NO ARBITRAGE AND CLOSURE RESULTS FOR TRADING CONES WITH TRANSACTION COSTS

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Abstract. In this paper, we consider trading with proportional transaction costs as in Schachermayer's paper of 2004. We give a necessary and sufficient condition for $\mathcal{A}$, the cone of claims attainable from zero endowment, to be closed. Then we show how to define a revised set of trading prices in such a way that firstly, the corresponding cone of claims attainable for zero endowment, $\tilde{\mathcal{A}}$, does obey the Fundamental Theorem of Asset Pricing and secondly, if $\tilde{\mathcal{A}}$ is arbitrage-free then it is the closure of $\mathcal{A}$. We then conclude by showing how to represent claims.

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1. Introduction, notation and main results

1.1. **Introduction.** Recollect the Fundamental Theorem of Asset Pricing in finite discrete time (see, for example, Schachermayer [10]): the fact that $\mathcal{A}$, the set of claims attainable for 0 endowment, is arbitrage-free implies and is implied by the existence of an Equivalent Martingale Measure; in addition, $\mathcal{A}$ is closed if it is arbitrage-free.

In [11], Schachermayer showed that the Fundamental Theorem of Asset Pricing fails in the context of trading with spreads/transaction costs, by giving an example of an $\mathcal{A}$ which is arbitrage-free, but whose closure does contain an arbitrage (see also Kabanov, Rasonyi and Stricker [7] and [8]). Consequently it is of interest to investigate further when the cone $\mathcal{A}$ is closed, and in cases when it is not, to find descriptions of its closure.

Schachermayer then established (Theorem 1.7 of [11]) the equivalence of two criteria associated with the no-arbitrage condition for the general set-up for trading with spreads/transaction costs: that robust no-arbitrage implies and is implied by the existence of a strictly consistent price process. Here, robust no-arbitrage means loosely that even with smaller bid-ask spreads there is no arbitrage, whilst a strictly consistent price process is one taking values in the relative interior of the set of consistent prices. In Theorem 2.1 of [11] he showed that the robust no-arbitrage condition implies the closure (in $\mathcal{L}^0$) of the set of attainable claims.

In this paper we shall first give, in Theorem 1.1, a simple necessary and sufficient condition for the set of attainable claims to be closed. We go on to show, in Theorem 1.2, how to amend the bid-ask spreads so that the new cone of attainable claims does satisfy the original Fundamental Theorem (i.e. is either arbitrage-free and closed or admits an arbitrage). Moreover, we show that in the arbitrage-free case the new cone is simply the closure of the original cone of attainable claims. Finally, in section 4,

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we consider representation of attainable claims and characterize claims attainable for a given initial endowment.

1.2. **Notation and main results.** We are equipped with a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t : t = 0, 1, \ldots, T), \mathbb{P})$. We denote the set of non-negative, real-valued $\mathcal{F}_t$-measurable random variables by $m\mathcal{F}_t^+$ and the bounded non-negative, real-valued $\mathcal{F}_t$-measurable random variables by $b\mathcal{F}_t^+$. We denote the set of $\mathbb{R}^d$-valued $\mathcal{F}_t$-measurable random variables by $\mathcal{L}_t^0$ and the non-negative $\mathbb{R}^d$-valued $\mathcal{F}_t$-measurable random variables by $\mathcal{L}_t^{0,+}$. More generally, we denote the set of $\mathcal{F}_t$-measurable random variables taking values in the (suitably measurable) random set S by $\mathcal{L}^0(S; \mathcal{F}_t)$.

We recall the setup from Schachermayer's paper [11] for trading with d assets. A $d \times d$ matrix, Π is said to be a *bid-ask matrix* if

- $\Pi^{ij} > 0$ for all i, j;
- $\Pi^{ii} = 1$;

and

- $\Pi^{ij}\Pi^{jk} \geq \Pi^{ik}$.

We interpret Π^{ij} as the number of units of asset i required to purchase one unit of asset j.

An adapted $\mathbb{R}^{d\times d}$ process $(\pi_t : t = 0, 1, \ldots, T)$ with each π_t being a bid-ask matrix is known as a *bid-ask process* and gives the time t price for one unit of each asset in terms of each other asset. We assume that we are given a fixed bid-ask process, π.

Next we define, for a fixed bid-ask matrix, Π, the solvency cone, $K(\Pi)$, as the convex cone in $\mathbb{R}^d$ spanned by the canonical basis vectors of $\mathbb{R}^d$, $(e_i)_{1\leq i\leq d}$, together with the vectors $\Pi^{ij}e_i - e_j$. The solvency cone thus consists of all those holdings which can be traded to a non-negative holding at the prices specified by Π.

The cone of portfolios available at price zero under the bid-ask matrix Π is $-K(\Pi)$.

The time t trading cone consists of all those portfolios (including those attainable by the "burning" of assets) which are available at time t from zero endowment. A moment's thought will show that the set of trades which will be available at time t is the convex cone $\mathcal{L}^0(-K(\pi_t); \mathcal{F}_t) \stackrel{def}{=} -\mathcal{K}_t$.

The fundamental object of study is the cone of claims attainable from zero endowment, which will be denoted by $\mathcal{A}$, and is defined to be

$$(-\mathcal{K}_0) + \ldots + (-\mathcal{K}_T).$$

We also consider

$$\mathcal{C}_t \stackrel{def}{=} \{X \in \mathcal{L}_t^0 : cX \in \mathcal{A} \text{ for all } c \in b\mathcal{F}_t^+\}.$$

We say a few words on the interpretation of $\mathcal{C}_t$ versus $-\mathcal{K}_t$. It is clear that $-\mathcal{K}_t \subseteq \mathcal{C}_t \subseteq \mathcal{A}$, thus we have the equality

$$\mathcal{A} = \mathcal{C}_0 + \ldots + \mathcal{C}_T.$$

We can think of $\mathcal{C}_t$ as consisting of those trades which are available on terms that are known at time t but which may require trading at later times to be realised.

Although each $-\mathcal{K}_t$ is closed in $\mathcal{L}_t^0$, this is not enough to ensure that $\mathcal{A}$ is closed in $\mathcal{L}_T^0$. In contrast we find the following necessary and sufficient condition for the closure of $\mathcal{A}$:

Theorem 1.1. *$\mathcal{A}$ is closed in $\mathcal{L}_T^0$ if and only if each $\mathcal{C}_t$ is closed.*

Let $\bar{\mathcal{A}}$ denote the closure of $\mathcal{A}$ in $\mathcal{L}^0_T$. Unlike in a classical market, $\mathcal{A}$ can be arbitrage-free, that is to say

$$\mathcal{A} \cap \mathcal{L}^{0,+}_T = \{0\},$$

yet not closed. It is then natural to ask for a description of the closure, $\bar{\mathcal{A}}$.

Theorem 1.2. *There is an adjusted bid-ask process $\tilde{\pi}$ (see Definition 3.6) such that the associated cone of claims $\tilde{\mathcal{A}}$ satisfies $\mathcal{A} \subseteq \tilde{\mathcal{A}} \subseteq \bar{\mathcal{A}}$. Moreover, either $\tilde{\mathcal{A}}$ contains an arbitrage or it is arbitrage-free and closed. In the former case, $\bar{A}$ also contains an arbitrage, while in the latter case*

$$\bar{\mathcal{A}} = \tilde{\mathcal{A}}.$$

2. Results on the closedness of $\mathcal{A}$

As we have remarked already, $\mathcal{A}$ can be arbitrage-free but not closed. Recall that Schachermayer gives a sufficient condition for the closedness of $\mathcal{A}$ in terms of robust arbitrage.

Schacheramyer defines the bid-ask spreads as the (random) intervals $[\frac{1}{\pi^{j,i}_t}, \pi^{ij}_t]$, for $i, j \in \{1, \ldots, d\}$ and $t = 0, \ldots, T$, and defines robust no-arbitrage as follows:

- the bid-ask process π satisfies *robust no-arbitrage* if there is a bid-ask process $\tilde{\pi}$ with smaller bid-ask spreads than π (i.e. one whose bid-ask spreads almost surely fall in the relative interiors, in $\mathbb{R}$, of the bid-ask spreads for π) whose cone of admissible claims is arbitrage-free.

