Minimising the time to shuttle a diffusion between two points

Saul Jacka, Warwick University

UBC
22 July, 2013

Problem: Shuttling in minimal time:

- problem is to control the drift of a diffusion (which reflects at 0 and 1) so as to minimise the time it takes to travel from 0 to 1 and back again.
  Can only choose drift once at each level.
- problem models one arising in simulated tempering (a form of MCMC) – with the level corresponding to temperature in a "heat bath". Idea is that when simulating a draw from a highly multimodal distribution we use a reversible Markov Process to increase the temperature (and thus smear out the modes temporarily) and then move around the statespace, then the temperature MP reduces again to allow us to sample.

Model

- suppose that $dX^\mu_s = \sigma(X^\mu_s)dB_s + \mu(X^\mu_s)ds,$ for each μ.
- letting τ_z denote first hitting time of z by X, the shuttle time $\mathbf{S}$ is defined by $\mathbf{S} = \inf\{t > \tau_1 \ : X_t = 0\}$
  and we seek a function μ to

  Problem 1: (for f a positive cost function) minimise $E_0[\int_0^{\mathbf{S}} f(X_t)dt]$;

  Problem 2: (for α a positive discount function) maximise $E_0[\exp(-\int_0^{\mathbf{S}} \alpha(X_t)dt)]$,

- There is an obvious guess for the first problem: $\mu \equiv 0$! Correct in the case where $f = \sigma = 1$, but *not* in general.
- To formulate problem, we allow ourselves to choose μ dynamically — but only once for each level: i.e., letting X^* denote the running supremum of the controlled process, we set

$$dX_s^\mu = \sigma(X_s^\mu) dB_s + \mu_{\tau_{X_s^*}} ds.$$

To describe solution, we now define the (random) scale function s: so, for each control μ, we define

- $s' = s'_\mu(x) = \exp(-2\int_0^x \frac{\mu_{\tau_z}}{\sigma^2(z)} dz)$, so that $\frac{s''(x)}{s'(x)} = -2\frac{\mu_{\tau_x}}{\sigma^2(x)}$
- and define $s = s_\mu(x) = \int_0^x s'_\mu(u) du$ and $m^f = m^f_\mu(x) = 2\int_0^x \frac{f(u)du}{\sigma^2(u)s'_\mu(u)}$.

Notice that $s'(X_t)$ and $m^f(X_t)$ are adapted.

Theorem 1: Assume that $\frac{\sqrt{f}}{\sigma}$ is C^1 then the optimal payoff process for Problem 1 is given by

$$V_t^\mu \stackrel{def}{=} \text{ess inf}_{\mu^*:\ \mu^*|_{[0,t]}=\mu|_{[0,t]}} E[\int_0^{\mathbf{S}} f(X_u)du|\mathcal{F}_t]$$
$$= \int_0^t f(X_s)ds + \phi_\mu^*(X_t, X_t^*),$$

where

$$\phi^*(x,y) = \begin{cases} 2\Big(\sqrt{s(y)m^f(y)} + \int\limits_y^1 \frac{\sqrt{f(v)}}{\sigma(v)}dv\Big)^2 - 2\int\limits_0^x s'(v)m^f(v)dv & y < 1, \\ 2s(x)m^f(1) - \int\limits_0^x s'(v)m^f(v)dv & y = 1. \end{cases}$$

Proof: Only do static version.

- Lemma: if $a, b > 0$, then

$$\inf_{x>0} ax + b\frac{1}{x} = 2\sqrt{ab}, \text{ attained at } x = \sqrt{\frac{b}{a}}.$$

Now payoff starting at 0 if use control s is
p(s)=$\phi^s(0,1) + \phi^s(1,0)$ where

$$\phi^s(x,y) \stackrel{def}{=} E_x[\int_0^{\tau_y} f(X_t^s)dt].$$

Easy to check that

$$\phi(x,y) = \begin{cases} \int_x^y dvs'(v) \int_0^v \frac{2f(u)}{\sigma^2(u)s'(u)}du & x \le y, \\ \int_y^x dvs'(v) \int_v^1 \frac{2f(u)}{\sigma^2(u)s'(u)}du & x \ge y. \end{cases}$$

so

$$p(s) = \int_0^1 s'(v) \int_0^1 \frac{2f(u)}{\sigma^2(u)s'(u)} du dv = s(1)m^f(1).$$

Now suppose that s is fixed on $[0, y]$ and rewrite p by dividing domain of integration into four rectangles:

$$\begin{aligned} p(y) = \quad & s(y)m^f(y) + \int_y^1 \Big(s(y) \tfrac{2f(u)}{\sigma^2(u)s'(u)} + m^f(y)s'(u) \Big) du \\ & + \int_y^1 \int_y^1 \left(\frac{s'(v)}{s'(u)} \frac{f(u)}{\sigma^2(u)} + \frac{s'(u)}{s'(v)} \frac{f(v)}{\sigma^2(v)} \right) du dv \end{aligned}$$

2nd term in $p(y)$ minimised by setting

$$s'(u) = \sqrt{\frac{s(y)}{m^f(y)}}\sqrt{\frac{2f(u)}{\sigma^2(u)}}$$

and a quick check shows this minimises 3rd term also. Plugging static optimum into dynamic problem gives candidate solution.

