The noisy veto-voter model: a Recursive Distributional Eqn.

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Cambridge
9 March, 2010

• Model: random number, M of independent voters each get a veto on a decision: $Y_i = 0$ or 1, where veto=0. Final result, Y, is recorded as a 1 or 0 with error probability $1 - p$. Thus

$$Y = \xi \prod_{i=1}^{M} Y_i + (1 - \xi)(1 - \prod_{i=1}^{M} Y_i), \tag{1}$$

where ξ is Ber(p) and the Y_is, M and ξ are all independent.

• Alternative interpretation: model for a noisy distributed error-reporting system. Here a 0 represents an error report from a sub-system. Noise can reverse the binary (on-off) report from any sub-system.

Interest is centred on iterations of this structure.
In particular, we seek a stationary distribution, ν, such that if Y_i are iid with distribution ν and are independent of (M, ξ), then Y also has distribution ν.

So seek distributions ν on $[0,1]$ such that $(Y_i; 1 \le i)$ are iid with distribution $\nu \Rightarrow Y$ satisfying (1) also has distribution ν.
More precisely, with $\mathcal{P} =$ p.m.s on $[0,1]$, suppose that M has distribution d on $\overline{\mathbb{Z}}_+$ and define the map $\mathcal{T} \equiv \mathcal{T}_d : \mathcal{P} \to \mathcal{P}$ by

$$\mathcal{T}(\nu) = \text{Law}(\xi \prod_{i=1}^{M} Y_i + (1-\xi)(1 - \prod_{i=1}^{M} Y_i))$$

when the Y_i are iid $\sim \nu$ and are independent of M, and seek dynamics and fixed points of $\mathcal{T}$.

The existence and uniqueness of fixed points of this type of map, together with properties of the solutions, are addressed by Aldous and Bandhapadhyay in [1]), though we are dealing with a non-linear case to which the main results do not apply.

Generalisation of setting is so-called *tree-indexed* problem or Recursive Tree Process (RTP), in which we think of the Y_i as being marks associated with the daughter nodes of the root of T, a family tree of a Galton-Watson branching process. Start at level m of the random tree. Each vertex v in level $m-1$ of the tree has M_v daughter vertices, where the M_v are i.i.d. with common distribution d, and has associated with it noise ξ_v, where the $(\xi_u; u \in T)$ are iid and are independent of the $(M_u; u \in T)$.

By associating with daughter vertices independent random variables Y_{vi} having distribution ν, we see that Y_v and $Y_{vi}; 1 \leq i \leq M_v$ satisfy equation (1).

In this setting get the notion of endogeny.

Loosely speaking, a solution to the tree-indexed problem is said to be endogenous if it is a function of the noise alone so that no additional randomness is present.

- Work on a rooted tree with infinite branching factor. Random tree is embedded within it. An initial ancestor (in level zero), which we denote $\emptyset$, gives rise to a countably infinite number of daughter vertices (which form the members of the first generation), etc.

- Assign each vertex an address: members of the first generation are denoted $1, 2, ...$, the second generation by $11, 12, \ldots 21, 22, \ldots, 31, 32, \ldots$ etc.

- Write $uj, j = 1, 2, ...$ for daughters of a vertex u.

• Write $\mathbf{T}$ for the collection of all vertices or nodes (i.e. $\mathbf{T} = \bigcup_{n=0}^{\infty} \mathbb{N}^n$) partitioned by depth.