Theorem 2.1 of Schachermayer [11] then states that robust no-arbitrage implies that the cone $\mathcal{A}$ is closed — as the remark after the proof states, the proof relies only on the collection of *null strategies* (see Definition 2.5) being a closed vector space. However it is easy to find an example where $\mathcal{A}$ is closed and arbitrage-free but robust no-arbitrage fails.

Consider the following example:

Example 2.1. *Suppose that $T = 1$, $d = 2$, $\pi^{1,2}_0 = 1$, $\pi^{2,1}_0 = 2$ whilst $\pi^{ij}_1 = 1$ for each pair i, j. Take $\Omega = \mathbb{N}$, $\mathcal{F}_0$ trivial and $\mathcal{F}_1 = 2^{\mathbb{N}}$ with $\mathbb{P}$ given by $\mathbb{P}(n) = 2^{-n}$.*

It is immediately clear that robust no-arbitrage cannot hold, since any bid-ask process $\tilde{\pi}$ with smaller bid-ask spreads than π must have $\tilde{\pi}^{1,2}_0 \in (\frac{1}{2}, 1)$ and $\tilde{\pi}^{2,1}_1 = 1$. There is then an arbitrage in the corresponding cone $\tilde{\mathcal{A}}$ since $e_2 - \tilde{\pi}^{1,2}_0 e_1 + e_1 - \tilde{\pi}^{2,1}_1 e_2$ must be a positive multiple of e_1.

Remark 2.2. *With the setup of Example 2.1, it is clear from the bid-ask prices that*

$$-\mathcal{K}_0 = \{(x, y) : \ x + y \le 0 \text{ and } x + 2y \le 0\}$$

and

$$-\mathcal{K}_1 = \{(X, Y) \in \mathcal{L}^0_1 : \ X + Y \le 0 \ \mathbb{P} \text{ a.s.}\}$$

and so (since $-\mathcal{K}_0 \subset -\mathcal{K}_1$ and $\mathcal{A} = -\mathcal{K}_0 + -\mathcal{K}_1$)

$$\mathcal{A} = \{(X, Y) \in \mathcal{L}^0_1 : \ X + Y \le 0 \ \mathbb{P} \text{ a.s.}\}.$$

We can then see that $\mathcal{C}_0 = \{(x, y) : \ x + y \le 0\}$, while $\mathcal{C}_1 = \mathcal{A} = \{(X, Y) \in \mathcal{L}^0_1 : X + Y \le 0 \ \mathbb{P}$ a.s.$\}$.

It is tempting to speculate that if $\mathcal{A}$ is not closed, then $\bar{\mathcal{A}}$ contains an arbitrage. The following example (compare with example 1.3 in Grigoriev [4]) shows that this is false.

Example 2.3. *Suppose that* $T = 1$, $d = 2$, $\pi_1^{1,2} = 1$, $\pi_1^{2,1} = 2$ *whilst* $\pi_0^{ij} = 1$ *for each pair* i, j. *Take* $\Omega = \mathbb{N}$, $\mathcal{F}_0$ *trivial and* $\mathcal{F}_1 = 2^{\mathbb{N}}$ *with* $\mathbb{P}$ *given by* $\mathbb{P}(n) = 2^{-n}$.

Then we have

$$\bar{\mathcal{A}} = \{(X, Y) \in \mathcal{L}_1^0 : \; X + Y \leq 0 \; \mathbb{P} \; a.s.\},$$

whereas

$$\mathcal{A} = \{(X, Y) \in \mathcal{L}_1^0 : \; X + Y \leq 0 \; \mathbb{P} \; a.s. \text{ and } 2X + Y \text{ is a.s. bounded above}\}.$$

Lemma 2.4. *For each* t, $\mathcal{C}_t$ *is a convex cone in* $\mathcal{L}_t^0$ *and*

$$\mathcal{A} = \mathcal{C}_0 + \ldots + \mathcal{C}_T.$$

Proof. Convexity for $\mathcal{C}_t$ is inherited from $\mathcal{A}$ as is stability under multiplication by positive scalars. The decomposition result follows from the fact that

$$-\mathcal{K}_t \subseteq \mathcal{C}_t$$

and the fact that $\mathcal{C}_t \subseteq \mathcal{A}$. □

Definition 2.5. *For any decomposition of* $\mathcal{A}$ *as a sum of convex cones:*

$$\mathcal{A} = \mathcal{M}_0 + \ldots + \mathcal{M}_T \, ,$$

we call elements of $\mathcal{M}_0 \times \ldots \times \mathcal{M}_T$ *which almost surely sum to 0, null-strategies (with respect to the decomposition* $\mathcal{M}_0 + \ldots + \mathcal{M}_T$*) and denote the set of them by* $\mathcal{N}(\mathcal{M}_0 \times \ldots \times \mathcal{M}_T)$. *For convenience we denote* $(-\mathcal{K}_0) \times \ldots \times (-\mathcal{K}_T)$ *by* $\mathbb{K}$ *and* $\mathcal{C}_0 \times \ldots \times \mathcal{C}_T$ *by* $\mathbb{C}$.

In what follows we shall often use the lemma below (Lemma 2 in Kabanov et al [8]):

Lemma 2.6. *Suppose that*

$$\mathcal{A} = \mathcal{M}_0 + \ldots + \mathcal{M}_T$$

is a decomposition of $\mathcal{A}$ *into convex cones with* $\mathcal{M}_t \subseteq \mathcal{L}_t^0$ *and* $b\mathcal{F}_t^+ \mathcal{M}_t \subseteq \mathcal{M}_t$ *for each* t*; then* $\mathcal{A}$ *is closed if* $\mathcal{N}(\mathcal{M}_0 \times \ldots \times \mathcal{M}_T)$ *is a vector space and each* $\mathcal{M}_t$ *is closed.*

Lemma 2.7. *Suppose that* $\mathcal{A} = \mathcal{M}_0 + \ldots + \mathcal{M}_T$, *where for each* t, $\mathcal{M}_t \subseteq \mathcal{L}_t^0$ *and* $b\mathcal{F}_t^+ \mathcal{M}_t \subseteq \mathcal{M}_t$, *then*

$$\mathcal{M}_t \subset \mathcal{C}_t.$$

Moreover, for each $0 \leq t \leq T$,

$$\mathcal{A}_t(\mathcal{C}) \stackrel{def}{=} \mathcal{C}_0 + \ldots + \mathcal{C}_t = \mathcal{A} \cap \mathcal{L}_t^0. \tag{2.1}$$

Proof. The inclusion $\mathcal{M}_t \subset \mathcal{C}_t$ follows immediately from the fact that $\mathcal{M}_t \subset \mathcal{A}$; the stability under multiplication by $b\mathcal{F}_t^+$; and the definition of $\mathcal{C}_t$.

To prove the equality (2.1), suppose $X \in \mathcal{A} \cap \mathcal{L}_t^0$. Let

$$X = \xi_0 + \ldots + \xi_T,$$

be a decomposition of X with $\underline{\xi} \in \mathbb{C}$. It follows from the fact that $X \in \mathcal{L}_t^0$ and $\xi_s \in \mathcal{L}_t^0$ for each $s < t$ that

$$Y = \xi_t + \ldots + \xi_T \in \mathcal{L}_t^0.$$

Therefore, it is sufficient to show that

$$(\mathcal{C}_t + \ldots + \mathcal{C}_T) \cap \mathcal{L}_t^0 \subset \mathcal{C}_t.$$

Now take $Y \in (\mathcal{C}_t + \ldots + \mathcal{C}_T) \cap \mathcal{L}_t^0$ and $c \in b\mathcal{F}_t^+$: clearly $cY \in \mathcal{A} \cap \mathcal{L}_t^0$ and hence, by definition, $Y \in \mathcal{C}_t$. □

We may now give the
Proof of Theorem 1.1
First assume that $\mathcal{A}$ is closed and $(X_n)_{n\geq 1}$ is a sequence in $\mathcal{C}_t$ converging in $\mathcal{L}^0$ to X. It follows immediately from the assumption that $cX_n \xrightarrow{\mathcal{L}^0} cX \in \mathcal{A}$ for all $c \in b\mathcal{F}_t^+$, hence $X \in \mathcal{C}_t$.

