov, Stricker, Rasonyi (2001), Schachermayer, Campi (2004, 2006), Jacka, Berkaoui, Warren (2008), Viens (2010), Bayraktar (2017)

Coherent Risk Measures: Artzner, Delbaen, Eber and Heath (1999)

Convex Risk Measures: Follmer and Schied (2004)

Dynamic Coherent Risk Measures: Fritelli, Penner, Riedel, Detlefsen, Scandolo, Cheridito, Kupper, Delbaen (2004 ...)

- Coherent risk measures (CRMs) were introduced by Artzner, Delbaen, Eber and Heath.
- A key example was Chicago Mercantile Exchange's margin requirements.
- Amendments to Basel III accords (dubbed Basel 3.5 by Embrechts) mandate the use of Average Value at Risk (a coherent risk measure, unlike the widely-used Value at Risk (VaR) measure, which is not coherent) for reserving risk-capital
- Many financial institutions have regulatory or other reasons for testing their reserves and a dynamic version of coherent risk measures is a model for this process.

- CRMs normally defined by certain properties. For our purposes we simply define a CRM, ρ, (with the Fatou property) on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

$$\rho : L^{\infty}(\mathcal{F}) \to \mathbb{R}$$

  by

$$\rho : X \mapsto \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[-X],$$

  where $\mathcal{Q}$ is some non-empty collection of p.m.s absolutely cts wrt $\mathbb{P}$ (which can be assumed to be convex and closed wrt to the topology induced by the L^1 norm on densities).
- Conditional version:

$$\rho_t : X \mapsto \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[-X|\mathcal{F}_t]$$

- Examples of CRMs include superhedging prices in incomplete frictionless markets and (as we shall see) minimal hedging endowments in markets with proportional transaction costs.
- Drawback of reserving with CRMs is problem of time-consistency (see Bielecki, Cialenco, and Pitera (2017)): can view time-t reserve for a liability payable at a later time T as itself a liability, payable at time t. So it can be argued that one actually needs an initial reserve of $\rho_0 \circ \cdots \circ \rho_{T-1}(X)$.
- Delbaen (2006) gave a necessary and sufficient condition, m-stability, for this to equal $\rho_0(X)$— not in general true, although

  $$\rho_0 \circ \cdots \circ \rho_{T-1}(X) \geq \rho_0(X).$$

  In particular, Average- or Tail-Value at Risk (also known as Expected Shortfall) is not, in general, time-consistent.

- Assume liabilities are expressed in terms of time-T units and so at time 0, the risk or reserve is expressed in terms of units of a zero-coupon bond (or currency) payable at T.
- Assets need not just correspond to the unit of account and we allow holding multiple currencies or assets to perform the reserving function.
- Envisage a set of assets numbered $0, 1, \ldots, d$ with random terminal values $\mathbf{V} = (v^0 \equiv 1, v^1, \ldots, v^d)$ (given in the distinguished unit of account). Assume that each v^i is bounded and bounded away from 0. We term these assets *numéraires*.

- Recall models for trading with proportional transaction costs introduced by Jouini and Kallal , developed by Cvitanic and Karatzas , Kabanov , Kabanov and Stricker and further studied by Schachermayer, Campi and Jacka, Berkaoui and Warren amongst others. For more recent developments see Bielecki, Cialenco and Rodriguez (2017).
- There are $d+1$ assets traded at bid-ask prices $\pi_t^{i,j}$ (so $\pi_t^{i,j}$ units of asset i buy 1 unit of asset j at time t.

- In discrete time, we can define the claims/holdings attainable for zero endowment as

$$\mathcal{A}^{\pi} = \oplus K_t^{\pi}$$

where K_t^{π} is the cone of $\mathcal{F}_t$-measurable vector holdings available by trading (and consumption) from 0 at time t, so it consists of the convex cone of elements of $\mathcal{L}_t^0 \stackrel{def}{=} L^0(\mathbb{R}^{d+1}, \mathcal{F}_t, \mathbb{P})$ generated by positive, $\mathcal{F}_t$-measurable multiples of $\{e_j - \pi_t^{i,j} e_i : \ i \neq j\}$ and non-positive elements of $\mathcal{L}_t^0$.
- Improving on the results of Scachermayer and others, Jacka, Berkaoui and Warren were able to show that $(\mathcal{A}^{\pi})^0$, the closure in $\mathcal{L}_T^0$ of $\mathcal{A}^{\pi}$, is arbitrage-free iff there exists a consistent price process Z i.e a process with the following properties:

