The Hausdorff dimension of some snowflake-like recursive constructions

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Abstract

Fractal subsets of $\mathbb{R}^n$ with highly regular structure are often constructed as a limit of a recursive procedure based on contractive maps.

The Hausdorff dimension of recursively constructed fractals is relatively easy to find when the contractive maps associated with each recursive step satisfy the Open Set Condition (OSC). We present a class of random recursive constructions which resemble snowflake structures and which break the OSC. We calculate the associated Hausdorff dimension and conjecture that an a.s. deterministic exact Hausdorff function does not exist.

1 Introduction

A recursive construction is defined as the limit of a sequence of sets, each of which is derived from its predecessor in the sequence by a simple step or rule. When the recursive step is deterministic, the resulting construction will often be self-similar, such as the Sierpinski gasket or carpet. The recursive step may, however, contain some randomisation. If the randomisation is conducted in a certain way, the resulting construction can be called statistically self-similar (see [12] for details).

Mandelbrot [16] pointed out the use of such sets as models of naturally occurring phenomena, such as snowflake crystals.

Restricting our attention to $\mathbb{R}^n$ from now on, in its simplest form the recursive step may be formulated in terms of a set of contractive maps $\{\phi_i\}_{i\in I}$. If we begin the recursion with a non-empty, compact set K we can write each step in the construction as

$$K_j = \bigcup_{\underline{i}\in\{1,\cdots,n\}^j} \phi_{\underline{i}}(K) \tag{1}$$

where $\phi_{(i_1,i_2,\cdots,i_j)} = \phi_{i_j}\circ\phi_{i_{j-1}}\circ\cdots\circ\phi_{i_1}$. Hence the limiting set can be written as

$$K = \bigcap_{j=1}^{\infty} K_j$$

The OSC is of great use when calculating the Hausdorff dimension of the limiting set, see [7] for examples.

The Hausdorff dimension of deterministic OSC recursive constructions was calculated by Hutchinson [14] and a good description of the exact Hausdorff functions was obtained by, for example, Graf in [12].

In Mauldin and Williams [18] (also in [13]) the recursive step is randomised in a heterogeneous way. That is a random (according to a law μ say) i.i.d. selection of contractive maps (satisfying the OSC) is applied to each scaled copy of the original set in every recursive step. Here it is possible to ensure the a.s. existence not only of the Hausdorff dimension α but also of a (deterministic) exact Hausdorff function h for the resulting set F where α and h satisfy

$$\int \sum_i \mathrm{Lip}(\phi_i)^\alpha d\mu = 1$$

and

$$h(t) = t^\alpha (\log|\log(t)|)^\theta$$

with $\theta \leq 1 - \frac{\alpha}{n} < 1$.

In Sec. 4 we introduce a collection of homogeneous random recursive models for snowflake crystals which breaks the OSC. Homogeneous refers to the application of the *same* random selection of contractive maps to each copy of the original set at each recursive step. Recursive constructions of a related homogeneous type were studied by Bedford in [3].

No general theory exists for finding the Hausdorff dimension of non-OSC constructions. We use special properties of the collection to find a random exact Hausdorff function. We can then use this function to calculate the Hausdorff dimension.

In Sec. 5 a condition is given (the *rotation* or R-condition) which splits the class into those models for which we can find the Hausdorff dimension explicitly and those for which we cannot. Under the condition, the Hausdorff dimension is found to be

$$\mathbb{E}(B_1) + 1$$

where B_i is a random variable associated with the number of contractive maps in the selection for stage i.

We then use a triangle covering approach to refine the proof of the above result to apply to those models which break the condition. The Hausdorff dimension of these models is given by the largest characteristic exponent of a random matrix product.

In sec. 7 we comment that the given results are in some sense strong, that is they do not depend on any nice properties of the random sequence of recursive steps (independence, stationarity, etc) but merely on the sequence itself. We also comment on other similar models which break the OSC and conjecture that no similar results are possible.

