The Iterated Auxiliary Particle Filter

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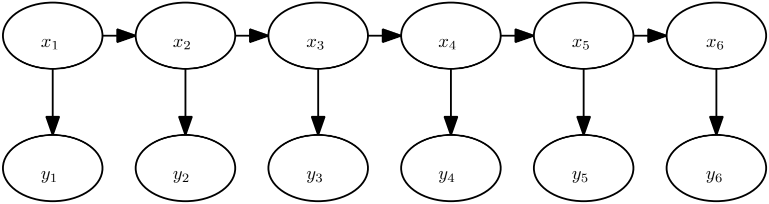
Outline

- Background: SMC and PMCMC
- Iterative Lookahead Methods
  - Motivation
  - Methodology
  - Applications: linear Gaussian and stochastic volatility
  - Ongoing work: diffusion bridges
- Conclusions

Discrete Time Filtering

Online inference for Hidden Markov Models:



- Given *transition* $f_\theta(x_{n-1}, x_n)$,
- and *likelihood* $g_\theta(x_n, y_n)$,
- use $p_\theta(x_n|y_{1:n})$ to characterize latent state, but,

$$p_\theta(x_n|y_{1:n}) = \frac{\int p_\theta(x_{n-1}|y_{1:n-1}) f_\theta(x_{n-1}, x_n) dx_{n-1} g_\theta(x_n, y_n)}{\int \int p_\theta(x_{n-1}|y_{1:n-1}) f_\theta(x_{n-1}, x'_n) dx_{n-1} g_\theta(x'_n, y_n) dx'_n}$$

isn't often tractable.

Particle Filtering

A (sequential) Monte Carlo (SMC) scheme to approximate the filtering distributions.

A Simple Particle Filter [4]

At $n = 1$:

- Sample $X_1^1, \ldots, X_1^N \sim \mu_\theta$.

For $n > 1$:

- Sample

$$X_n^1, \ldots, X_n^N \sim \frac{\sum_{j=1}^N g_\theta(X_{n-1}^j, y_{n-1}) f_\theta(X_{n-1}^j, \cdot)}{\sum_{k=1}^n g_\theta(X_{n-1}^k, y_{n-1})}$$

- Approximate $p_\theta(dx_n|y_{1:n}), p_\theta(y_{1:n})$ with

$$\widehat{p_\theta}(\cdot|y_{1:n}) = \frac{\sum_{j=1}^N g_\theta(X_n^j, y_n)\delta_{X_n^j}}{\sum_{k=1}^N g_\theta(X_n^k, y_n)}, \frac{\widehat{p_\theta}(y_{1:n})}{\widehat{p_\theta}(y_{1:n-1})} = \frac{1}{n}\sum_{j=1}^N g_\theta(X_n^k, y_n)$$

Online Particle Filters for Offline Parameter Estimation

Particle Markov chain Monte Carlo (PMCMC) [2]

- Embed SMC within MCMC,
- justified via explicit auxiliary variable construction,
- or in some cases by a pseudomarginal [1] argument.
- Very widely applicable,
- but prone to poor mixing when SMC performs poorly for some θ [7, Section 4.2.1].
- Is valid for *very* general SMC algorithms.

Twisting the HMM (a complement to [8])

Given (μ, f, g) and $y_{1:T}$, introducing $\boldsymbol{\psi} := (\psi_1, \psi_2, \ldots, \psi_T), \psi_t \in \mathcal{C}_b(\mathsf{X}, (0, \infty))$ and

$$\tilde{\psi}_0 := \int_{\mathsf{X}} \mu(x_1)\, \psi_1(x_1)\, dx_1 \quad \tilde{\psi}_t(x_t) := \int_{\mathsf{X}} f(x_t, x_{t+1})\, \psi_{t+1}(x_{t+1})\, dx_{t+1}$$

we obtain $(\mu_1^{\boldsymbol{\psi}}, \{f_t^{\boldsymbol{\psi}}\}, \{g_t^{\boldsymbol{\psi}}\})$, with