1. $Z \geq 0$
2. Z a $\mathbb{P}$-martingale
3. $Z_t^j - Z_t^i \pi_t^{i,j} \leq 0$ a.s.

so that Z_t is "in the polar cone (in $\mathbb{R}^{d+1}$) of K_t" for each t. Moreover, in this case, there is an adjusted bid-ask process $\tilde{\pi}_\cdot^{i,j}$ such that $(\mathcal{A}^\pi)^0 = \mathcal{A}^{\tilde{\pi}}$

Under our assumptions on $\mathbf{V}$, to value any portfolio Y of holdings in the elements in $\mathbf{V}$, we take the inner product $Y \cdot \mathbf{V}$ and any bounded X may be written in the form $Y \cdot \mathbf{V}$ with Y bounded .

A CRM is (can be thought of as) a reserving mechanism:

- assume insurer is reserving for risk according to a conditional CRM ρ_t, at each time t.
- Insurer reserves $\rho_t(-X)$ for a random claim X.
- Aggregate position of holding the risky claim X and reserving adequately should be acceptable.
- The set $\mathcal{A}_t$ of acceptable claims at time t consists of those $\mathcal{F}_T$-measurable bounded random variables with non-positive ρ_t.
- We say the portfolio Y_t reserves at time t for a claim X if
$$-X + Y_t \cdot \mathbf{V} \in \mathcal{A}_t.$$

- We define a conditional Coherent Pricing Measure (CPM) as follows:

Definition

$\rho_t : L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_t)$ is a conditional CPM if

$$\rho_t : X \mapsto \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t].$$

This corresponds to the usual definition of a conditional CRM (with the Fatou property) with a sign change on the argument.

$\mathcal{A}_t \stackrel{def}{=} \{X : \rho_t(X) \leq 0\}$ defines acceptable risks at time t; so want to arrive at a holding of X at time T (to pay the claim) by:

- hold $\rho(X)$ at time zero
- successively assume (entering into a contract for) acceptable risks at each intermediate time until we hold X at time T.
- So want holding $W.\mathbf{V}$ at time t which is a reserve for X (so that $\rho_t(X - W.\mathbf{V}) \leq 0$ and $W \in \mathcal{L}_t^\infty$), make contract for some adjustment Y in our portfolio which gives a t acceptable risk $Y.\mathbf{V}$ at time t
- so arrive at time $t+1$ with a net risk of $X - (W+Y).\mathbf{V}$ which is then $t+1$-acceptable (and known, so we need $Y \in \mathcal{L}_{t+1}^\infty$).

May only be true in some limiting sense and so we define (predictable) **V**-representability as follows:

Definition

The CPM ρ is (predictably) $\mathbf{V}$-representable if

$$\mathcal{A} = \overline{\oplus K_t.\mathbf{V}},$$

where

$$K_t \stackrel{def}{=} \{Y \in \mathcal{L}^{\infty}_{t+1} : \ Y.\mathbf{V} \in \mathcal{A}_t\}$$

and the closure is in the weak* topology.

Given a dynamic CPM ρ with conditional acceptance sets $\mathcal{A}_t$, define $\sigma^{p,\rho}$, or just σ^p_t, by

$$\sigma^{p,\rho}_t : X \mapsto \text{ess inf}\{\rho_t(Y.\mathbf{V}) : \; Y \in \mathcal{L}^\infty_{t+1} \text{ and } X - Y.\mathbf{V} \in \mathcal{A}_{t+1}\}.$$

It's not hard to show that σ^p_t is a conditional CPM and $\sigma^p_t \geq \rho_t$. We **define** (predictable) $\mathbf{V}$-time-consistency as follows:

Definition

A dynamic CPM ρ is (predictably) $\mathbf{V}$-time-consistent if $\sigma^{p,\rho}_t = \rho_t$ for each t.

It's not too hard to show the equivalence of $\mathbf{V}$-representability and $\mathbf{V}$-time-consistency.

Remark

The ess inf in the definition of σ^p_t is "sequentially attained" so this shows we do not need to consider nets in taking the closure of $\oplus K_t$.