2 Hausdorff dimension and exact Hausdorff function

Let $U \subseteq \mathbb{R}^n$. If $\{U_i\}_{i\geq 1}$ is a countable (or finite) collection of sets then we say that $\{U_i\}_{i\geq 1}$ is a δ-cover of U if

$$U \subset \bigcup_i U_i \quad \text{and} \quad |U_i| \leq \delta \quad \text{for all } i \geq 1$$

where $|U_i| = \text{diam}(U_i)$ using the usual metric on $\mathbb{R}^n$. Let $s > 0$ and define

$$H^s_\delta(U) = \inf\{\sum_i |U_i|^s \ : \ \{U_i\} \text{ is a } \delta\text{-cover of } F\}.$$

Now set

$$H^s(U) = \lim_{\delta\to 0} H^s_\delta(U).$$

It is easily shown that the limit exists and also that $H^s(\cdot)$ is indeed a measure (Hausdorff s-measure).

For a set $U \subset \mathbb{R}^n$ define the *Hausdorff Dimension* to be the unique number $dim_H(U)$ such that

$$H^s(U) = \begin{cases} \infty & s < dim_H(U) \\ 0 & s > dim_H(U) \end{cases}$$

The definition of Hausdorff measure above can be generalized to include any positive increasing function $f : [0,\infty) \to [0,\infty)$, (rather than just powers) of the diameters of the covering sets with the property that $f(x) \to 0$ as $x \to 0$;

$$H^f_\delta(U) = \inf\{\sum_i f(|U_i|) \quad : \ \{U_i\} \text{ is a } \delta\text{-cover of } U\}$$

The analogue of the dimension here is an *exact Hausdorff function*, a function h such that

$$0 < H^h(U) < \infty$$

The theory of these measures (which are in general non σ-finite) has proceeded largely due to the work of A. S. Besicovitch and his students. A fine technical work on the theory of Hausdorff measures is Rogers [23].

The contractive maps most commonly used in the formulation of recursive constructions are called similitudes, essentially just scaled isometries.

(Similitude) A function $S : X \to X$ on a metric space (X, d) is called a *similitude* if $d(S(x), S(y)) = rd(x, y)$ for all $x, y \in X$ and for some fixed r.

3 Random Recursions

In this sec. we will outline the formulation of a homogeneous random recursion.

Let $\mathcal{S} = (S_i)_{i=1}^N$ be a set of similitudes on $\mathbb{R}^M$, let μ be a probability measure on the power set of $\{1, \ldots, N\}$ and let K be a non-empty, compact subset of $\mathbb{R}^M$ with a non-empty interior. We specify the construction as follows; $K_0 = K$, now we choose an subset $T_1 \subseteq \{1, \ldots, N\}$ according to μ and set

$$K_1 = \bigcup_{j \in T_1} S_j(K_0)$$

Now choose another set $T_2 \subseteq \{1, \ldots, N\}$ according to μ and independently of T_1, we can now write

$$K_2 = \bigcup_{l \in T_2} \bigcup_{j \in T_1} S_l(S_j(K_0))$$

Continuing the recursion yields a sequence of sets $(K_n)_{n \geq 1}$ and we may define our recursive construction to be F where

$$F = \limsup K_n = \bigcap_{n \geq 1} \bigcup_{j \geq n} K_j.$$

4 A class of random recursive constructions

Let H be a regular hexagon of radius 1, let D be its 'diameter' and $v_1, \ldots, v_6$ its vertices as shown in Fig 1. Define the similitudes $\phi_1, \ldots, \phi_6$ by requiring that ϕ_i maps H to H_i as shown in Fig 1.

Figure 1 here

We will call a random recursive construction procedure a *GH* (general hexagonal) procedure if (in the notation of the previous Sec.);

1. K is the unit hexagon H centred at the origin

2. the set of similitudes is $\{\phi_1, \ldots, \phi_6\}$:

3. the selection law satisfies $\mu(\{1,2\} \subseteq T_i) = 1$.

Figure 2 *here*

An example of a GH construction appears in Fig. 2. The associated selection law μ is given by

$$\mu(\{1,2\}) = 1 - \mu(\{1,2,3,6\}) = p$$

for $0 < p < 1$. We refer to this particular model as SH (special hexagonal). Despite the simplicity of this model, simulations produce snowflake-like figures of great variety and complexity. Some examples are shown in Fig. 3.