The dynamic result follows from Bellman's principle, i.e. V_t is a submartingale under each admissible control and is a martingale under some admissible control.

- in general, the "optimal control" starting at level y has a jump in s' so the "optimal control" will have a singular drift at level y — this corresponds to imposing partial reflection at level y;
- same general form works for discounted time to shuttle.

Suppose that we have run dynamic problem for a while (suboptimally) and have *fixed* μ *at every level above* y. How should we now proceed?

We can plan in advance to use a fixed μ at level x if X reaches this level (from above) before hitting 1. If not, then clearly we can use infinite downward drift (with zero cost).

The corresponding static payoff (starting at level y) is

$$p(y) = \tilde{s}(y)\Big[\tilde{m}^f(y)(1+\int_0^y \frac{s'(z)}{\tilde{s}(s)}dz)+m^f(y)+\int_0^y dz\frac{s'(z)}{\tilde{s}(z)}(\tilde{m}^f(z)-\tilde{m}^f(y))$$

where $\tilde{s}(z) \stackrel{def}{=} s[z,1]$ and $\tilde{m}^f(z) = \int_z^1 f(u)dm(u)$.

Then, if we set $\tilde{m}^f(y) = a$, $\tilde{s}(y) = b$ and $\tilde{s}(0) = c$ and let

$$H(z) = 1 + \ln c - \ln \tilde{s}(z),$$

then

$$p(y) = ab + ab\ln\frac{c}{b} + \frac{2b}{ce}\int_0^y \frac{He^H f}{\sigma^2 H'}(u)du$$

Euler-Lagrange equation is, setting $F(z, H, H') = \frac{He^H f}{\sigma^2 H'}(z)$, $F_H - \frac{\partial}{\partial z} F_{H'} = 0$, which gives

$$(1 + \frac{1}{H})H' = \frac{H''}{H'} + \frac{1}{2}(\ln \frac{\sigma^2}{f})'. \qquad (1)$$

Amazingly, we can solve (1) explicitly to get

$$H' = KHe^H \sqrt{\frac{f}{\sigma^2}}$$

and then,

$$\phi(H(z)) = \phi(1 + \ln \frac{c}{b}) \frac{\beta(z)}{\beta(y)}$$

where

$$\phi(x) = \int_1^x \frac{dt}{te^t} \text{ and } \beta(x) = \int_0^x \sqrt{\frac{f(u)}{\sigma^2(u)}} du.$$

Substituting back in $p(y)$ and optimising in the choice of c gives a value of

$$\hat{p}(y) = \tilde{s}(y)\tilde{m}^f(y) + \frac{2\beta^2(y)}{e}\psi^*(r(y))$$

where

$$r = \frac{e\tilde{s}\tilde{m}^f}{2\beta^2}$$

and ψ^* is the Fenchel conjugate of ψ, i.e.

$$\psi^*(x) = \inf_{t\geq 0} tx + \psi(t),$$

where $\psi(t) = \frac{1}{t\phi(1+t)}$.

To establish that we have the optimal dynamic control, we consider the corresponding payoff

$$\begin{aligned} V(x,i,s) &= E_x \int_0^{\tau_i} f(X_t)dt + \frac{\tilde{s}(x)}{\tilde{s}(i)}\hat{p}(i) \\ &= \int_0^x s'(v)\tilde{m}^f(v)dv + \tilde{s}(x)\tilde{m}^f(i)(1 + \frac{\psi^*(r(i))}{r(i)}). \end{aligned}$$

Bellman's principle then requires us to show thatif I is the running infimum of X then S_t^s, given by

$$S_t^s = \int_0^t f(X_u)du + V(X_t, I_t, s),$$

is a submg for each choice of s and a a mg for some choice of s.

A little more work (!) shows that it is sufficient to show that

$$\inf_{s'(i)} \tilde{s}\tilde{m}^f \frac{\psi^*(r)'}{r} - s'(i)\tilde{m}^f - \tilde{s}\frac{\beta'}{s'}(1 + \frac{\psi^*(r)}{r}) = 0.$$

This is true.

- Open problem: solve dynamic case when μ is fixed on some interval $[x_*, x^*]$.