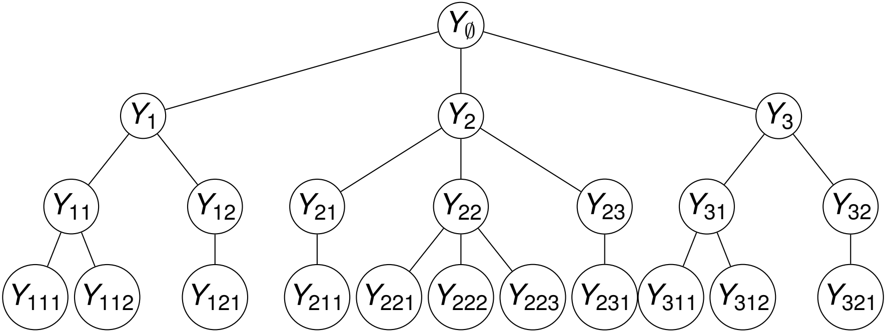
• Define the depth function $|\cdot|$ by $|u| = n$ if vertex u is in level n of tree. Associate to each vertex u iid random variables M_u with distribution d, giving the number of offspring produced by u. The vertices $u1, u2, ..., uM_u$ are thought of as being alive (relative to u) and the $\{uj : j > M_u\}$ as dead.

• Write original equation as a recursion on the vertices of $\mathbf{T}$:

$$Y_u = \xi_u \prod_{i=1}^{M_u} Y_{ui} + (1-\xi_u)(1-\prod_{i=1}^{M_u} Y_{ui}), \;\; u \in \mathbf{T}. \qquad (2)$$

Advantage of the embedding now becomes clear: we can talk about the RDE at any vertex in the infinite tree and yet, because the product only runs over the live daughters relative to u, the random Galton-Watson family tree is encoded into the RDE as noise.

Live descendants of root node to depth 3



Easy to transform the RDE (2) into the following, simpler, RDE:

$$X_u = 1 - \prod_{i=1}^{N_u} X_{ui}, \;\; u \in \mathbf{T}. \qquad (3)$$

• Colour red all the nodes, v, for which $\xi_v = 0$.

• Proceed down each line of descent from a node u until we hit a red node.

• In this way, we either "cut" the tree at collection of nodes which we regard as revised family of u, or not, in which case u has an infinite family.

• Denote new random family size by N_u then

$$Y_u = 1 - \prod_{i=1}^{N_u} Y_{\hat{ui}},$$

if u is red, where $\hat{ui}$ denotes the ith red node in the revised family of u.

• Condition on node u being red, then with this revised tree we obtain the RDE (3).

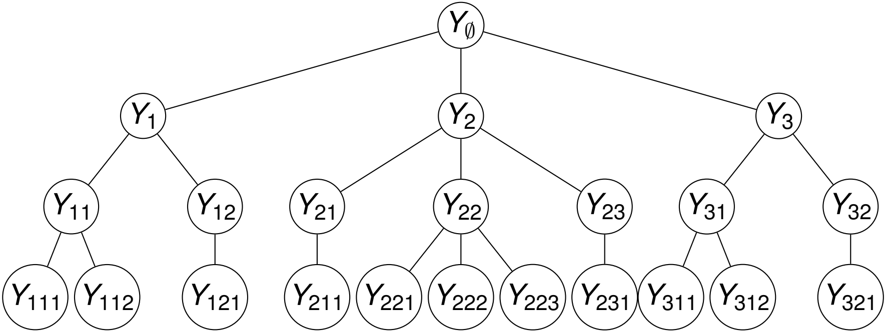
• Family size in new tree corresponds to total number of deaths in the original tree when it is independently thinned, with the descendants pruned with probability q.

• PGF, H, of the family size N_u on the new tree H is minimal positive solution of

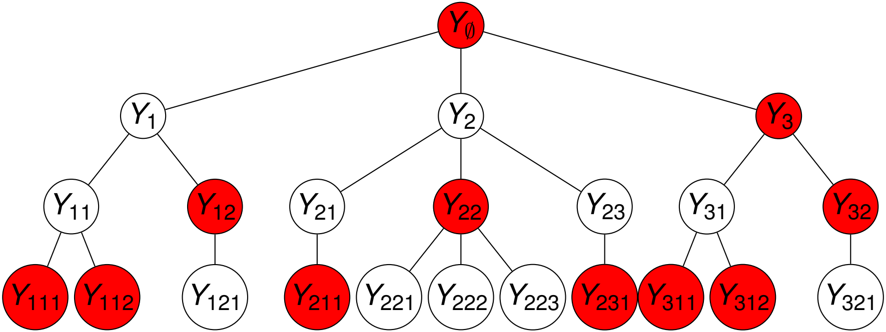
$$H(z) = G(pH(z) + qz), \tag{4}$$

where original tree has family size PGF G.