For the reverse implication we shall show that $\mathcal{N}(\mathbb{C})$ is a vector space and the result will then follow from Lemma 2.6.

Now suppose $(\xi_0, \ldots, \xi_T) \in \mathcal{N}(\mathbb{C})$ and $c \in b\mathcal{F}_t^+$ with almost sure upper bound B: then, defining

$$\zeta_s = B\xi_s$$

for $s \neq t$ and

$$\zeta_t = (B - c)\xi_t,$$

it is clear (from the definition of $\mathcal{C}_s$) that

$$(\zeta_0, \ldots, \zeta_T) \in \mathbb{C},$$

with

$$\sum_0^T \zeta_s = -c\xi_t.$$

It follows that

$$-c\xi_t \in \mathcal{A}, \forall c \in b\mathcal{F}_t^+$$

and so $-\xi_t \in \mathcal{C}_t$ for each t so that $\mathcal{N}(\mathbb{C})$ is a vector space as required. ☐

Remark 2.8. *In the proof above we used the fundamental property of null strategies: if $(\xi_s)_{0\leq s\leq T}$ is a null strategy then $-\xi_t \in \mathcal{C}_t$. A null strategy allows one to eliminate friction in any of its component trades. In what follows we shall generalize this idea to more general sequences of strategies.*

3. A revised fundamental theorem of asset pricing

We return to Example 2.3:

Example 3.1. *Recall that $T = 1$, $d = 2$, $\pi_1^{1,2} = 1$, $\pi_1^{2,1} = 2$ whilst $\pi_0^{ij} = 1$ for each pair i, j; $\Omega = \mathbb{N}$, $\mathcal{F}_0$ is trivial and $\mathcal{F}_1 = 2^{\mathbb{N}}$ with $\mathbb{P}$ given by $\mathbb{P}(n) = 2^{-n}$.*

We leave it as an exercise for the reader to show, as claimed above, that $\bar{\mathcal{A}} = \{(X, Y) \in \mathcal{L}_1^0 : X + Y \leq 0 \ \mathbb{P}$ a.s.$\}$ and hence corresponds to an adjusted bid-ask process, *which is identically equal to 1. To do so, one may consider the null strategy ξ given by $\xi_0 = e_1 - e_2$ and $\xi_1 = e_2 - e_1$.*

In this section we shall show that $\bar{\mathcal{A}}$, if arbitrage-free, can always be represented by some adjusted bid-ask process. However, the next example, which is a minor adaptation of one of the key examples in Schachermayer [11], shows that it is necessary to consider more than just null strategies when seeking the appropriate adjusted prices.

Definition 3.2. *We define $\mathcal{C}_t(\bar{\mathcal{A}})$ by analogy with $\mathcal{C}_t(\mathcal{A})$:*

$$\mathcal{C}_t(\bar{\mathcal{A}}) \stackrel{def}{=} \{X \in \mathcal{L}_t^0 : cX \in \bar{\mathcal{A}} \text{ for all } c \in b\mathcal{F}_t^+\}.$$

Example 3.3. *Suppose that $T = 1$, $d = 4$, $\Omega = \mathbb{N}$, $\mathcal{F}_0$ is trivial and $\mathcal{F}_1 = 2^{\Omega}$. The bid-ask process at time 0 satisfies $\pi_0^{2,1} = 1$, $\pi_0^{4,3} = 1$ whilst $\pi_0^{ij} = 3$ for each other pair*

i, j with $i \neq j$. At time 1, we have $\pi_1^{1,4} = \omega = \frac{1}{\pi_1^{4,1}}$ and $\pi_1^{2,3} = \omega = \frac{1}{\pi_1^{3,2}}$, whilst $\pi_1^{4,3} = 1$ and $\pi_1^{3,4} = 3$. All other entries are defined implicitly by the criterion

$$\pi_1^{ij} = \min_{i=i_0,\ldots,i_n=j} \pi_1^{i_0 i_1} \ldots \pi_1^{i_{n-1} i_n}.$$

We shall show that $e_4 - e_3, e_2 - e_1, e_1 - e_2 \in \mathcal{C}_1(\bar{\mathcal{A}})$ even though there is no **null** *strategy, ξ, with $\xi_0 = e_1 - e_2$ or with $\xi_0 = e_2 - e_1$ or with $\xi_0 = e_3 - e_4$.*

First, define a sequence of strategies ξ^N as follows: $\xi_0^N = N(e_1 - e_2)$ and

$$\xi_1^N = \frac{N}{\omega}(e_4 - \omega e_1) + (\frac{N}{\omega} - 1_{(N \geq \omega)})(e_3 - e_4) + N(e_2 - \frac{1}{\omega} e_3),$$

which means that $\xi_1^N = N(e_2 - e_1) + 1_{(N \geq \omega)}(e_4 - e_3)$.

Notice that $\sum_{t=0}^{1} \xi_t^N = 1_{(N \geq \omega)}(e_4 - e_3) \xrightarrow{\mathcal{L}^0} e_4 - e_3$ as $N \to \infty$, so we conclude that $e_4 - e_3 \in \mathcal{C}_0(\bar{\mathcal{A}})$. However, $e_3 - e_4 \in -\mathcal{K}_1$ and so $((e_4 - e_3), (e_3 - e_4))$ is null for $\mathbb{C}(\bar{\mathcal{A}})$ and hence $e_4 - e_3 \in \mathcal{C}_1(\bar{\mathcal{A}})$.

Now, given an element X of $b\mathcal{F}_1^+$ with a.s. bound B, consider the strategy $((N + B)(e_1 - e_2) + (e_3 - e_4), (N + (B - X)(e_2 - e_1) + 1_{(N+(B-X) \geq \omega)}(e_4 - e_3))$, which sums to $X(e_1 - e_2) - 1_{(N+(B-X)<\omega)}(e_4 - e_3) \xrightarrow{\mathcal{L}^0} X(e_1 - e_2)$ as $N \to \infty$. This shows that $e_1 - e_2 \in \mathcal{C}_1(\bar{\mathcal{A}})$ and so is also in $\mathcal{C}_0(\bar{\mathcal{A}})$.

Lastly, consider the strategy

$$(N(e_1 - e_2) + (e_3 - e_4), (N + X))(e_2 - e_1) + 1_{(N+X \geq \omega)})(e_4 - e_3)),$$

which sums to $X(e_2 - e_1) - 1_{(N+X<\omega)}(e_4 - e_3) \xrightarrow{\mathcal{L}^0} X(e_2 - e_1)$ as $N \to \infty$. This shows that $e_2 - e_1 \in \mathcal{C}_1(\bar{\mathcal{A}})$ and so is also in $\mathcal{C}_0(\bar{\mathcal{A}})$.

It follows that $\bar{A}$ corresponds to the adjusted bid-ask process $\tilde{\pi}$ given, for $t = 0$, by: $\tilde{\pi}_0^{1,2} = \tilde{\pi}_0^{2,1} = \tilde{\pi}_0^{3,4} = \tilde{\pi}_0^{4,3} = 1$, $\tilde{\pi}_0^{i,j} = \tilde{\pi}_0^{j,i} = 3$ for $i \in \{1, 2\}$ and $j \in \{3, 4\}$; and for $t = 1$ by: $\tilde{\pi}_1^{1,4} = \omega = \frac{1}{\tilde{\pi}_1^{4,1}} = \tilde{\pi}_1^{2,3} = \frac{1}{\tilde{\pi}_1^{3,2}}$, whilst $\tilde{\pi}_1^{4,3} = \tilde{\pi}_1^{3,4} = \tilde{\pi}_1^{1,2} = \tilde{\pi}_1^{2,1} = 1$. To see this, notice that the inclusion $\mathcal{A} \subset \tilde{\mathcal{A}}$ is obvious, while $\tilde{\mathcal{A}}$ is closed (by robust no-arbitrage) and the inclusion $\tilde{\mathcal{A}} \subset \bar{\mathcal{A}}$ follows from the arguments above.

