Auxiliary Particle Methods

Perspectives & Applications

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Introduction

Outline

- Background: Particle Filters
  - Hidden Markov Models & Filtering
  - Particle Filters & Sequential Importance Resampling
  - Auxiliary Particle Filters
- Interpretation
  - Auxiliary Particle Filters are SIR Algorithms
  - Theoretical Considerations
  - Practical Implications
- Applications
  - Auxiliary SMC Samplers
  - The Probability Hypothesis Density
  - On Stratification
- Conclusions

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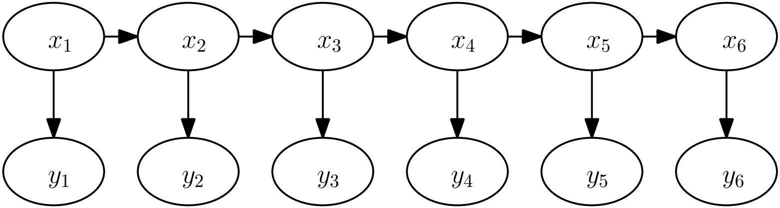
Hidden Markov Models



- X_n is a $\mathcal{X}$-valued Markov Chain with transition density f:

$$X_n | \{X_{n-1} = x_{n-1}\} \sim f(\cdot | x_{n-1})$$

- Y_n is a $\mathcal{Y}$-valued stochastic process:

$$Y_n | \{X_n = x_n\} \sim g(\cdot | x_n)$$

Optimal Filtering

- The *filtering distribution* may be expressed recursively as:

$$p(x_n|y_{1:n-1}) = \int p(x_{n-1}|y_{1:n-1}) f(x_n|x_{n-1}) dx_{n-1} \quad \text{Prediction}$$

$$p(x_n|y_{1:n}) = \frac{p(x_n|y_{1:n-1}) g(y_n|x_n)}{\int p(x_n|y_{1:n-1}) g(y_n|x_n) dx_n} \quad \text{Update.}$$

- The *smoothing distribution* may be expressed recursively as:

$$p(x_{1:n}|y_{1:n-1}) = p(x_{1:n-1}|y_{1:n-1}) f_n(x_n|x_{n-1}) \quad \text{Prediction}$$

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Particle Filters: Sequential Importance Resampling

At time $n = 1$:

- Sample $X^i_{1,1} \sim q_1(\cdot)$.
- Weight
$$W^i_1 \propto f(X^i_{1,1})g(y_1|X^i_{1,1})/q_1(X^i_{1,1})$$
- Resample.

At times $n > 1$, iterate:

- Sample $X^i_{n,n} \sim q_n(\cdot|X^i_{n-1,n-1})$. Set $X^i_{n,1:n-1} = X^i_{n-1}$.
- Weight
$$W^i_n \propto \frac{f(X^i_{n,n}|X^i_{n,n-1})g(X^i_{n,n}|y_n)}{q_n(X_{n,n}|X_{n,1:n-1})}$$
- Resample.

Auxiliary [v] Particle Filters (Pitt & Shephard '99)

If we have access to the next observation before resampling, we could use this structure:

- Pre-weight every particle with $\lambda_n^{(i)} \propto \hat{p}(y_n|X_{n-1}^{(i)})$.
- Propose new states, from the mixture distribution

$$\sum_{i=1}^{N} \lambda_n^{(i)} q(\cdot|X_{n-1}^{(i)}) / \sum_{i=1}^{N} \lambda_n^{(i)}.$$

- Weight samples, correcting for the pre-weighting.

$$W_n^i \propto \frac{f(X_{n,n}^i|X_{n,n-1}^i)g(X_{n,n}^i|y_n)}{\lambda_n^i q_n(X_{n,n}|X_{n,1:n-1})}$$

- Resample particle set.

Some Well Known Refinements

We can tidy things up a bit:

1. The auxiliary variable step is equivalent to multinomial resampling.
2. So, there's no need to resample before the pre-weighting.