Delbaen showed that (predictable, cash) time-consistency is equivalent to *m*-stability:

Definition

A CPM ρ corresponding to the collection (convex, a.c. w.r.t $\mathbb{P}$) $\mathcal{Q}$ is *m*-stable if, whenever $\tau \leq T$ is a stopping time and $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathcal{Q}$ then $\tilde{\mathbb{Q}}$, defined by the two properties

1. $\tilde{\mathbb{Q}}\big|_{\mathcal{F}_\tau} = \mathbb{Q}_1\big|_{\mathcal{F}_\tau}$
2. $\tilde{\mathbb{Q}}(\cdot|\mathcal{F}_\tau) = \mathbb{Q}_2(\cdot|\mathcal{F}_\tau)$

is in $\mathcal{Q}$.

We say that $\tilde{\mathbb{Q}}$ is the (predictable) τ-pasting of $\mathbb{Q}_1$ and $\mathbb{Q}_2$.

Remark

Compare with Girsanov approach to stochastic control with zero "cost of control".

Given a p.m. $\mathbb{Q}$, define the process $\mathbf{V}_t^{\mathbb{Q}} \stackrel{def}{=} \mathbb{E}_{\mathbb{Q}}[\mathbf{V}|\mathcal{F}_t]$.

We extend the definition to $\mathbf{V}$-m-stability as follows

Definition

$\mathcal{Q}$ (or ρ) is (predictably) $\mathbf{V}$-m-stable if whenever $\tau \leq T$ is a stopping time and $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathcal{Q}$ then their τ-pasting, $\tilde{\mathbb{Q}}$, is in $\mathcal{Q}$ *provided that*

$$\mathbf{V}_\tau^{\mathbb{Q}_1} = \mathbf{V}_\tau^{\mathbb{Q}_2}.$$

Example

Given a $\mathbb{P}$-martingale X, setting $\mathcal{Q}$ to be the collection of EMMs for X, $\mathcal{Q}$ is X_T-m-stable.

Theorem (To appear in J. Convex Analysis 2019)

A CPM ρ is V-time consistent iff it is V- represntable iff it is V-m-stable.

Remark

The proof uses the Bipolar Theorem (hence the closure in $\overline{\oplus K_t}$). The polar cone (in $\mathcal{L}^1$) of the component K_t is given by

$$K_t^* = \{Z \in \mathcal{L}^1 :\ \mathbb{E}[Z|\mathcal{F}_{t+1}] = \alpha \mathbf{V}_{t+1}^{\mathbb{Q}} \text{ for some } \mathbb{Q} \in \mathcal{Q}, \alpha \in L_+^0(\mathcal{F}_t)\}.$$

The trick is to show that

$$\cap K_t^* = \mathbb{R}_+\{\frac{d\mathbb{Q}}{d\mathbb{P}}\mathbf{V} :\ \mathbb{Q} \in \mathcal{Q}\}$$

iff ρ is $\mathbf{V}$-time-consistent.

In the optional case we consider a stronger version of multi-asset time-consistency which corresponds to explicitly adjusting portfolios (and which therefore seems appropriate to situations where trading of the assets held as reserves is possible).

The same equivalence (paper on arXiv) holds with definitions changed as follows:

Definition

1. Optional time consistency: ρ is optionally time consistent wrt $\mathbf{V}$ if $\sigma_t^{o,\rho} = \rho_t$ for each t, where

$$\sigma_t^o \overset{def}{=} \operatorname{ess\,inf}\{\rho_t(Y.\mathbf{V}) : \ Y \in \mathcal{L}_t^\infty \ \text{ and } X - Y.\mathbf{V} \in \mathcal{A}_{t+1}\}.$$

2. Optional representability: define $K_t \overset{def}{=} \{Y \in \mathcal{L}_t^\infty \ \text{ and } Y.\mathbf{V} \in \mathcal{A}_t\}$ then ρ is optionally representable if $\mathcal{A} = \overline{\oplus_t K_t}.\mathbf{V}$.
3. Given τ, we say that $\tilde{\mathbb{Q}}$ is *an* optional τ-pasting of $\mathbb{Q}_1$ and $\mathbb{Q}_2$ if $\tilde{\mathbb{Q}}\big|_{\mathcal{F}_\tau} = \mathbb{Q}_1\big|_{\mathcal{F}_\tau}$ and $\tilde{\mathbb{Q}}(\cdot|\mathcal{F}_{(\tau+1)\wedge T}) = \mathbb{Q}_2(\cdot|\mathcal{F}_{(\tau+1)\wedge T})$.
4. We say $\mathcal{Q}$ (or ρ) is (optionally) $\mathbf{V}$-m-stable if for any $\tau \leq T$ and $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathcal{Q}$, any τ-pasting, $\tilde{\mathbb{Q}}$, is in $\mathcal{Q}$ *provided* $\mathbf{V}_\tau^{\mathbb{Q}_1} = \mathbf{V}_\tau^{\tilde{\mathbb{Q}}}$.