In order to prove our new version of the Fundamental Theorem we first define the *adjusted* bid-ask process, $\tilde{\pi}$. This process will either be equal to the original bid-ask process or frictionless (ω by ω and for a given pair (i, j)).

Definition 3.4. *Given a bid-ask process π, we define for each (i, j, t),*

$$z_t^{i,j} \stackrel{def}{=} e_j - \pi_t^{ij} e_i$$

and

$$R_t^{i,j} \stackrel{def}{=} \{B \in \mathcal{F}_t \ : \ -z_t^{i,j} 1_B \in \bar{\mathcal{A}}\}. \tag{3.1}$$

Lemma 3.5. *If $B \in \mathcal{F}_t$ then*

$$-z_t^{i,j} 1_B \in \bar{\mathcal{A}} \Leftrightarrow -z_t^{i,j} 1_B \in \mathcal{C}_t(\bar{\mathcal{A}}).$$

Proof. Clearly the RHS implies the LHS *a fortiori*.

To prove the reverse implication, first note that, by definition of $-\mathcal{K}_t$,

$$kz_t^{i,j} \in -\mathcal{K}_t \text{ for any } k \in m\mathcal{F}_t^+,$$

which in turn implies that

$$kz_t^{i,j} \in \mathcal{C}_t \text{ for any } k \in m\mathcal{F}_t^+, \tag{3.2}$$

since $-\mathcal{K}_t \subset \mathcal{C}_t$. Now suppose that $c \in b\mathcal{F}_t^+$ with bound M, and set

$$Z \stackrel{def}{=} c(-z_t^{i,j} 1_B) = M(-z_t^{i,j} 1_B) + (M - c) z_t^{i,j} 1_B. \tag{3.3}$$

The first term on the right hand side of (3.3) is in $\bar{\mathcal{A}}$ since M is a positive constant, $-z_t^{i,j} 1_B$ is in $\bar{\mathcal{A}}$ by assumption and $\bar{\mathcal{A}}$ is a cone. The second term is in $\bar{\mathcal{A}}$ by (3.2) and, since $\bar{\mathcal{A}}$ is a convex cone, $Z \in \bar{\mathcal{A}}$. The result follows. □

Now observe that the collection $R_t^{i,j}$ is closed under countable unions. To see this, observe first that, since $\bar{\mathcal{A}}$ is a closed cone, $R_t^{i,j}$ is closed under countable, *disjoint*, unions. Now notice that, from Lemma 3.5, if $B \in R_t^{i,j}$ and $D \in \mathcal{F}_t$ then $B \cap D \in R_t^{i,j}$. It follows that if $(B_n)_{n \geq 1}$ is a sequence in $R_t^{i,j}$ then $B_n \setminus (\bigcup_{k=1}^{n-1} B_k) = B_n \cap (\bigcup_{k=1}^{n-1} B_k)^c \in R_t^{i,j}$ and hence $\bigcup_n B_n \in R_t^{i,j}$. We then deduce, by the usual exhaustion argument, that there exists a $\mathbb{P}$-a.s. maximum, which we denote by $B_t^{i,j}$; that is to say that

$$B \in R_t^{i,j} \text{ and } B_t^{i,j} \subseteq B \Rightarrow \mathbb{P}(B \setminus B_t^{i,j}) = 0.$$

Definition 3.6. *We define the* **adjusted** *bid-ask process $\tilde{\pi}$ as follows :*

$$\text{for each pair } i \neq j \text{ and for each } t,\ \tilde{\pi}_t^{j,i} \stackrel{def}{=} \frac{1}{\pi_t^{ij}} 1_{B_t^{i,j}} + \pi_t^{ji} 1_{(B_t^{i,j})^c}\,.$$

Remark 3.7. *$\tilde{\pi}$ need not satisfy the condition:*

$$\tilde{\pi}^{ik} \leq \tilde{\pi}^{ij} \tilde{\pi}^{jk},$$

but we may still define the corresponding trading cone and apply Lemma 2.6.

We denote the corresponding trading cones and cone of attainable claims by $(-\tilde{\mathcal{K}}_t)_{0 \leq t \leq T}$ and $\tilde{\mathcal{A}}$ respectively. Throughout the rest of the paper we denote $e_j - \tilde{\pi}_t^{i,j} e_i$ by $\tilde{z}_t^{i,j}$.

We now give the

Proof of Theorem 1.2

We first show that

$$\mathcal{A} \subseteq \tilde{\mathcal{A}} \subseteq \bar{\mathcal{A}},$$

and then show that $\tilde{\mathcal{A}}$ is closed if it is arbitrage-free.

Proof that ($\mathcal{A} \subseteq \tilde{\mathcal{A}}$):

Since $\pi_t^{ij} \pi_t^{ji} \geq 1$, it follows from the definition that $\tilde{\pi}_t \leq \pi_t$ for each t and so

$$-\mathcal{K}_t \subseteq -\tilde{\mathcal{K}}_t,$$

and hence

$$\mathcal{A} \subseteq \tilde{\mathcal{A}}.$$

Proof that ($\tilde{\mathcal{A}} \subseteq \bar{\mathcal{A}}$):

we show this by demonstrating that

$$-\tilde{\mathcal{K}}_t \subseteq \bar{\mathcal{A}}$$

for each $0 \leq t \leq T$.

This, in turn, is achieved by showing that

$$d\, \tilde{z}_t^{j,i} \in \bar{\mathcal{A}}, \text{ for all } d \in m\mathcal{F}_t^+. \tag{3.4}$$

From the definition of the adjusted bid-ask process, we obtain :

$$\tilde{z}_t^{j,i} = -\tilde{\pi}_t^{j,i}\, z_t^{i,j}\, 1_{B_t^{i,j}} + z_t^{j,i}\, 1_{(B_t^{i,j})^c}\,.$$

Observe that $-z_t^{i,j}\, 1_{B_t^{i,j}} \in \mathcal{C}_t(\bar{\mathcal{A}})$ by definition of the set $B_t^{i,j}$ and (3.3), so

$$-d\tilde{\pi}_t^{j,i}\; z_t^{i,j}\, 1_{B_t^{i,j}} \in \mathcal{C}_t(\bar{\mathcal{A}}) \subset \bar{\mathcal{A}},$$

and

$$d\, z_t^{j,i}\, 1_{(B_t^{i,j})^c} \in -\mathcal{K}_t \subseteq \bar{\mathcal{A}}$$

by definition of $-\mathcal{K}_t$, so that $d\, \tilde{z}_t^{j,i} \in \bar{\mathcal{A}}$ as required.

Proof that ($\tilde{\mathcal{A}}$ is closed if $\tilde{\mathcal{A}}$ is arbitrage-free):

We prove this by showing that the nullspace $\tilde{\mathcal{N}} \overset{def}{=} \mathcal{N}\Big((-\tilde{\mathcal{K}}_0) \times \ldots \times (-\tilde{\mathcal{K}}_T)\Big)$ is a vector space and then appealing to Lemma 2.6.

Let $\xi \in \tilde{\mathcal{N}}$. Then, defining $\mathcal{C}_t(\tilde{\mathcal{A}})$ analogously to $\mathcal{C}_t(\mathcal{A})$, for each t we have, by Remark 2.8, $-\xi_t \in \mathcal{C}_t(\tilde{\mathcal{A}})$, because ξ is null for $\tilde{A}$.