Figure 3 *here*

In fact, simulations from any model in the class GH, even one with asymmetric replacements, appear to produce snowflake-like figures. Fig. 4 contains simulations from different models within the GH class.

Figure 4 *here*

Let $F(H)$ be the limiting set obtained using a particular GH procedure as given above and let $F(D)$ be the closure of the limiting set obtained using the identical procedure applied to an initial set $K = D$, the diameter of the hexagon H.

Lemma 4.1 (Equivalence lemma)

$$F(H) = F(D)$$

Proof

Clearly $D \subset H$ and $F(H)$ is closed hence

$$F(D) \subseteq F(H).$$

Conversely if $x \in F(H)$ take $i_1, \cdots, i_n$ such that $i_j \in T_j$ for $1 \leq j \leq n$ *and* $x \in \phi_{i_n} \circ \phi_{i_n - 1} \cdots \phi_{i_1}(H)$ Now $d(x, \phi_{i_n} \circ \phi_{i_n - 1} \cdots \phi_{i_1}(0)) \leq 2^{-n}$ and the origin $0 \in D$. Since x is arbitrary in $F(H)$ and $F(D)$ is closed then $x \in F(D)$ and hence $F(D) = F(H)$.

For convenience we will use the initial set D in all subsequent constructions.

5 A covering using triangles

The most important consideration in calculating the Hausdorff dimension of a recursive construction is keeping track of the number of scaled copies of the original set (D) that are present in any particular stage. This is simple when the OSC holds as no elements in any stage overlap. In a general GH model there is non-trivial overlapping of the form shown in Fig. 5 for the SH model.

Figure 5 *here*

However, with this particular model the largest number of copies of D (lines) which may overlap (coincide) is 2. In fact there is a condition which identifies this property in GH models.

A GH construction is said to satisfy the *R-condition* if no replacement used in the construction overlaps with its image under a rotation of π about its centre, except at the line being replaced.

Under the R-condition the maximum overlap (coincidence) of elements (lines) is 2, without the R-condition the maximum overlap is unbounded.

We will now introduce a method to keep track of the total number of elements (lines) present at any particular stage based on covering with triangles.

We will use triangles of side length 2^{-n} to cover D_n (the nth stage in the recursive construction with initial set D), in which the elements are themselves of length 2^{-n}. Note the following facts

1. elements of D_n will fill the edges of each covering triangle
2. all future branching from the elements in a triangle will clearly not be contained within the triangle, but
3. the future branching *within* a particular triangle is dependent on the composition of the edges of that triangle and no other

The possible arrangements of elements of D_n on the edges of an appropriately sized triangle (up to rotation and reflection) are shown in Fig. 6.

Figure 6 here

Note that at each stage each covering triangle generates at most 4 covering triangles of the next stage. The types of the new triangles depend only on the previous triangle type and the form of the replacement. Note that we ignore any ‘empty’ triangles that may be generated.

Also, in any particular GH procedure certain configurations of line segments may never occur. For instance in the SH example the triangles numbered 20 through to 23 are never needed to cover a part of any structure which may occur. In such a case, to simplify the notation, we assume the list of triangle types and the ‘transition’ matrices below are *trimmed* so that redundent types are omitted.

We may now analyse each D_n, calculating for example the number of elements or the number of directed elements (counting 2 or more for an overlapped element), just by counting the numbers of each type of covering triangle at each stage. These are dependent only on the initial distribution of triangle types[1] and the sequence of choices of contractive maps (be it deterministic or random).

Figure 7 here

If we let t_n be the column vector of counts of each (non-redundant) triangle-type needed to cover D_n in the method described above then

$$t_n = M_n t_0$$

where $M_n = P^{T_n} P^{T_{n-1}} \cdots P^{T_1}$ and $P^{T_i}_{k,l}$ is the number of triangles of type k generated from a single triangle of type l by the application of the similitudes

[1]For a GH procedure the initial distribution is given by the vector $(1, 1, 0, \cdots, 0)$

indexed by T_i.