Now we have:

- Pre-weight every particle with $\lambda_n^{(i)} \propto \hat{p}(y_n|X_{n-1}^{(i)})$.
- Resample
- Propose new states
- Weight samples, correcting for the pre-weighting.

$$W_n^i \propto \frac{f(X_{n,n}^i|X_{n,n-1}^i)g(X_{n,n}^i|y_n)}{\lambda_n^i q_n(X_{n,n}|X_{n,1:n-1})}$$

Interpretation

General SIR Algorithms

The SIR algorithm can be used somewhat more generally.
Given $\{\pi_n\}$ defined on $E_n = \mathcal{X}^n$:

- Sample $X_{n,n}^i \sim q_n(\cdot|X_{n-1}^i)$. Set $X_{n,1:n-1}^i = X_{n-1}^i$.
- Weight

$$W_n^i \propto \frac{\pi_n(X_n^i)}{\pi_{n-1}(X_{n,1:n-1})q_n(X_{n,n}|X_{n,1:n-1})}$$

- Resample.

An Interpretation of the APF

If we move the first step at time $n+1$ to the last at time n, we get:

- Resample
- Propose new states
- Weight samples, correcting earlier pre-weighting.
- Pre-weight every particle with $\lambda_{n+1}^{(i)} \propto \hat{p}(y_{n+1}|X_n^{(i)})$.

which is an SIR algorithm targetting the sequence of distributions

$$\eta_n(x_n) \propto p(x_{1:n}|y_{1:n})\hat{p}(y_{n+1}|x_n)$$

which allows estimation under the actually interesting distribution via importance sampling.

Theoretical Considerations

- Direct analysis of the APF is largely unnecessary.
- Results can be obtained by considering the associated SIR algorithm.
- SIR has a (discrete time) Feynman-Kac interpretation.

For example. . .

Proposition. Under standard regularity conditions

$$\sqrt{N}\left(\widehat{\varphi}_{n,APF}^{N}-\overline{\varphi}_{n}\right)\rightarrow\mathcal{N}\left(0,\sigma_{n}^{2}\left(\varphi_{n}\right)\right)$$

where,

$$\sigma_{1}^{2}\left(\varphi_{1}\right)=\int\frac{p\left(\left.x_{1}\right|y_{1}\right)^{2}}{q_{1}\left(x_{1}\right)}\left(\varphi_{1}\left(x_{1}\right)-\overline{\varphi}_{1}\right)^{2}dx_{1}$$

$$\sigma_{n}^{2}(\varphi_{n})=\int\frac{p(x_{1}|y_{1:n})^{2}}{q_{1}(x_{1})}\left(\int\varphi_{n}(x_{1:n})p(x_{2:n}|y_{2:n},x_{1})dx_{2:n}-\bar{\varphi}_{n}\right)^{2}dx_{1}$$
$$+\sum_{k=2}^{t-1}\int\frac{p(x_{1:k}|y_{1:n})^{2}}{\widehat{\mathbf{p}}(\mathbf{x_{1:k-1}}|\mathbf{y_{1:k}})q_{k}(x_{k}|x_{k-1})}\left(\int\varphi_{n}(x_{1:n})p(x_{k+1:n}|y_{k+1:n},x_{k})dx_{k+1:n}-\bar{\varphi}_{n}\right)^{2}dx_{1:k}$$
$$+\int\frac{p(x_{1:n}|y_{1:n})^{2}}{\widehat{\mathbf{p}}(\mathbf{x_{1:n-1}}|\mathbf{y_{1:n}})q_{n}(x_{n}|x_{n-1})}\left(\varphi_{n}(x_{1:n})-\bar{\varphi}_{n}\right)^{2}dx_{1:n}.$$

Practical Implications

- It means we're doing importance sampling.
- Choosing $\widehat{p}(y_n|x_{n-1}) = p\left(y_n|x_n = \mathbb{E}\left[X_n|x_{n-1}\right]\right)$ is dangerous.
- A safer choice would be ensure that

$$\sup_{x_{n-1},x_n} \frac{g(y_n|x_n)f(x_n|x_{n-1})}{\widehat{p}(y_n|x_{n-1})q(x_n|x_{n-1})} < \infty$$

- Using APF doesn't *ensure* superior performance.