Now, since $\xi_t \in -\tilde{\mathcal{K}}_t$ we may write it as

$$\xi_t = \sum_{i,j} \alpha_t^{i,j}\, \tilde{z}_t^{i,j} - \sum_k \beta_t^k e_k,$$

for suitable $\alpha_t^{i,j}$ and β_t^k in $b\mathcal{F}^+$. Moreover, $-\xi_t \in \mathcal{C}_t(\tilde{\mathcal{A}})$ and since $\sum_{i,j} \alpha_t^{i,j}\, \tilde{z}_t^{i,j} \in \tilde{\mathcal{A}}$ we conclude that $\sum_k \beta_t^k e_k \in \tilde{\mathcal{A}}$. Now, since, by assumption, $\tilde{\mathcal{A}}$ is arbitrage-free, we conclude that $\sum_k \beta_t^k e_k = 0$ a.s., so

$$\xi_t = \sum_{i,j} \alpha_t^{i,j}\, \tilde{z}_t^{i,j},$$

and consequently $-\sum_{i,j} \alpha_t^{i,j}\, \tilde{z}_t^{i,j} \in \mathcal{C}_t(\tilde{\mathcal{A}})$. Since $\mathcal{C}_t(\tilde{\mathcal{A}})$ is a convex cone and $\alpha_t^{i,j}\, \tilde{z}_t^{i,j} \in -\tilde{\mathcal{K}}_t \subset \mathcal{C}_t(\tilde{\mathcal{A}})$ for each (i,j), we may deduce that, for each pair (i,j):

$$-\alpha_t^{j,i}\, \tilde{z}_t^{j,i} \in \mathcal{C}_t(\tilde{\mathcal{A}}).$$

Now, multiplying by the positive, bounded and $\mathcal{F}_t$-measurable r.v. $\frac{1}{\alpha_t^{j,i}} 1_{\left(\{\alpha_t^{j,i} > \frac{1}{n}\} \cap (B_t^{i,j})^c\right)}$, we see that

$$-z_t^{j,i}\, 1_{\left(\{\alpha_t^{j,i} > \frac{1}{n}\} \cap (B_t^{i,j})^c\right)} = -\tilde{z}_t^{j,i}\, 1_{\left(\{\alpha_t^{j,i} > \frac{1}{n}\} \cap (B_t^{i,j})^c\right)} \in \tilde{\mathcal{A}} \subset \bar{\mathcal{A}}.$$

Then, by definition of the set $B_t^{j,i}$, for each n the subset $D_t^{i,j}(n) \overset{def}{=} \{\alpha_t^{j,i} > \frac{1}{n}\} \cap (B_t^{i,j})^c \subset B_t^{j,i}$. Now, by taking the union over n, we see that

$$D_t^{i,j} \overset{def}{=} \{\alpha_t^{j,i} > 0\} \cap (B_t^{i,j})^c = \cup_n D_t^{ij}(n) \subset B_t^{j,i},$$

and we obtain therefore that

$$\tilde{\pi}_t^{j,i} = \pi_t^{ji} = \frac{1}{\tilde{\pi}_t^{i,j}}$$

on the subset $D_t^{i,j}$. We deduce that

$$-\tilde{z}_t^{j,i}\, 1_{D_t^{i,j}} = -z_t^{j,i}\, 1_{D_t^{i,j}} = \tilde{\pi}_t^{j,i}\, z_t^{i,j}\, 1_{D_t^{i,j}} \in -\tilde{\mathcal{K}}_t\,,$$

and

$$-\tilde{z}_t^{j,i}\, 1_{\left(\{\alpha_t^{j,i} > 0\} \cap B_t^{i,j}\right)} = \tilde{\pi}_t^{j,i}\, z_t^{i,j}\, 1_{\left(\{\alpha_t^{j,i} > 0\} \cap B_t^{i,j}\right)} \in -K_t \subset -\tilde{\mathcal{K}}_t\,.$$

Hence $-\xi_t \in -\tilde{\mathcal{K}}_t$. It follows that $\tilde{\mathcal{N}}$ is a vector space as claimed. □

4. Decompositions of $\mathcal{A}$, representation and dual cones

4.1. **Decompositions of $\mathcal{A}$ and consistent price processes.** We have given a necessary and sufficient condition for $\mathcal{A}$ to be closed in terms of the $\mathcal{C}_t(\mathcal{A})$ and we have shown how to amend the bid-ask prices so that the new cone attainable with zero endowment is $\bar{\mathcal{A}}$ (if $\bar{\mathcal{A}}$ is arbitrage-free). It is natural to ask whether the resulting trading cones $(-\tilde{\mathcal{K}}_t)_{0\le t\le T}$ coincide with the $\mathcal{C}_t(\tilde{\mathcal{A}})$'s. The following example shows that this is far from the case.

Example 4.1. *Suppose that $T=1$, $d=4$, $\Omega=\{1,2\}$, $\mathcal{F}_0$ is trivial and $\mathcal{F}_1=2^\Omega$. The bid-ask process at time 0 satisfies $\pi_0^{4,3}=\pi_0^{4,2}=1$ whilst, for all other pairs $i\neq j$, $\pi_0^{ij}=4$; the bid-ask process at time $t=1$ satisfies $\pi_1^{2,1}(1)=4/3=2-\pi_1^{3,1}(1)=2-\pi_1^{2,1}(2)=\pi_1^{3,1}(2)$ whilst, for all other pairs $i\neq j$, $\pi_1^{ij}=4$. By considering the strategy ξ given by $\xi_0=\frac{1}{2}(e_3+e_2)-e_4$ and $\xi_1=e_1-\frac{1}{2}(e_3+e_2)$, we see that $e_1-e_4\in\mathcal{A}$ and hence is in $\mathcal{C}_0$. Now Ω is finite so $\mathcal{A}$ is closed and it is now easy to check that $\tilde{\pi}=\pi$, yet $e_1-e_4\notin -K_0$ and so $-\tilde{\mathcal{K}}_0\neq\mathcal{C}_0$.*

In the rest of this section we shall show that nevertheless, the $\mathcal{C}_t$'s and their 'duals' behave like the original trading cones.

Whereas each trading cone, being generated by a finite set of random vectors, can clearly be identified as $\mathcal{L}^0(S;\mathcal{F}_t)$ for a suitable random cone S, the same is not evidently true of the $\mathcal{C}_t$s. Thus, we first need some abstract results relating to cones of random variables.

Remark 4.2. *We denote by $\mathcal{D}$, the collection of all closed subsets of $\mathbb{R}^d$. The standard Borel σ-algebra on $\mathcal{D}$, known as the Effros σ-algebra, and denoted $\mathcal{B}(\mathcal{D})$, is as follows: for any set B in $\mathbb{R}^d$ define $\mathcal{D}(B)$ by*

$$\mathcal{D}(B)=\{C\in\mathcal{D}:\, C\cap B\neq\emptyset\},$$

then $\mathcal{B}(\mathcal{D})=\sigma(\pi)$, where

$$\pi=\{\mathcal{D}(B):\, B \text{ open in } \mathbb{R}^d\}.$$

Definition 4.3. *We denote by Υ, the set of all maps measurable with respect to the Effros σ-algebra. We refer to any $\Lambda\in\Upsilon$ as a* random closed set.

Lemma 4.4. *For any $X\in\mathcal{L}^0(\mathbb{R}^d;\mathcal{F})$ and $\Lambda\in\Upsilon$,*

$$(X\in\Lambda)\overset{def}{=}\{\omega:\, X(\omega)\in\Lambda(\omega)\}\in\mathcal{F}. \tag{4.1}$$

Proof. First, by the fundamental measurability theorem of Himmelberg [5], there is a sequence of $\mathbb{R}^d$-valued random variables $(X_n)_{n\ge1}$ such that a.s

$$\Lambda(\omega)=\overline{\{X_n(\omega):\, n\ge1\}}.$$

Then, the set $\{\omega:\, X(\omega)\in\Lambda(\omega)\}=\bigcap_n\bigcup_i\{\omega:\, |X_i(\omega)-X(\omega)|<\frac{1}{n}\}\in\mathcal{F}$. □

Remark 4.5. *In what follows we call a map $D\in\Upsilon$ with values in the set of closed convex cones in $\mathbb{R}^d$ a* random closed cone.