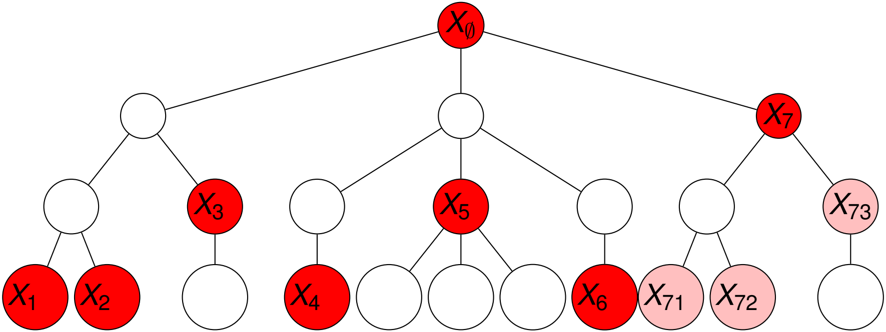
Live descendants of root node to depth 3

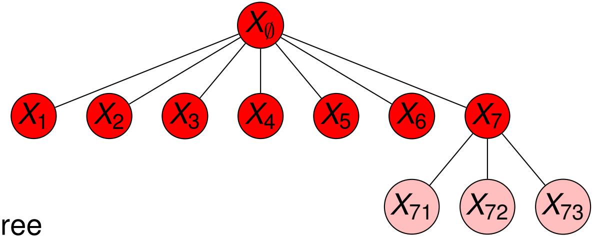


Tree with red nodes to depth 3



Tree with red nodes to depth 3





Pruned tree

From now on, assume that

$$\mathbb{P}(2 \leq N < \infty) > 0 \text{ which } \Leftrightarrow H \text{ is strictly convex,}$$

and we will consider non-negative solutions to (3) (easy to see these must lie in $[0,1]$)

Rewrite (3) as

$$1 - X = \prod_{i=1}^{N} X_i.$$

Then

$$(1-X)^n = \prod_{i=1}^{N} X_i^n \Rightarrow E[(1-X)^n] = E[\prod_{i=1}^{N} X_i^n] = H(E[X^n]).$$

Denote the nth moment of a generic solution to (3) by m_n, then

$$H(m_n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k m_k$$

or

$$H(m_n) + (-1)^{n-1} m_n = \sum_{k=0}^{n-1} \binom{n}{k} (-1)^k m_k. \tag{5}$$

Define

$$\phi : t \mapsto H(t) + t \text{ and } \psi : t \mapsto H(t) - t.$$

Clearly moment equation $\Rightarrow m_1$ solves $\phi(t) = 1$.

• Since $\phi(0) < 1 < 1 + \phi(1)$ and H is cts and strictly increasing, there is a unique solution μ_1 and so unique solution to RDE on $\{0, 1\}$ is Ber(μ_1).

• Result from [1] guarantees tree-indexed solution corresponding to a solution to the basic RDE and we denote such a solution by $\boldsymbol{S}$.

- Q: Are there other solutions?
- A: Sometimes!
- Q: Is this solution endogenous?
- A: Sometimes!

- Consider possible values for second moment, m_2.
- From moment equation, must solve

$$\psi(t) = 1 - 2\mu_1.$$

- Clearly μ_1 is a solution (***S*** has all moments equal to μ_1). Moreover ψ inherits strict convexity from H so *at most* two solutions.
- There is an acceptable candidate (i.e. a soln. less than μ_1) iff $\mu_1 > \mu_*$, the argmin of ψ, and this clearly happens iff $H'(\mu_1) > 1$. Iterating argument, see there are two candidate moment sequences iff $H'(\mu_1) > 1$.