$$\mu_1^{\boldsymbol{\psi}}(x_1) := \frac{\mu(x_1)\psi_1(x_1)}{\tilde{\psi}_0}, \qquad f_t^{\boldsymbol{\psi}}(x_{t-1}, x_t) := \frac{f(x_{t-1}, x_t)\, \psi_t(x_t)}{\tilde{\psi}_{t-1}(x_{t-1})}$$

and the sequence of non-negative functions

$$g_1^{\boldsymbol{\psi}}(x_1) := g(x_1, y_1)\, \frac{\tilde{\psi}_1(x_1)}{\psi_1(x_1)} \tilde{\psi}_0, \qquad g_t^{\boldsymbol{\psi}}(x_t) := g(x_t, y_t)\, \frac{\tilde{\psi}_t(x_t)}{\psi_t(x_t)}.$$

Proposition

For any sequence of bounded, continuous and positive functions $\boldsymbol{\psi}$, *let*

$$Z_{\boldsymbol{\psi}} := \int_{\mathsf{X}^T} \mu_1^{\boldsymbol{\psi}}(x_1)\, g_1^{\boldsymbol{\psi}}(x_1) \prod_{t=2}^{T} f_t^{\boldsymbol{\psi}}(x_{t-1}, x_t)\, g_t^{\boldsymbol{\psi}}(x_t)\, dx_{1:T}.$$

Then, $Z_{\boldsymbol{\psi}} = p_\theta(y_{1:T})$ *for any such* $\boldsymbol{\psi}$.

The optimal choice is:

$$\psi_t^*(x_t) := g(x_t, y_t)\, \mathbb{E}\left[\prod_{p=t+1}^{T} g(X_p, y_p) \,\middle|\, \{X_t = x_t\} \right], \quad x_t \in \mathsf{X},$$

for $t \in \{1, \ldots, T-1\}$. Then, $Z_{\boldsymbol{\psi}^*}^N = p(y_{1:T})$ with probability 1.

Towards Iterative Auxiliary Particle Filters [5]

ψ-Auxiliary Particle Filter

1. Sample $\xi_1^i \sim \mu^{\psi}$ independently for $i \in \{1, \ldots, N\}$.
2. For $t = 2, \ldots, T$, sample independently

$$\xi_t^i \sim \frac{\sum_{j=1}^N g_{t-1}^{\psi}(\xi_{t-1}^j) f_t^{\psi}(\xi_{t-1}^j, \cdot)}{\sum_{j=1}^N g_{t-1}^{\psi}(\xi_{t-1}^j)}, \qquad i \in \{1, \ldots, N\}.$$

Necessary features of ψ

1. It is possible to sample from f_t^{ψ}.
2. It is possible to evaluate g_t^{ψ}.
3. To be useful: $\mathbb{V}(\widehat{Z}_{\psi}^N)$ must be small.

A Recursive Approximtion

Proposition

The sequence $\boldsymbol{\psi}^*$ *satisfies* $\psi_T^*(x_T) = g(x_T, y_T)$, $x_T \in \mathsf{X}$ *and*

$$\psi_t^*(x_t) = g(x_t, y_t) f\left(x_t, \psi_{t+1}^*\right), \quad x_t \in \mathsf{X}, \quad t \in \{1, \ldots, T-1\}.$$

Algorithm 1 Recursive function approximations

For $t = T, \ldots, 1$:

1. Set $\psi_t^i \leftarrow g\left(\xi_t^i, y_t\right) f\left(\xi_t^i, \psi_{t+1}\right)$ for $i \in \{1, \ldots, N\}$.
2. Choose ψ_t as a member of Ψ on the basis of $\xi_t^{1:N}$ and $\psi_t^{1:N}$.