Example

1. Suppose that $\mathcal{Q}$ is the set of EMM's for the $\mathbb{P}$-martingale M and $\mathbf{V} = (1, M_T)$, then $\mathcal{Q}$ is optionally $\mathbf{V}$-m-stable.
2. Suppose that $\mathcal{F}_0$ is $\mathbb{Q}$-trivial, then every optionally 1-m-stable $\mathcal{Q}$ containing $\mathbb{Q}$ contains every p.m. $\mathbb{Q}'$ s.t. $\mathbb{Q}' << \mathbb{Q}$, so that $\rho(X) \geq ||X^+||_\infty$.
3. Suppose that $\mathcal{Q} = \{\mathbb{Q} : \ V_T^{\mathbb{Q}} \in I_t \text{ for each } t\}$, where each I_t is a random closed convex set in $\mathbb{R}^{d+1}$, then $\mathcal{Q}$ is optionally $\mathbf{V}$-m-stable.

Notice that $K_t = \{Y \in \mathcal{L}_t^\infty \text{ and } Y.\mathbf{V} \in \mathcal{A}_t\}$
$= \{Y \in \mathcal{L}_t^\infty \text{ with } \mathbb{E}_{\mathbb{Q}}[Y.\mathbf{V}|\mathcal{F}_t] \leq 0 \text{ for all } \mathbb{Q} \in \mathcal{Q}\}$
$= \{Y \in \mathcal{L}_t^\infty \text{ with } Y.\mathbf{V}_t^{\mathbb{Q}} \leq 0 \text{ for all } \mathbb{Q} \in \mathcal{Q}\}$
$= \{Y \in \mathcal{L}_t^\infty \text{ with } Y.(\frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_t}\mathbf{V}_t^{\mathbb{Q}}) \leq 0 \text{ for all } \mathbb{Q} \in \mathcal{Q}\}$, and that $\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_t}\mathbf{V}_t^{\mathbb{Q}})\right)_{t=0,\dots,T}$ is a $\mathbb{P}$-martingale.

Given an acceptance cone $\mathcal{A}$ for an optionally **V**-m-stable CPM ρ we know that we can write it as

$$\mathcal{A} = \overline{\oplus K_t.\mathbf{V}}$$

It is tempting to think of the K_t as cones of admissible trades and to ask whether (by analogy with the transaction costs case) $\mathcal{A}^0$, the closure in L^0 of $\mathcal{A}$, can be written as

$$\mathcal{A}^0 = \oplus K_t^0, \mathbf{V} \text{ i.e. } \oplus K_t^0 \text{ is closed in } \mathcal{L}^0.$$

Lemma

Suppose $C \subseteq \mathcal{L}^0(\mathbb{R}^{d+1}, \mathcal{F})$ is a closed (in $\mathcal{L}^0$) cone, closed under multiplication by non-negative elements of $L^\infty(\mathcal{F})$, then there is a random closed cone M^C such that

$$C = \{Y : \ Y \in M^C a.s.\}.$$

Easy to see that K_t^0 has this property wrt $\mathcal{F}_t$ and it follows that $K_t^0 = \{Y \in \mathcal{L}^0(\mathbb{R}^{d+1}, \mathcal{F}_t) : \ Y.V_t^{\mathbb{Q}} \leq 0 \text{ for all } \mathbb{Q} \in \mathcal{Q}\}$.

We can now appeal to a result of Kabanov's. In our context it says

Lemma

If the null strategies $\mathcal{N}$, given by $\mathcal{N} \stackrel{def}{=} \{(\xi_0 \ldots, \xi_T) \in K_0^0 \times \ldots \times K_T^0 : \ \sum \xi_t = 0\}$ form a vector space then $\oplus K_t^0$ is closed.