Theorem 4.6. Abstract closed convex cones theorem. *Let $\mathcal{C}$ be a closed convex cone in $\mathcal{L}^0(\mathbb{R}^d;\mathcal{F})$, then*

$$\mathcal{C} \text{ is stable under multiplication by (scalar) elements of } b\mathcal{F}^+ \tag{4.2}$$

iff there is a map $\Lambda\in\Upsilon$ such that

$$\mathcal{C}=\mathcal{L}^0(\Lambda;\mathcal{F}). \tag{4.3}$$

In this case, the map Λ is a random closed cone.

Proof. The implication (4.3)$\Rightarrow$ (4.2) is obvious.

To prove the direct implication: we consider the family:

$$\Upsilon_{\mathcal{C}} = \{\Gamma \in \Upsilon : \ \mathcal{L}^0(\Gamma;\mathcal{F}) \subset \mathcal{C}\}.$$

From Valadier [13] and [14], there is an essential supremum $\Lambda \in \Upsilon$ of this family $\Upsilon_{\mathcal{C}}$, i.e.:

(1) for all $\Gamma \in \Upsilon_{\mathcal{C}}$, we have $\Gamma \subset \Lambda$ a.s.;
(2) if $\Sigma \in \Upsilon$ is such that for all $\Gamma \in \Upsilon_{\mathcal{C}}$, we have $\Gamma \subset \Sigma$ a.s, then $\Lambda \subset \Sigma$ a.s.

Moreover there is a countable subfamily $(\Gamma^n)_{n\geq 1} \subset \Upsilon_{\mathcal{C}}$ such that $\Lambda = \overline{\bigcup_{n\geq 1}\Gamma^n}$ a.s. We want to prove that $\mathcal{C} = \mathcal{L}^0(\Lambda;\mathcal{F})$. To do this, first we remark that $\mathcal{C}(\Lambda) = \overline{\bigcup_{n\geq 1}\mathcal{C}(\Gamma^n)}$. Then $\mathcal{L}^0(\Lambda;\mathcal{F}) \subset \mathcal{C}$ and so $\Lambda \in \Upsilon_{\mathcal{C}}$. Now let $\xi \in \mathcal{C}$ and define the map $\Gamma(\omega) = \Lambda(\omega) \cup \{\xi(\omega)\}$. For $X \in \Gamma$ a.s and $B = \{\xi = X\}$ we have $X1_{B^c} \in \Lambda$ and then $X1_{B^c} \in \mathcal{C}$ and $X1_B = \xi 1_B \in \mathcal{C}$. So $X \in \mathcal{C}$. We deduce that $\mathcal{L}^0(\Gamma;\mathcal{F}) \subset \mathcal{C}$ and then $\Gamma \in \Upsilon_{\mathcal{C}}$. By the essential supremum property of Λ, we have $\Gamma \subset \Lambda$ and then $\xi \in \Lambda$ a.s.

Now suppose that (4.3) is satisfied and consider the sequence $(X_n)_{n\geq 1}$ that generates Λ. For any $\alpha \in \mathbb{R}^n$, define

$$Y_{n,\alpha} = \sum_{i=1}^{n} \alpha_i X_i.$$

Notice that, denoting the non-negative rationals by $\mathbb{Q}_+$, the collection

$$S \stackrel{def}{=} \{Y_{n,\alpha} : \ \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Q}_+^n\}$$

is countable.

Define the map $\tilde{\Lambda}$ by:

$$\tilde{\Lambda}(\omega) = \overline{\{Y(\omega) : \ Y \in S\}},$$

where the closure is in $\mathbb{R}^d$. From the convex cone property of $\mathcal{C}$, we have each $Y \in \mathcal{C}$ and then, from (4.3), $\mathbb{P}(Y \in \Lambda) = 1$. We deduce that $\tilde{\Lambda} \subset \Lambda$ a.s and then (since $X_n \in S$ for each n) that $\Lambda = \tilde{\Lambda}$ a.s. □

Definition 4.7. *Given a closed convex cone $\mathcal{C}$ in $\mathcal{L}^0_t$ satisfying (4.2) (with respect to the σ-algebra $\mathcal{F}$) we denote the corresponding random convex cone in (4.3) by $\Lambda(\mathcal{C};\mathcal{F})$.*

Corollary 4.8. *Suppose that $0 \leq p \leq \infty$ and let $\mathcal{C}$ be a convex cone in $\mathcal{L}^p(\mathbb{R}^d;\mathcal{F})$ with $\mathcal{C}$ closed in $\mathcal{L}^p(\mathbb{R}^d;\mathcal{F})$ if $0 \leq p < \infty$, and with $\mathcal{C}$ $\sigma(\mathcal{L}^\infty(\mathbb{P}),\mathcal{L}^1(\mathbb{P}))$-closed if $p = \infty$. Then, $\mathcal{C}$ is stable under multiplication by (scalar) elements of $b\mathcal{F}^+$ iff there exists a random closed cone D such that*

$$\mathcal{C} = \mathcal{L}^p(D;\mathcal{F}).$$

Proof. First suppose that $0 \leq p < \infty$ and consider $\overline{\mathcal{C}}^0 \stackrel{def}{=} \overline{\mathcal{C}}^{\mathcal{L}^0}$, the closure of $\mathcal{C}$ in $\mathcal{L}^0$. It is clear that $\overline{\mathcal{C}}^0$ inherits stability under multiplication by $b\mathcal{F}^+$ from $\mathcal{C}$ so, by Theorem 4.6,

$$\overline{\mathcal{C}}^0 = \mathcal{L}^0(D;\mathcal{F}),$$

where $D = \Lambda(\overline{\mathcal{C}}^0;\mathcal{F})$. It suffices then to prove that $\mathcal{C} = \overline{\mathcal{C}}^0 \cap \mathcal{L}^p$. The inclusion $\mathcal{C} \subset \overline{\mathcal{C}}^0 \cap \mathcal{L}^p$ is obvious. Now let $X \in \overline{\mathcal{C}}^0 \cap \mathcal{L}^p$, so there exists a sequence $Y^n \in \mathcal{C}$ which converges a.s to X. Take a sequence $(\phi_m)_{m\geq 1}$ of continuous functions on $\mathbb{R}$ with compact support such that ϕ_m tends pointwise to 1 as $m \to \infty$, then, by the Bounded

Convergence Theorem, $Y^n_m \overset{def}{=} Y^n f_m(|Y^n|) \in \mathcal{C}$ converges to $Y_m \overset{def}{=} X\,\phi_m(|X|)$ in $\mathcal{L}^p$. So $Y_m \in \mathcal{C}$ and, by letting $m \uparrow \infty$, we obtain the result that $X \in \mathcal{C}$.

In the case where $p = \infty$, given $X \in \overline{\mathcal{C}}^0 \cap \mathcal{L}^\infty$ again take a sequence (Y^n) in $\mathcal{C}$ such that $Y^n \xrightarrow{\text{a.s.}} X$. Then, for any $f \in \mathcal{L}^1(\mathbb{R}^d;\mathcal{F})$ and any m, we have that $f.Y^n\phi_m(|Y^n|) \xrightarrow{\text{a.s.}} f.X\phi_m(|X|)$, and then $f.Y^n\phi_m(|Y^n|) \xrightarrow{\mathcal{L}^1} f.X\phi_m(|X|)$ by the Dominated Convergence Theorem. We conclude that $X\phi_m(|X|) \in \mathcal{C}$ and hence, again letting $m \uparrow \infty$, we obtain the inclusion $\overline{\mathcal{C}}^0 \cap \mathcal{L}^\infty \subset \mathcal{C}$, since $\mathcal{C}$ is closed in $\sigma(\mathcal{L}^\infty, \mathcal{L}^1)$ and hence in $\mathcal{L}^\infty$. □

Lemma 4.9. *Let $\mathcal{C}$ be a closed convex cone in $\mathcal{L}^0(\mathbb{R}^d;\mathcal{F})$, stable under multiplication by (scalar) elements of $b\mathcal{F}^+$, let $1 \le p < \infty$, and $\Lambda = \Lambda(\mathcal{C};\mathcal{F})$ be as defined before, then defining*

$$\mathcal{C}^p = \mathcal{C} \cap \mathcal{L}^p,$$

the polar of $\mathcal{C}^p$ is given by

$$(\mathcal{C}^p)^* = \mathcal{L}^q(\Lambda^*;\mathcal{F}),$$

where q is the conjugate of p and Λ^* is the polar of Λ in $\mathbb{R}^d$.