• Still can't *guarantee* two different solutions in this case but since we're working on a bounded domain, moment sequences are distribution-determining so only a singular solution in case where $H'(\mu_1) \leq 1$.

• Suppose we take conditional expectations in RDE (conditional on all noise in the tree). We get

$$\begin{aligned} C_u &= E[S_u|\sigma(N_v:\ v\in\mathbf{T})] = E[1-\textstyle\prod_{i=1}^{N_u} S_{ui}|\sigma(N_v:\ v\in\mathbf{T})] \\ &= 1-\textstyle\prod_{i=1}^{N_u} E[S_{ui}|\sigma(N_v:\ v\in\mathbf{T})] \\ &= 1-\textstyle\prod_{i=1}^{N_u} C_{ui}, \end{aligned}$$

i.e. $\boldsymbol{C}$ also solves the RDE! This is not as special as it looks.

• Follows that, when $H'(\mu_1 \leq 1)$, $\boldsymbol{C}=\boldsymbol{S}$ and this is unique solution and endogenous.

- Q: What about the case where $H'(\mu_1) > 1$?
- A: It turns out that in this case $\boldsymbol{C}$ and $\boldsymbol{S}$ are distinct and give the only solutions!

The proof is tortuous but works like this:

1. Use a martingale argument to show that $\boldsymbol{C}$ is the unique endogenous solution
2. Use a result of Warren to show that in case where N is bounded, $\boldsymbol{S}$ is endogenous iff $H'(\mu_1) \leq 1$.
3. Take limits and conclude that when $H'(\mu_1) > 1$, $\boldsymbol{S} \neq \boldsymbol{C}$ and deduce there are exactly two solutions in this case.

Sketch proof of 1: Fix an endogenous solution X and define for each vertex u: $C_u^{[n]} = E[X_u|(N_v : |v| \leq n + |u|)]$. Clearly C is a bounded martingale so converges a.s. and in L^2 to X_u (since X is endogenous). But

$$\begin{aligned} C_u^{[n]} &= E[1 - \prod_{i_1=1}^{N_u} X_{ui_1}|(N_v : |v| \leq n + |u|)] \\ &= 1 - \prod_{i_1=1}^{N_u} C_{ui_1}^{[n-1]} \\ &= 1 - \prod_{i_1=1}^{N_u} \left(1 - \prod_{i_2=1}^{N_{ui_1}} (\ldots(1 - \prod_{i_n=1}^{N_{ui_1 i_2 \ldots i_{n-1}}} (1 - \mu_1^{N_{ui_1 i_2 \ldots i_n}}))\ldots)\right) \end{aligned}$$

and this is clearly independent of the choice of X.

Sketch proof of 2:

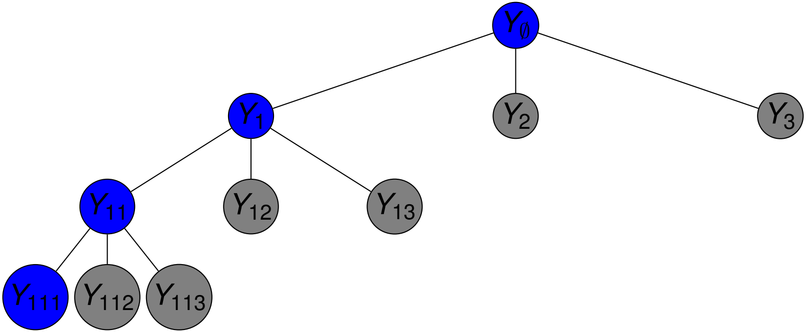
• Warren's result is for solutions to symmetric RDEs on a rooted d-ary tree:

$$Y_u = h(Y_{u1}, \dots, Y_{ud}; \xi_u), \tag{6}$$

where the Ys live on a finite space $\mathcal{S}$ with law π and ξ has law ν.