A Contrived Illustration

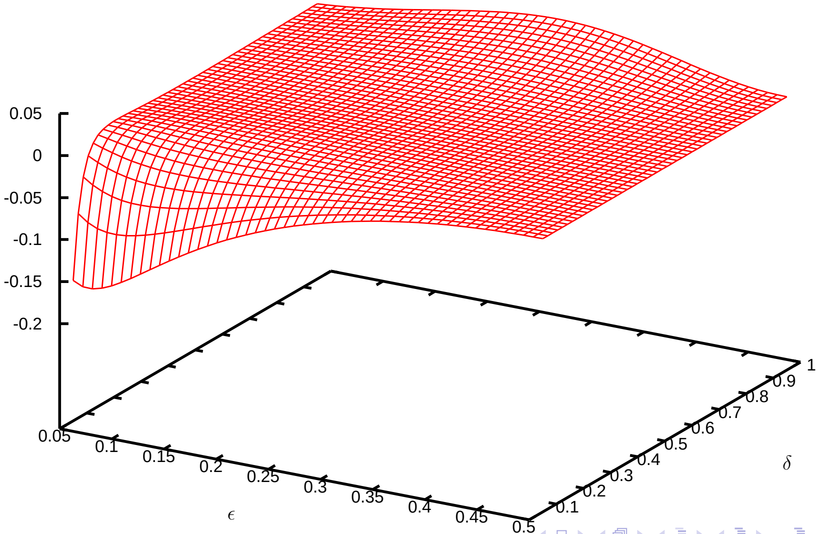
Consider the following binary state-space model with common state and observation spaces:

$$\begin{aligned} \mathcal{X} &= \{0,1\} & p(x_1 = 0) &= 0.5 & p(x_n = x_{n-1}) &= 1-\delta \\ \mathcal{Y} &= \mathcal{X} & & & p(y_n = x_n) &= 1-\varepsilon. \end{aligned}$$

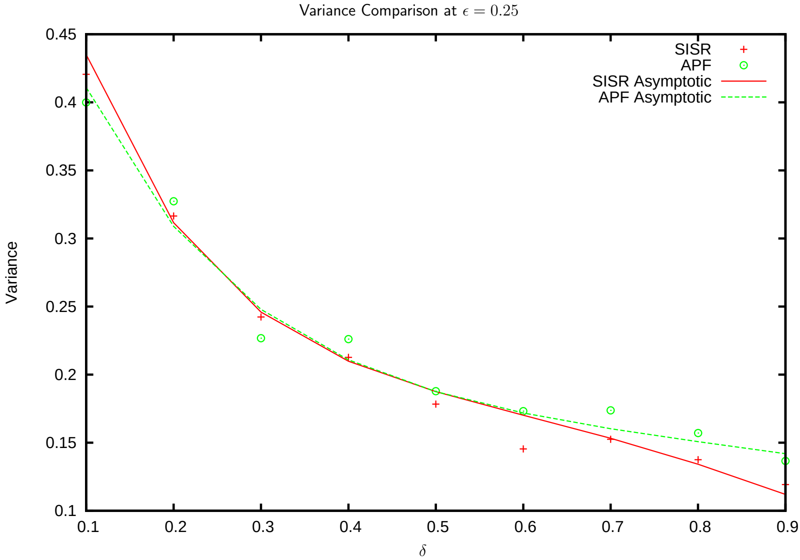
- δ controls ergodicity of the state process.
- ϵ controls the information contained in observations.

Consider estimating $\mathbb{E}(X_2|Y_{1:2} = (0,1))$.

Variance of SIR - Variance of APF



Practical Implications



Applications & Extensions

SMC Samplers (Del Moral, Doucet & Jasra, 2006)

SIR Techniques can be adapted for any sequence of distributions $\{\pi_n\}$.