Proof. This parallels the second half of the proof of Theorem 4.6. □

Definition 4.10. *An adapted sequence of random closed cones in $\mathbb{R}^d$, $(M_t)_{t=0,\dots,T}$, is called a trading decomposition of $\mathcal{A}$ if*

$$\mathcal{A} = \mathcal{L}^0(M_0;\mathcal{F}_0) + \ldots + \mathcal{L}^0(M_T;\mathcal{F}_T).$$

For such a decomposition, set $\mathcal{M}_t = \mathcal{L}^0(M_t;\mathcal{F}_t)$ and, recalling that $\mathbb{M}$ denotes $\mathcal{M}_0 \times \ldots \times \mathcal{M}_T$, set

$$\mathcal{A}_t(\mathbb{M}) \overset{def}{=} \mathcal{M}_0 + \ldots + \mathcal{M}_t.$$

For any trading decomposition $(M_t)_{t=0,\dots,T}$, we define a consistent price process *(with respect to $(M_t)_{t=0,\dots,T}$) to be a martingale, Z, with Z_t taking values in $M^*_t \setminus \{0\}$ for each t. Thus, a consistent price process is nothing but a martingale selection of the set-valued process $(M^*_t \setminus \{0\})$.*

Let $\phi : \Omega \to (0,1]$ be an $\mathcal{F}_T$-measurable positive random variable. We denote by $\mathcal{L}^1_\phi$ the Lebesgue space associated to the norm defined by

$$||f||_{\mathcal{L}^1_\phi} \overset{def}{=} \mathbb{E}\{\phi\,|f|_{\mathbb{R}^d}\}\,.$$

Its dual, denoted by $\mathcal{L}^\infty_\psi$, with $\psi = \frac{1}{\phi}$, is associated with the norm

$$||f||_{\mathcal{L}^\infty_\psi} = \text{ess sup}\{\psi\,|f|_{\mathbb{R}^d}\}\,.$$

Theorem 4.11. *$\bar{\mathcal{A}}$, the closure of $\mathcal{A}$ in $\mathcal{L}^0$, is arbitrage-free iff there is a consistent (for some and then for any trading decomposition $(M_t)_{t=0,\dots,T}$ of $\mathcal{A}$) price process Z, and in this case, for every strictly positive $\mathcal{F}_T$-measurable $\phi : \Omega \to (0,1]$ we may find a consistent price process Z such that $|Z_T| \le c\phi$ for some positive constant c. In particular, taking $\phi = 1$, we can find a* bounded *consistent price process iff $\bar{\mathcal{A}}$ is closed.*

Proof. This follows very closely the proof of Theorem 1.7 (assuming Theorem 2.1) of Schachermayer [11], ignoring references to 'robust' and 'strict'. A sketch proof is as follows: under the assumption that $\bar{\mathcal{A}}$ is arbitrage-free, an exhaustion argument (see [15]), establishes the existence of a strictly positive element, Z, of the polar to

$\bar{A} \cap \mathcal{L}^1_\phi$, whilst Lemma 4.9 and the fact that $\mathcal{M}_t \subset \mathcal{A}$ establishes that $Z_t \overset{def}{=} \mathbb{E}[Z|\mathcal{F}_t] \in \Lambda^*(\mathcal{M}_t; \mathcal{F}_t)$. Conversely, given a consistent Z, we define a frictionless bid-ask process $\hat{\pi}$ by

$$\hat{\pi}^{ij}_t = \frac{Z^j_t}{Z^i_t}.$$

Taking Z^1 as numéraire and observing that $\mathbb{Q}$ given by $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is an EMM for the corresponding discounted asset prices, we see, by applying the fundamental theorem for frictionless trading, that $\hat{\mathcal{A}}$ is closed and arbitrage-free. Now it is clear, since Z is a consistent price process, that $\mathcal{M}_t \subset -\hat{\mathcal{K}}_t = \{X \in \mathcal{L}^0_t : Z_t.X \le 0 \text{ a.s.}\}$ and hence it follows that $\bar{A}$ is arbitrage-free. □

Similar results were proved in Stricker [12], Jouini and Kallal [9], Schachermayer [11] and Grigoriev [4].

We denote $\mathcal{A} \cap \mathcal{L}^1_\phi$ by $\mathcal{A}^\phi$ and by $\mathcal{A}^{*,\psi}$ its polar cone. We denote the consistent price processes with $Z_T \in \mathcal{A}^{*,\psi}$ by $\mathcal{A}^{o,\psi}$, and the sets $\{X : X = Z_t \text{ for some } Z \in \mathcal{A}^{*,\psi}\}$ and $\{X : X = Z_t \text{ for some } Z \in \mathcal{A}^{o,\psi}\}$ by $\mathcal{A}^{*,\psi}_t$ and $\mathcal{A}^{o,\psi}_t$ respectively.

Remark 4.12. *Notice that if $\mathcal{A}^{o,\psi}$ is non-empty, then, identifying martingales with their terminal values, $\mathcal{A}^{*,\psi}$ is the closure in $\mathcal{L}^\infty_\psi$ of $\mathcal{A}^{o,\psi}$. This is a standard argument, following from the fact that if $X \in \mathcal{A}^{*,\psi}$ and $Y \in \mathcal{A}^{o,\psi}$, then $X + \epsilon Y \in \mathcal{A}^{o,\psi}$ for every $\epsilon > 0$. It also follows that $\mathcal{A}^{*,\psi}_t$ is the closure in $\mathcal{L}^\infty_\psi$ of $\mathcal{A}^{o,\psi}_t$.*

Remark 4.13. *Note that in Theorem 4.11, we do not need to assume that $\mathcal{A}$ is decomposed as a sum of $-\mathcal{K}_t$'s, but merely that it admits a trading decomposition.*

Lemma 4.14. *Let $X \in \mathcal{L}^1_\phi$. Then the following assertions are equivalent.*

(1) $X \in \mathcal{C}^\phi_t \overset{def}{=} \mathcal{C}_t \cap \mathcal{L}^1_\phi$.
(2) $X \in \mathcal{L}^1_\phi(\mathcal{F}_t)$ *and* $Z_t \cdot X \le 0$ *a.s. for all* $Z \in \mathcal{A}^o_\psi$.
(3) $\mathbb{E}[(W \cdot X)|\,\mathcal{F}_t] \le 0$ *for all* $W \in \mathcal{L}^{\infty,+}_\psi$ *such that* $\mathbb{E}[W|\mathcal{F}_t] \in \mathcal{A}^{0,\psi}_t$.

Proof. ((1) ⇒ (2))

Clearly, if $X \in \mathcal{C}^\phi_t$, $X \in \mathcal{L}^1_\phi(\mathcal{F}_t)$. Now, for $Z \in \mathcal{A}^o_\psi$ and $f \in b\mathcal{F}^+_t$ we have:

$$\mathbb{E}f(Z_t \cdot X) = \mathbb{E}Z_t \cdot (f\,X) = \mathbb{E}Z_T \cdot (f\,X) \le 0\,,$$

since $Z_T \in \mathcal{A}^*_\psi$ and $f\,X \in \mathcal{A}^\phi$. Since f is arbitrary it follows that $Z_t \cdot X \le 0$ a.s.