Using this cover, the number of triangles required at stage n is $1^t M_n t_0$.

For example in the SH model, the number of lines, counting each overlapped line as two, grows exactly as 2^{n+X_n} where X_n is the number of times the 4-line replacement is used up to stage n. If we let π be the fixed vector listing the number of edges in each triangle type (counting multiplicity),

$$\pi^t = (1, 1, 2, 1, 2, 2, 2, 3, 3, 3, 3, 4, 4, 5, 5, 3, 3, 6, 4, 3, 3, 4, 4)$$

it is clear that $\pi^t M_n t_0 = 2^{n+X_n} \pi^t t_0$ and that

$$\frac{\pi^t M_n t_0}{6} \leq 1^t M_n t_0 \leq \pi^t M_n t_0$$

Hence the number of triangles of diameter 2^{-n} required to cover D_n, and hence the number of lines contained in D_n, grows as $2^{(2-p)n+o(n)}$. It follows immediately that $(2-p)$ is an upper bound for the Hausdorff dimension any realisation of the SH model.

Figure 8 here

6 Exact Hausdorff function for SH

Theorem 6.1 *If F is a recursive construction formed via the SH model and we denote by X_n the number of times the set $\{1, 2, 3, 6\}$ is chosen up to stage n then the function ϕ, given by*

$$\phi(2^{-n}) = 2^{-(n+X_n)}$$

($\phi(x)$ being obtained by interpolation whenever $2^{-n} < x < 2^{-n+1}$), is an exact Hausdorff function.

Proof

Lower bound

Consider a set $U \in \mathbb{R}^2$ of diameter d with

$$2^{-(n+1)} < d \leq 2^{-n}$$

By considering all the hexagons of diameter $2^{-(n+1)}$ which can intersect U, we can see (Fig. 8) that the most line segments of length $2^{-(n+2)}$ in D_{n+2} which can be killed (i.e. they are covered and all their 'decendents' are covered) by U is 154.

Now suppose that $(U_i)_{i \in I}$ is a δ-cover of F with $2^{-(N+1)} < \delta \leq 2^{-N}$ and let

$$n_0 = \min\{n : \exists i \in I \ \text{ with } \ 2^{-(n+1)} < \operatorname{diam}(U_i) \leq 2^{-n}\}.$$

Now define n_i recursively by

$$n_{i+1} = \min\{n > n_i : \exists i \in I \ \text{ with } \ 2^{-(n+1)} < \operatorname{diam}(U_i) \leq 2^{-n}\}.$$

Let π_0 be the proportion of line segments of length $2^{-(n_0+2)}$ in $D_{(n_0+2)}$ killed by sets in the cover with diameters in the range $(2^{-(n_0+1)}, 2^{-n_0}]$, then it follows that the number of such sets in the cover is at least

$$\frac{2^{(n_0+2)+X_{(n_0+2)}}\pi_0}{154}$$

Define π_1 as the proportion of those line segments of length $2^{-(n_1+2)}$ in $D_{(n_1+2)}$ which have not already been killed by covering sets of diameter $> 2^{-n_1}$ but which are killed by sets of diameter in the range $(2^{-(n_1+1)}, 2^{-n_1}]$

The remaining number of lines not covered at stage n_1 is

$$(1-\pi_0)2^{(n_0+2)+X_{n_0+2}}2^{(n_1-n_0)+(X_{n_1+2}-X_{n_0+2})}.$$

Using the same arguments, the number of covering sets in the range $(2^{-(n_1+1)}, 2^{-n_1}]$ is at least

$$\frac{2^{(n_1+2)+X_{n_1+2}}\tilde{\pi_1}}{154}$$

where $\tilde{\pi_1} = (1-\pi_0)\pi_1$ and $0 \leq \pi_1 \leq 1$. Continuing for $n_2 < n_3 < \cdots$ we obtain

$$\begin{aligned} H_\phi^\delta(F) \quad \geq \quad & \liminf_{N\to\infty} \inf_{\tilde{\pi}}((\tfrac{2^{(n_0+2)+X_{n_0+2}}}{154}\tilde{\pi_0}\phi(2^{-(n_0+1)}))+ \\ & (\tfrac{2^{(n_1+2)+X_{n_1+2}}}{154}\tilde{\pi_1}\phi(2^{-(n_1+1)})) + \cdots) \end{aligned}$$

where the infimum is taken over $\tilde{\pi} = (\tilde{\pi_0}, \tilde{\pi_1}, \tilde{\pi_2}, \cdots)$ such that $\Sigma_i\tilde{\pi_i} = 1$ and each $\tilde{\pi_j} \geq 0$.