- Define
$$\widetilde{\pi_n}(x_{1:n}) = \pi_n(x_n) \prod_{k=1}^{n-1} L_k(x_{k+1}, x_k).$$
- Sample at time n,
$$X_n^i \sim K_n(X_{n-1}^i, \cdot)$$
- Weight
$$W_n^i \propto \frac{\pi_n(X_n^i) L_{n-1}(X_n^i, X_{n-1}^i)}{\pi_{n-1}(X_{n-1}^i) K_n(X_{n-1}^i, X_n^i)}$$
- Resample.

Auxiliary SMC Samplers

An *auxiliary SMC sampler* for a sequence of distributions π_n comprises:

- An SMC sampler targeting some auxiliary sequence of distributions μ_n.
- A sequence of importance weight functions

$$\widetilde{w_n}(x_n) \propto \frac{\mathrm{d}\pi_n}{\mathrm{d}\mu_n}(x_n).$$

Generic approaches:

- Choose $\mu_n(x_n) \propto \pi_n(x_n)V_n(x_n)$.
- Incorporate as much information as possible prior to resampling.

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Resample-Move Approaches

A common approach in SMC Samplers:

- Choose $K_n(x_{n-1}, x_n)$ to be π_n-invariant.
- Set
$$L_{n-1}(x_n, x_{n-1}) = \frac{\pi_n(x_{n-1})K_n(x_{n-1}, x_n)}{\pi_n(x_n)}.$$
- Leading to
$$W_n(x_{n-1}, x_n) = \frac{\pi_n(x_{n-1})}{\pi_{n-1}(x_{n-1})}.$$
- It would make more sense to resample and then move…

An Interpretation of Pure Resample-Move

It's SIR with auxiliary distributions…

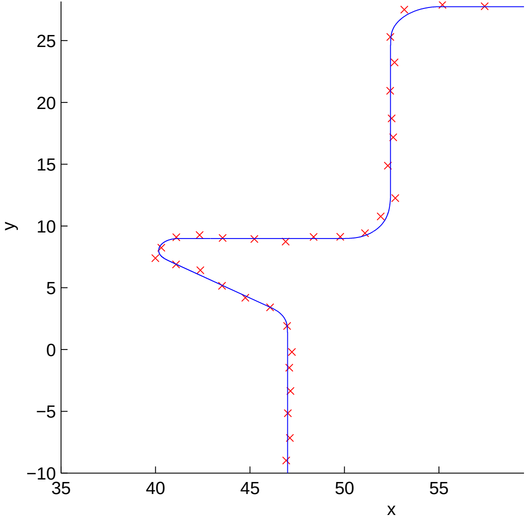
$$\mu_n(x_n) = \pi_{n+1}(x_n)$$

$$L_{n-1}(x_n, x_{n-1}) = \frac{\mu_{n-1}(x_{n-1})K_n(x_{n-1}, x_n)}{\mu_{n-1}(x_n)} = \frac{\pi_n(x_{n-1})K_n(x_{n-1}, x_n)}{\pi_n(x_n)}$$

$$w_n(x_{n-1:n}) = \frac{\mu_n(x_n)}{\mu_{n-1}(x_n)} = \frac{\pi_{n+1}(x_n)}{\pi_n(x_n)}$$

$$\widetilde{w}_n(x_n) = \frac{\mu_{n-1}(x_n)}{\mu_n(x_n)} = \frac{\pi_n(x_n)}{\pi_{n+1}(x_n)}.$$

Piecewise-Deterministic Processes



"Filtering" of Piecewise-Deterministic Processes (Whiteley, Johansen & Godsill, 2007)

- $X_n = (k_n, \tau_{n,1:k_n}, \theta_{n,0:k_n})$ specifies a continuous time trajectory $(\zeta_t)_{t \in [0,t_n]}$.
- ζ_t observed in the presence of noise, e.g. $Y_n = \zeta_{t_n} + V_n$
- Target sequence of distributions $\{\pi_n\}$ on nested spaces:

$$\pi_n(k, \tau_{1:k}, \theta_{0:k}) \propto q(\theta_0) \prod_{j=1}^{k} f(\tau_j | \tau_{j-1}) q(\theta_j | \tau_j, \theta_{j-1}, \tau_{j-1}) \times S(t_n, \tau_k) \prod_{p=1}^{n} g(y_p | \zeta_{t_p})$$