• Now look at a single line of descent, e.g. $Y_\emptyset, Y_1, Y_{11} \dots$ and rename as $Y_0, Y_{-1}, \dots$. This is clearly a Markov chain and (6) gives us an innovations description. Now couple two copies, Y and Y' by using the same innovations to generate both, to get a MC on $\mathcal{S}^2$.

Line of descent ($d = 3$)



Now kill this MC on coupling and denote the reduced matrix by P^-. The coupling time has a decay rate of ρ where ρ is the Perron- Frobenius eigenvalue of $\mathbb{P}^-$. Warren's result is:

Y is endogenous iff either

- Case 1: $d\rho < 1$
  or
- Case 2: $d\rho = 1$ and $\mathbb{P}^-$ is irreducible and $L^2(Y_\emptyset) \cap L^2(\xi_u : u \in \mathcal{T})^\perp = \{0\}$.

Note: this is a nice improvement on a key result in [1], which looks at $\mathcal{T}^{(2)}$, corresponding to coupling all lines of descent simultaneously.

• Apply this to our problem by imposing upper bound of n to branching factor/family size.

• Quick calculation then shows that $P^-_{(1,0),(1,0)} = 0$ and $P^-_{(1,0),(0,1)} = E[\frac{N}{n}\mu_1^{N-1}] = H'(\mu_1)/n$. So this is also ρ.

• Quick check shows that conditions are satisfied in case 2, so $\boldsymbol{S}$ is endogenous in case of bounded branching factor, iff $H'(\mu_1^{(n)}) \leq 1$, where n refers to imposed upper bound on random branching factor.

Sketch proof of 3.

• $H_n \downarrow H$ so $\mu_1^{(n)} \uparrow \mu_1$ and $H_n'(\mu_1^{(n)}) \to H'(\mu_1)$, so if $H'(\mu_1) > 1$ then $H_n'(\mu_1^{(n)}) > 1$ for large n.

• Similarly, $C_u^{(n)} \stackrel{L^2}{\to} C_u$, but for large n $\boldsymbol{C}^{(n)} \neq \boldsymbol{S}^{(n)}$ (because $H_n'(\mu_1^{(n)}) > 1$) so $\mu_2^{(n)} \to \mu_2 < \mu_*$ and hence $\mu_2 \neq \mu_1$.

• Thus we have:

$$\text{singular solution is endogeneous iff } H'(\mu_1) \leq 1$$

Example

N has generating function $H(x) = x^2$ (i.e. $N \equiv 2$). Then moment equation tells us that

$$m_1^2 + m_1 - 1 = 0$$

so that $m_1 = (\sqrt{5} - 1)/2$. For m_2 we have

$$m_2^2 - m_2 - (2 - \sqrt{5}) = 0$$

so that $m_2 = m_1$ or m_1^2 and so on. In fact the two possible moment sequences turn out to be $m_0 = 1, m_n = (\sqrt{5} - 1)/2$ for $n \geq 1$ or $m_0 = 1, m_1 = (\sqrt{5} - 1)/2, m_n = m_1^n$ for $n \geq 2$. they correspond to the singular solution and the endogenous one (the latter is constant! This is expected because there is no noise in the tree.)

Example

$N\ \sim$Geometric(α) so $H(s) = \frac{\alpha s}{1-\beta s}$ (with $\beta = 1-\alpha$).
It follows that $\mu_1 = 1-\sqrt{\alpha}$ and then $H'(\mu_1) = 1$, so unique endogenous solution to the original RDE is discrete and value at root is a.s. limit of

$$1 - \prod_{i_1=1}^{N_\emptyset}(1 - \prod_{i_2=1}^{N_{i_1}}(\dots(1-(1-\sqrt{\alpha})^{N_{i_1,\dots,i_n}})\dots)).$$

Let ζ be law of endogenous solution. For any initial distribution ν, get $\mathcal{T}^n(\nu)$ by inserting iid random variables with law ν at level n of the tree and applying the recursion to obtain the corresponding solution $X_u^n(\nu)$ (with law $\mathcal{T}^{n-|u|}(\nu)$) at vertex u.