Clearly, for any $\epsilon > 0$ there will be a $k \in \mathbb{N}$ such that

$$H_\phi^\delta(F) \geq \frac{2^{(n_k+2)+X_{n_k+2}}}{154}\phi(2^{-(n_k+1)}) - \epsilon$$

Since n_k is bounded below by N which tends to ∞ as $\delta \to 0$,

$$H_\phi(F) \geq \liminf_{n\to\infty} 2^{n+X_n}\phi(2^{-n})/154$$

Upper bound

By considering the simple coverings of F given by placing a regular hexagon of diameter 2^{-n} over each line segment at stage n, it is clear that

$$H_\phi(F) \leq \liminf_{n\to\infty} \; 2^{n+X_n}\phi(2^{-n})$$

Hence to identify an exact Hausdorff function ϕ it is enough to ensure that

$$0 < \liminf_{n\to\infty} 2^{n+X_n}\phi(2^{-n}) < \infty \tag{2}$$

Now notice that this may be guaranteed by setting

$$\phi(2^{-n}) = 2^{-(n+X_n)}$$

and interpolating for $\phi(x)$ when $2^{-n} < x < 2^{-n+1}$. This completes the proof of Theorem 6.1. □

We can now obtain $dim_H(F)$ via the exact Hausdorff function ϕ in Theorem 6.1, using the facts that,

$$\frac{x^r}{\phi(x)} \to \infty \;\text{ as }\; x \to 0 \Rightarrow dim_H(F) \geq r$$

and

$$\frac{x^r}{\phi(x)} \to 0 \;\text{ as }\; x \to 0 \Rightarrow dim_H(F) \leq r$$

Corollary 6.2

$$dim_H(F) = 2 - p$$

Proof

Let $r < 2 - p$. Using the above remark we simply require that

$$\frac{(2^{-n})^r}{\phi(2^{-n})} \to \infty.$$

but

$$\frac{(2^{-n})^r}{\phi(2^{-n})} = 2^{-(n(r-1)-X_n)}$$

and the Strong Law of Large Numbers states that $\frac{X_n}{n} \to 1 - p$ with probability 1. Hence

$$2^{-(n(r-1)-X_n)} \sim 2^{-n(r-1-1+p)} = 2^{n(r-(2-p))} \to \infty$$

as $n \to \infty$. Hence r is a lower bound for dim_H for any $r < 2 - p$. The reverse follows similarly.

Hausdorff Dimension for GH

Theorem 6.3 *If F is a set constructed by a GH scheme satisfying the R-condition then an exact Hausdorff function, h, is given by*

$$h(2^{-n}) = R_n^{-1}$$

and $h(t)$ is calculated by linear interpolation for $2^{-n} \geq t > 2^{-(n+1)}$ where R_n is the number of distinct lines at stage n of the construction.

Proof

We simply mimic the proof of Theorem 6.1.

Theorem 6.4 *If F is a set constructed by a GH scheme satisfying the R-condition*

$$d_H(F) = E(B) + 1$$

where B is a random variable whose distribution is given by

$$B = \begin{cases} 0 & w.p. \;\; p_2 \\ \log(3)/\log(2) - 1 & w.p. \;\; p_3 \\ 1 & w.p. \;\; p_4 \end{cases}$$

and p_2, p_3, p_4 are the probabilities of choosing replacements with $2, 3$ or 4 lines respectively.

Proof

Notice that we can write

$$R_n = 2^{n+\sum_{i=1}^{n} B_i}$$

where the B_i are i.i.d. random variables each distributed as B. The result now follows in a similar manner to corollary 6.2.

