Warwick Statistics

Pricing American options with stochastic volatility and model uncertainty

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Problem: solving the optimal stopping problem We want to find the payoff (and stopping time) for the following (stochastic volatility) optimal stopping problem:

$$v(x, y, T) = \sup_{\tau \leq T} \mathbb{E}_{x,y}[e^{-q\tau} g(X_\tau)]$$

or

$$v(x, y, T) = \sup_{\tau \leq T} \mathbb{E}_{x,y}[e^{-r\tau} g(e^{r\tau} X_\tau)]$$

where

$$X_t = x + \int_0^t \sigma(X_s) Y_s dB_s,$$

Y is independent of B and either

$$Y_t = y + \int_0^t \eta(Y_s) dW_s + \int_0^t \mu(Y_s) ds$$

or Y is a skip-free Markov chain on E, a countable subset of $(0, \infty)$

Motivated by Jobert and Rogers (2006), where they show the optimal continuation region in the perpetual American put/infinite problem is of the form

$$C = \{(x, y) \in \mathbb{R} \times E : \; x > b(y)\} \tag{1}$$

and give an algorithm to find b.

When E is large, the algorithm can become very intensive if the ordering of the values of $\{b(e) : \; e \in E\}$ is not known.

Our aim is first to show that, under fairly general conditions, $v(x, \cdot, T)$ is increasing and hence if (1) holds then b is decreasing.

We do this by a coupling argument.

Hobson makes very simlar arguments for comparison in the European case.

From now on specialise to stochastic volatility case.

The idea: timechange X to G which solves the sde

$$G_t = x + \int_0^t \sigma(G_s) d\tilde{B}_s$$

using the timechange $\Gamma^y = (A^y)^{-1}$ where $A_t^Y = \int_0^t (Y_s^y)^2 ds$.

Notice that, since Y is skip-free, $y' > y$ implies $A^{y'} \geq A^y$ and $\Gamma^{y'} \leq \Gamma^y$.

It follows that

$$v(x,y,t) = \sup_{\rho \leq A^y_T} \mathbb{E}_x[e^{-q\Gamma^y_\rho} g(G_\rho)] \tag{2}$$

or

$$v(x,y,t) = \sup_{\rho \leq A^y_T} \mathbb{E}_x[e^{-r\Gamma^y_\rho} g(e^{r\Gamma^y_\rho} G_\rho)]. \tag{3}$$

In the first case, increasing y increases the index set and decreases the discount. In the second case we need g decreasing since the argument of g increases when y increases.

The correct coupling argument starts the construction in reverse, by first constructing G and time-changed versions of Y^y and $Y^{y'}$

Recall that Y satisfies

$$Y_t = y + \int_0^t \eta(Y_s)dW_s + \int_0^t \mu(Y_s)ds.$$

Drift rates are hard to estimate, so suppose we only know $\mu_* \leq \mu \leq \mu^*$ and we wish to price the American option. The superhedging price will be

$$V^s(x, y, T) = \sup_{\mu \in \mathcal{M},\, \tau \leq T} \mathbb{E}_{x,y}[e^{-q\tau} g(X_\tau)]$$

where

$$\mathcal{M} = \{\text{adapted processes } \mu \text{ such that } \mu_*(Y_t) \leq m_t \leq \mu^*(Y_t)\}.$$

Conversely, the client's price will be

$$V^b(x, y, T) = \inf_{\mu \in \mathcal{M}} \sup_{\tau \leq T} \mathbb{E}_{x,y}[e^{-q\tau} g(X_\tau)]$$

Point is that as soon as we know that V is increasing in y the candidate drift control is obvious: choose maximum drift to achieve supremum and minimal drift for infimum!

Sketch proof (superhedging case): look at HJB equation for stochastic control + optimal stopping problem

$$\max\Big(\sup_{m\in[\mu_*,\mu^*]} [\frac{1}{2}y^2\sigma^2(x)V^s_{xx} + \frac{1}{2}\eta^2(y)V^s_{yy} + mV^s_y - V^s_t - qV^s], \quad g - V^s\Big) = 0 \tag{4}$$

If we take V^s to be the corresponding value of v with $\mu = \mu^*$ then, since v is increasing in y, $V^s_y \geq 0$ and so the sup in (4) is attained at $m = \mu^*(y)$.

So, since v solves the optimal stopping problem, $e^{-qt}v(X_t, Y_t, T-t)$ is a martingale on the continuation region and equals g on the stopping region.

It follows that $\frac{1}{2}y^2\sigma^2(x)V^s_{xx} + \frac{1}{2}\eta^2(y)V^s_{yy} + \mu^* V^s_y - V^s_t - qV^s = 0$ on the continuation region and $g = V^s$ on the stopping region so that V^s satisfies the HJB equation.

Now, *what happens if we are only 95% certain that* μ *lies in the interval* $[\mu_*, \mu^*]$*?*

If we assume that the payoff is zero when this constraint is broken and denote the stopping time at which the constraint is broken is σ, then the Lagrangian for the superhedging/pricing problem is

$$V(x,y,T) = \sup_{m\in\mathcal{M}} \sup_{\tau\leq T} \sup_{\sigma} \mathbb{E}_{x,y}[e^{-q\tau} g(X_\tau) 1_{\tau<\sigma} + \lambda 1_{\sigma\leq\tau}].$$

It's (fairly) obvious that this means that

$$V^s(x,y,T) = \sup_{m\in\mathcal{M}} \sup_{\tau\leq T} \mathbb{E}_{x,y}[\max(e^{-q\tau} g(X_\tau), \lambda)].$$

Similarly, get

$$V^b(x, y, T) = \inf_{m \in \mathcal{M}} \sup_{\tau \leq T} \mathbb{E}_{x,y}[\min(e^{-q\tau} g(X_\tau), \lambda)].$$

In either case, presence of max or min does not affect monotonicty argument for V and hence for optimal choice of m. Continuity of V in λ allows calibration in λ to obtain the appropriate constrained optimum.