The basin of attraction $B(\pi)$ of any solution is given by

$$B(\pi) = \{\nu \in \mathbb{P} : \mathcal{T}^n(\nu) \stackrel{\text{weak}^*}{\to} \pi\},$$

which is, of course, equivalent to the set of distributions ν for which $X_u^n(\nu)$ converges in law to a solution X of the RDE, with law π.

Theorem

Let δ denote the discrete distribution on $\{0, 1\}$ with mean μ_1. Then

$$B(\zeta) = \{\nu \in \mathbb{P} : \int x d\nu(x) = \mu_1 \text{ and } \nu \neq \delta\}.$$

That is, $B(\zeta)$ is precisely the set of distributions on $[0, 1]$ with the correct mean (except the discrete distribution with mean μ_1).

Theorem

In the stable case where $H'(\mu_1) \leq 1$, let $b(\mu_1)$ be the basin of attraction of μ_1 under the iterative map for the first moment, $f : t \mapsto 1 - H(t)$. Then

$$B(\zeta) = \{\nu \in \mathbb{P} : \int x d\nu(x) \in b(\mu_1)\}.$$

• Both theorems are proved by analysis of 2nd moments to show L^2 convergence.

- Q: What happens outside these basins of attraction?
- A: get convergence to limit cycles of length 2!

• It is easily seen that the map for the first moment $f : t \mapsto 1 - H(t)$ can have only one- and two-cycles.

• This is because the iterated map $f^{(2)} : t \mapsto 1 - H(1 - H(t))$ is increasing in t and hence can have only one-cycles. Notice also that the fixed points (or one-cycles) of $f^{(2)}$ come in pairs: if p is a fixed point then so too is $1 - H(p) = f(p)$.

We consider the iterated RDE:

$$X = 1 - \prod_{i=1}^{N_\emptyset}(1 - \prod_{j=1}^{N_i} X_{ij}). \tag{7}$$

This corresponds to the iterated map on laws on [0,1], $\mathcal{T}^2$.
Denote a generic two-cycle of the map f by the pair (μ_1^+, μ_1^-).

Theorem

Suppose that (μ_1^+, μ_1^-) is a two-cycle of f. There are at most two solutions of the RDE (7) with mean μ_+^1. There is a unique endogenous solution C^+, and a (possibly distinct) discrete solution, S^+, taking values in $\{0, 1\}$. The endogenous solution C^+ is given by $P(S^+ = 1|\mathbf{T})$ (just as in the non-iterated case). The solutions are distinct if and only if $H'(\mu_-^1)H'(\mu_+^1) > 1$, i.e. if and only if μ_+^1 (or μ_-^1) is an unstable fixed point of $f^{(2)}$.

• Proof is again derived by looking at second moments and proving L^2 convergence.

Example

Recall: if N is Geometric(α), $H(s) = \frac{\alpha s}{1-\beta s}$ (with $\beta = 1 - \alpha$). It follows that

$$f^{(2)}(s) = s,$$

so that every pair $(s, \frac{1-s}{1-\beta s})$ is a two-cycle of f and the unique fixed point of f is $1 - \sqrt{\alpha}$. Follows that s is a neutrally stable fixed point of $f^{(2)}$ for each $s \in [0, 1]$.

For any s, there is a unique solution to the iterated RDE with mean s and it is discrete and endogenous and is the a.s. limit of $1 - \prod_{i_1=1}^{N_\emptyset}(1 - \prod_{i_2=1}^{N_{i_1}}(\ldots(1 - s^{N_{i_1,\ldots,i_{2n-1}}})\ldots))$.