We can't show this directly but can get around the problem. Then the closure result follows since the L^0-topology is coarser than the weak*-topology.

Have attempted to emphasise the analogies with the proportional transaction costs model. There are two issues:

1. There are no numéraires in the transaction costs model, since the final period is not frictionless;
2. trading costs are unbounded

Both of these problems can be overcome!

- add an extra trading period $(T, T+1]$
- impose numéraire risks that are so disadvantageous as to force the agent to sell up in the preceding time period.
- To generate the final, frictionless prices, we add on a simple "coin toss" for each other asset.
- Change basis/trade baskets of assets

Fix $0 < \epsilon < 1$ small. Define the $\mathbb{R}^{d+1}$-valued random variable $\tilde{\mathbf{V}} = (\tilde{v}^0, \tilde{v}^1, \dots, \tilde{v}^d)$, for $\omega \in \Omega$, $\omega' \in \{0,1\}^d$, by

$$\tilde{v}^0(\omega, \omega') = 1, \tag{1}$$

$$\tilde{v}^i(\omega, \omega') = (1 - \omega'_i)(1 - \epsilon)\frac{1}{\pi_T^{i,0}(\omega)} + \omega'_i(1 + \epsilon)\pi_T^{0,i}(\omega), \tag{2}$$

where the ω' are i.i.d. Bernoulli $(\frac{1}{2})$ under $\mathbb{P}$.

Now we define the frictionless bid-ask matrix at time $T+1$ by

$$\pi^{ij}_{T+1} := \frac{\tilde{v}^j}{\tilde{v}^i}.$$

The trading cone $\tilde{K}^0_{T+1}(\pi_{T+1})$ is generated by positive $\mathcal{F}_{T+1}$-measurable multiples of the vectors $-e_i$ and $e_j - \pi^{ij}_{T+1} e_i$, for $i, j \in \{0, 1, \ldots, d\}$. Define the cone $\mathcal{A}_{T+1}(\pi) = \mathcal{A}_T(\pi) + \tilde{K}^0_{T+1}(\pi_{T+1})$.

Since $\mathcal{A}_T(\pi)$ is closed and has no arbitrage, there exists at least one consistent price process Z for $\mathcal{A}_T(\pi)$.

We can extend any consistent price process for $\mathcal{A}_T(\pi)$, Z say, to a consistent price process for $\mathcal{A}_{T+1}(\pi)$ since consistency tells us that $Z^i_T(\omega) \in (\tilde{v}^i(\omega, 0), \tilde{v}^i(\omega, 1))$ and so (just using convexity), we can find a λ^Z s.t. with $Z_{T+1} \stackrel{def}{=} Z^0_T \lambda^Z \tilde{\mathbf{V}}$, with $Z_T = \mathbb{E}[Z_{T+1} | \mathcal{F}_T]$.

- Final prices $\tilde{\mathbf{V}}$ above are, in general unbounded, so need to adjust setting.
- Define new commodities corresponding to change of basis $\hat{e}_i = e_i + \sum_j e_j$. Corresponding frictionless (final) prices given by $\hat{v}_i = \tilde{v}_i + \sum_j \tilde{v}_j$,
- transform these by normalising, setting

$$\mathbf{V} = (v_0, \dots, v_d) \text{ where } v_i := \frac{\tilde{v}_i + \sum_j \tilde{v}_j}{1 + \sum_j \tilde{v}_j}.$$

- Finally, define set of probability measures $\mathcal{Q} = \{\mathbb{Q}^Z : \ Z \text{ is a consistent price process for } \mathcal{A}_T(\pi)\}$, where

$$\frac{d\mathbb{Q}^Z}{d\widetilde{\mathbb{P}}} := \frac{Z^0_{T+1} + \sum_j Z^j_{T+1}}{Z^0_0 + \sum_j Z^j_0}.$$

- Easy to check these are probability measures since Z's are consistent price processes so positive, vector-valued m.g.s.

The proof of the result is now clear; with this choice of $\mathcal{Q}$ and $\mathbf{V}$: $\mathcal{A}_{\mathcal{Q}}(\mathbf{V}) = \mathcal{A}_{T+1}(\tilde{\pi})$.

1. Bellman's principle: how does this work with **V**-m-stability?
2. How does this relate to expanding incomplete market to include assets with friction;
3. How to extend this to a cts time setting?
4. How to extend to convex risk measures?