"Filtering" of Piecewise-Deterministic Processes (Whiteley, Johansen & Godsill, 2007)

- π_n yields posterior marginal distributions for recent history of ζ_t
- Employ auxiliary SMC samplers
- Choose $\mu_n(k, \tau_{1:k}, \theta_{0:k}) \propto \pi_n(k, \tau_{1:k}, \theta_{0:k}) V_n(\tau_k, \theta_k)$
- Kernel proposes new pairs (τ_j, θ_j) and adjusts recent pairs
- Applications in object-tracking and finance

Probability Hypothesis Density Filtering (Mahler, 2003; Vo et al. 2005)

- Approximates the optimal filter for a class of spatial point process-valued HMMs
- A recursion for intensity functions:

$$\alpha_n(x_n) = \int_E f(x_n|x_{n-1}) p_S(x_{n-1}) \widehat{\alpha}_{n-1}(x_{n-1}) \mathrm{d}x_{n-1} + \gamma(x_n)$$

$$\widehat{\alpha}_n(x_n) = \left[1 - p_D(x_n) + \sum_{r=1}^{m_n} \frac{\psi_{n,r}(x_n)}{\mathcal{Z}_{n,r}}\right] \alpha_n(x_n).$$

- Approximate the intensity functions $\{\widehat{\alpha}_n\}_{n \geq 0}$ using SMC

An ASMC Implementation (Whiteley et al., 2007)

- $\widehat{\alpha}_{n-1}(dx_{n-1}) \approx \widehat{\alpha}_{n-1}^N(dx_{n-1}) = \frac{1}{N}\sum_{i=1}^N W_{n-1}^i \delta_{X_{n-1}^i}(dx_{n-1})$
- In a simple case, target integral at nth iteration:

$$\int_E \int_E \sum_{r=1}^{m_n} \varphi(x_n) \frac{\psi_{n,r}(x_n)}{\mathcal{Z}_{n,r}} f(x_n|x_{n-1}) p_S(x_{n-1}) dx_n \widehat{\alpha}_{n-1}^N(dx_{n-1})$$

- Proposal distribution $q_n(x_n', x_{n-1}', r_n)$ built from $\widehat{\alpha}_{n-1}^N$, potential functions $\{V_{n,r}\}$ and factorises:

$$q_n(x_n'|x_{n-1}', r) \frac{\sum_{i=1}^N V_{n,r}(X_{n-1}^{(i)}) W_{n-1} \delta_{X_{n-1}^{(i)}}(dx_{n-1}')}{\sum_{i=1}^N V_{n,r}(X_{n-1}^{(i)}) W_{n-1}} q_n(r)$$

- and also estimate normalising constants $\{\mathcal{Z}_{n,r}\}$ by IS

Stratification...

- Back to HMM, let $(A_p)_{p=1}^{M}$ denote a partition of $\mathcal{X}$
- Introduce stratum indicator variable $R_n = \sum_{p=1}^{M} p\mathbb{I}_{A_p}(X_n)$
- Define extended model:

$$p(x_{1:n}, r_{1:n}|y_{1:n}) \propto g(y_n|x_n)f(x_n|r_n, x_{n-1})f(r_n|x_{n-1}) \times p(x_{1:n-1}, r_{1:n-1}|y_{1:n-1})$$

- An auxiliary SMC filter resamples from distribution with $N \times M$ support points, over *particles* $\times$ *strata*
- Applications in switching state space models

Conclusions

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- The APF is a standard SIR algorithm for nonstandard distributions.
- This interpretation...
  - allows standard results to be applied directly,
  - provides guidance on implementation,
  - and allows the same technique to be applied in more general settings.
- Thanks for listening...any questions?

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