6.1 Outside the R-Condition

When a GH construction breaks the R-condition, it is easy to see that the argument for the exact Hausdorff function is still valid with the number of lines at stage n, (2^{n+X_n}), replaced by R_n (just defined to be the random number of distinct line segments, overlapped or not, at each stage n).

Using the triangle covering we can express R_n simply as

$$\pi M_n x$$

where M_n is a random product of the branching matrices associated with each replacement. To assure the existence of a power law governing the limiting behaviour of this product we will employ a version of Oseledec's theorem (first seen in [22]) as given in an excellent review of the topic in [15].

Theorem 6.5 (Theorem 3.1 of [15]) *Consider a random stationary sequence* $\{A_n \ : \ n \geq 0\}$ *of* $d \times d$ *real-valued matrices (on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*) and form the product*

$$A^{(n)} = A_{n-1} \cdots A_0.$$

There exist positive constants $\mu_1 > \mu_2 > \cdots > \mu_r$ *and, for each* $\omega \in \Omega$*, an ordered decomposition of* $\mathbb{R}^d$ *into subspaces of strictly decreasing dimension*

$$\mathbb{R}^d = V_\omega^1 \supset V_\omega^2 \supset \cdots V_\omega^r = \{0\}$$

such that for $1 \leq j \leq r-1$ *and* $v \in V_\omega^j \setminus V_\omega^{j+1}$ *we have*

$$\lim_{n \to \infty} \frac{1}{n} \log ||A_\omega^{(n)} v|| = \mu_j$$

Theorem 6.6 *If F is formed using a GH procedure then $dim_H(F)$ is almost surely constant and is given by $\log_2 \mu_1$ where μ_1 is defined as in the previous theorem.*

Proof

Using the identities from Sec. 5 and Theorem 6.5 it is clear that R_n grows as $2^{n \log_2 \mu_1 + o(n)}$ and so again mimicking the proofs of 6.1 and 6.2, it is clear the Hausdorff dimension can be derived from one of the exponents μ_i of Theorem 6.5. To show the correct exponent is μ_1 we must show that the starting vector of triangle types t_0 ($= (1, 1, 0, \cdots, 0)$ for GH constructions as given in Sec. 4) is in $V_\omega^1 \setminus V_\omega^2$.
As the decomposition of $\mathbb{R}^d$ in theorem 6.5 is into subspaces there must exist $i \in \mathbb{N}$ such that the vector corresponding to one triangle type i, $b_i = (0, \cdots, 0, 1, 0, \cdots, 0) \in V_\omega^1 \setminus V_\omega^2$.
However, each triangle type can be considered as the superposition of up to 3 rotated copies of a type 1 and 3 rotated copies of type 2, hence

$$6\pi^t M_n t_0 \geq \pi^t M_n b_i$$

for any i and so $t_0 \in V_\omega^1 \setminus V_\omega^2$.

We can generalise the argument further since the exact Hausdorff function is given by $h(2^{-n}) = R_n^{-1}$ regardless of the scheme employed for choosing the form of the recursive step.

In particular we can calculate the Hausdorff dimension of any GH model (explicitly under the R-condition) with a deterministic recursive step or a deterministically repeating sequence of recursive steps.

7 Further comments

The models in GH are based on the radii of a regular hexagon. It is natural to ask whether it is possible to extend the earlier results to models based on the regular $2n$-agon for $n > 3$.

A simple argument show that this is not possible for constructions based on $4n$-agons with $n \geq 2$. In these cases no fixed upper bound exists for the number of stage n elements a set of diameter close to 2^{-n} can cover. We conjecture that the same problem persists for any construction based on a $2n$-agon with $n > 3$.

It is thought that because of the simple form of the matrix product M_n used to calculate R_n in Theorem 6.6, we may be able to evaluate (or closely bound) the exponent μ_1.

We also conjecture that no a.s. deterministic exact Hausdorff function exists due to the homogeneous randomisation in GH models. This method of randomisation does not produce enough 'averaging' in the long run to be able to apply the same arguments as those in [13].

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