((2) ⇒ (1))

Now let $f \in b\mathcal{F}^+_t$ and X satisfy (2). We need only prove that $fX \in \mathcal{A}$.

Let $Z \in \mathcal{A}^o_\psi$ then

$$\mathbb{E}Z_T \cdot (f\,X) = \mathbb{E}Z_t \cdot (f\,X) = \mathbb{E}f(Z_t \cdot X) \le 0\,.$$

Therefore, given $Z \in \mathcal{A}^*_\psi$, by taking a sequence $(Z_n)_{n\ge1}$ in $\mathcal{A}^o_\psi$ converging in $\mathcal{L}^\infty_\psi$ to Z we conclude that $\mathbb{E}Z_T.(f\,X) \le 0$ and hence $fX \in \mathcal{A}^\phi \subset \mathcal{A}$.

((2) ⇒ (3))

We remark that for X satisfying (2) we have, for every $W \in \mathcal{L}^{\infty,+}_\psi$ such that $\mathbb{E}[W|\mathcal{F}_t] \in \mathcal{A}^{o,\psi}_t$ and $f \in b\mathcal{F}^+_t$,

$$\mathbb{E}(f\,(W \cdot X)) = \mathbb{E}(f\,\mathbb{E}(W|\,\mathcal{F}_t) \cdot X) \le 0.$$

Since f is an arbitrary element of $b\mathcal{F}^+_t$,

$$\mathbb{E}[(W \cdot X)|\,\mathcal{F}_t] \le 0.$$

((3) ⇒ (2))

Take an X satisfying (3). We prove first that $X \in \mathcal{L}^1_\phi(\mathcal{F}_t)$.

From (3) we deduce that for every $W \in \mathcal{L}^{\infty,+}_\psi$ we have $\mathbb{E}[(W - \mathbb{E}(W|\,\mathcal{F}_t)) \cdot X] = 0$ since

$$\mathbb{E}[(W - \mathbb{E}(W|\,\mathcal{F}_t))|\,\mathcal{F}_t] = 0 \in \mathcal{A}^{*,\psi}_t.$$

Consequently for every $W \in \mathcal{L}^{\infty,+}_\psi$ we get

$$\mathbb{E}W \cdot (X - \mathbb{E}(X|\,\mathcal{F}_t)) = \mathbb{E}(W - \mathbb{E}(W|\,\mathcal{F}_t)) \cdot X = 0\,.$$

Since W is an arbitrary element of $\mathcal{L}^{\infty,+}_\psi$ we may deduce that $X = \mathbb{E}(X|\,\mathcal{F}_t)$. Let $Z_t \in \mathcal{A}^{o,\psi}_t$, then

$$Z_t \cdot X = \mathbb{E}(Z_t \cdot X|\,\mathcal{F}_t) \leq 0.$$

□

4.2. **Representation.** The following is an easy modification of Theorem 4.1 of Schachermayer [11] and Theorem 4.2 of Delbaen, Kabanov and Valkeila [3]:

Theorem 4.15. *Suppose that $\theta \in \mathcal{L}^0_T$ and $\mathcal{A}$ is closed and arbitrage-free. The following are equivalent:*

(i) *There is a self-financing process η such that*

$$\theta \leq \eta_T,$$

i.e. $\theta \in \mathcal{A}$.

(ii) *For every consistent pricing process Z such that the negative part $(\theta \cdot Z_T)_-$ of the random variable $\theta \cdot Z_T$ is integrable, we have*

$$\mathbb{E}[\theta \cdot Z_T] \leq 0.$$

Proof. The proof is a much simplified version of the proof of Theorem 4.1 of Schachermayer [11]. We give a sketch of the proof.

(i)⇒(ii)

It is easy to check that Remark 2.4 of Schachermayer [11] remains valid if we replace the assumption there that π satisfies the robust no-arbitrage assumption by the assumption that $\mathcal{A}$ is closed and arbitrage-free, or indeed, merely the assumption that there is a consistent price process. With this change, we have the forward implication.

(ii)⇒(i)

Fix θ and suppose that (i) does not hold. Now choose a ϕ such that $\theta \in \mathcal{L}^1_\phi$. Note that $\mathcal{A}^\phi$ is a closed, convex cone in $\mathcal{L}^1_\phi$. Since $\theta \notin \mathcal{A}^\phi$, there exists a separating continuous linear functional $Z \in \mathcal{L}^\infty_\psi$ such that $Z|_{\mathcal{A}^\phi} \leq 0$ and $< Z, \theta >= E[Z.\theta] > 0$. It follows from the first of these properties that $Z_t = E[Z|\mathcal{F}_t]$ is a consistent price process, and then the second shows that (ii) fails. □

We may now consider representation of elements of $\mathcal{A}$:

Theorem 4.16. *Suppose $\theta \in \mathcal{A}^\phi$ and η is an adapted $\mathbb{R}^d$-valued process in $\mathcal{L}^1_\phi$ with $\eta_T = \theta$, and define $\xi = (\xi_0, ..., \xi_T)$ by $\xi_t \stackrel{def}{=} \eta_t - \eta_{t-1}$ with $\eta_{-1} \equiv 0$. Then $\xi \in \prod_0^T \mathcal{C}^\phi_t$ if and only if for all $Z \in \mathcal{A}^o_\psi$, the process M^Z defined by $M^Z_t = \eta_{t-1} \cdot Z_t$, is a supermartingale and $M^Z_T \geq \theta \cdot Z_T$.*

Proof. Let $\xi \in \prod_0^T \mathcal{C}^\phi_t$ and $Z \in \mathcal{A}^o_\psi$. Then

$$\mathbb{E}(M^Z_{t+1}|\,\mathcal{F}_t) = \mathbb{E}(\eta_t \cdot Z_{t+1}|\,\mathcal{F}_t) = \eta_t \cdot Z_t = M^Z_t + \xi_t \cdot Z_t \leq M^Z_t\,,$$

since $\xi_t \in \mathcal{C}_t^\phi$ and $Z \in \mathcal{A}_\psi^o$. Moreover we have

$$M_T^Z = \eta_{T-1} \cdot Z_T = -\xi_T \cdot Z_T + \theta \cdot Z_T \geq \theta \cdot Z_T \,,$$

by the same argument. Conversely, we prove that for every t, $\xi_t \in \mathcal{C}_t^\phi$: by Lemma 4.14 we need to prove that $Z_t \cdot \xi_t \leq 0$ a.s for every $Z \in \mathcal{A}_\psi^o$ which is the case since, for $t \leq T-1$,

$$\xi_t \cdot Z_t = \mathbb{E}(M_{t+1}^Z | \mathcal{F}_t) - M_t^Z \leq 0 \,,$$

and for $t = T$ we have

$$\xi_T \cdot Z_T = \theta \cdot Z_T - M_T^Z \leq 0.$$

□

Problem 4.17. *We would like to show that*

$$\mathcal{A}^\phi = \mathcal{C}_0^\phi + \ldots + \mathcal{C}_T^\phi, \tag{4.4}$$

or just that

$$\mathcal{A}^\phi = \overline{\mathcal{C}_0^\phi + \ldots + \mathcal{C}_T^\phi},$$

(where the closure is in $\mathcal{L}^\phi$) but a proof of either statement eludes us.

We conjecture that (4.4) is true.

Remark 4.18. *We can consider η's only defined for $t \leq T-1$ in the theorem above to obtain the following:*

Corollary 4.19. *Suppose that η is adapted to $(\mathcal{F}_t : 0 \leq t \leq T-1)$. Then $\xi \in \prod_0^{T-1} \mathcal{C}_t^\phi$ if and only if the process M^Z is a supermartingale for all $Z \in D^{0,\psi}$. We may close η on the right by θ if and only if $M_T^Z \geq \theta \cdot Z$ for all $Z \in D^{0,\psi}$.*

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