Example

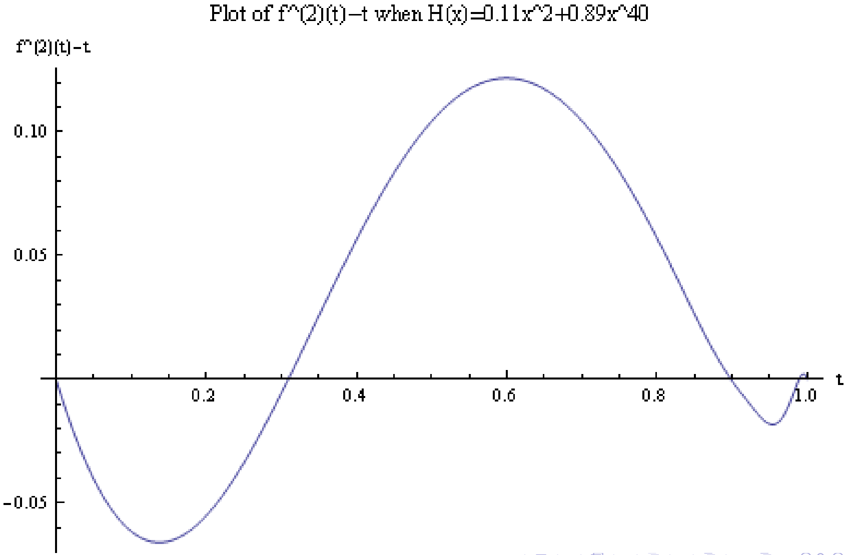
Consider original noisy veto-voter model on binary tree. It follows from (4) that

$$H(z) = (pH(z) + qz)^2 \Rightarrow H(z) = \frac{1 - 2pqz - \sqrt{1 - 4pqz^2}}{2p^2}.$$

This is non-defective if and only if $p \leq \frac{1}{2}$ (naturally), i.e. if and only if extinction is certain in the trimmed tree from the original veto-voter model. It is fairly straightforward to show that $H'(\mu_1) > 1 \Leftrightarrow p < \frac{1}{2}$. Thus, the endogenous solution is non-discrete precisely when the trimmed tree is sub-critical i.e. when modified family size is a.s. finite.

Example

In contrast to the case of the veto-voter model on the binary tree, the veto-voter model on a trinary tree can show a non-endogenous discrete solution even when the trimmed tree is supercritical. More precisely, the trimmed tree is supercritical precisely when $p > \frac{1}{3}$, but the discrete solution is non-endogenous if and only if $p < p_e^{(3)} \stackrel{def}{=} \frac{3.\sqrt{3}-4}{3.\sqrt{3}-2}$, and $p_e^{(3)} > \frac{1}{3}$.


Plot of f^(2)(t)−t when H(x)=0.11x^2+0.89x^40
f^(2)(t)-t
0.10
0.05
-0.05
0.2
0.4
0.6
0.8
1.0
t

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- A. Bandyopadhyay (2006): A necessary and sufficient condition for the tail-triviality of a recursive tree process, *Sankhya*, **68**, 1, 1–23.
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Theorem

(Aldous and Bandyopadhyay) Suppose $\mathcal{S}$ is a Polish space. Consider an invariant RTP with marginal distribution μ. Denoting by $\mu^{\nearrow}$ the diagonal measure on $\mathcal{S}^2$ with marginals μ then we have:
(a) If the endogenous property holds, then $\mu^{\nearrow}$ is the unique fixed point of $\mathcal{T}^{(2)}$.
(b) Conversely, suppose $\mu^{\nearrow}$ is the unique fixed point of $\mathcal{T}^{(2)}$. If also $\mathcal{T}^{(2)}$ is continuous with respect to weak convergence on the set of bivariate distributions with marginals μ, then the endogenous property holds.
(c) Further, the endogenous property holds if and only if
$$\mathcal{T}^{(2)^n}(\mu \otimes \mu) \overset{weak^*}{\rightarrow} \mu^{\nearrow}.$$