Algorithm 2 An iterated auxiliary particle filter with parameters (N_0, k, τ)

1. Initialize: set ${\psi_t}^0 = \mathbf{1}$. $l \leftarrow 0$.
2. Repeat:
   - 2.1 Run a $\boldsymbol{\psi}^l$-APF with N_l particles; set $\hat{Z}_l \leftarrow Z_{\boldsymbol{\psi}^l}^{N_l}$.
   - 2.2 If $l > k$ and $\mathrm{sd}(\hat{Z}_{l-k:l})/\mathrm{mean}(\hat{Z}_{l-k:l}) < \tau$, go to 3.
   - 2.3 Compute $\boldsymbol{\psi}^{l+1}$ using Algorithm 1.
   - 2.4 If $N_{l-k} = N_l$ and the sequence $\hat{Z}_{l-k:l}$ is not monotonically increasing, set $N_{l+1} \leftarrow 2N_l$.
     Otherwise, set $N_{l+1} \leftarrow N_l$.
   - 2.5 Set $l \leftarrow l + 1$. Go to 2a.
3. Run a $\boldsymbol{\psi}^l$-APF. Return $\hat{Z} := Z_{\boldsymbol{\psi}}^{N_l}$.

An Elementary Implementation

Function Approximation

- Numerically obtain:

$$(m_t^*, \Sigma_t^*, \lambda_t^*) = \arg\min_{(m,\Sigma,\lambda)} \sum_{i=1}^{N} (\mathcal{N}(x_t^i, m, \Sigma) - \lambda \psi_t^i)^2$$

- Set:

$$\psi_t(x_t) := \mathcal{N}(x_t; m_t^*, \Sigma_t^*) + c(N, m_t^*, \Sigma_t^*).$$

Stopping Rule

- $k = 3$ or $k = 5$ in the following examples
- $\tau = 0.5$

Resampling

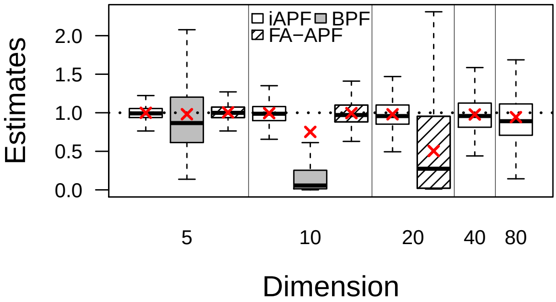
- Multinomial when ESS $< N/2$.

A Linear Gaussian Model: Behaviour with Dimension

$$\mu = \mathcal{N}(\cdot; \mathbf{0}, I_d) \qquad f(x, \cdot) = \mathcal{N}(\cdot; Ax, I_d)$$

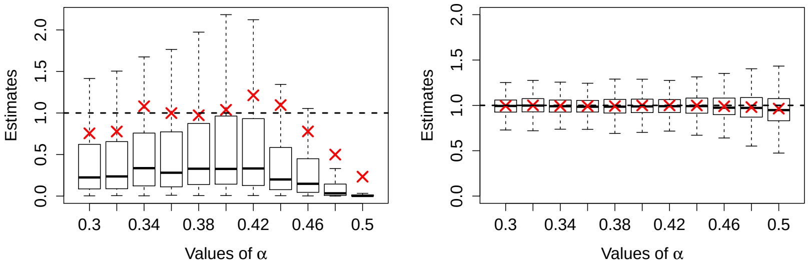
$$\text{and } g(x, \cdot) = \mathcal{N}(\cdot; x, I_d) \qquad \text{where } A_{ij} = 0.42^{|i-j|+1},$$

Box plots of $\hat{Z}/Z$ for different $|\mathsf{X}|$ (1000 replicates; $T = 100$).



Linear Gaussian Model: Sensitivity to Parameters

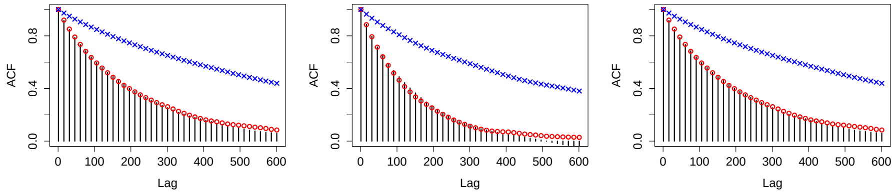
Fixing $d = 10$: Bootstrap ($N = 50,000$) / iAPF ($N_0 = 1,000$)



Box plots of $\frac{\hat{Z}}{Z}$ for different values of the parameter α using 1000 replicates.

Linear Gaussian Model: PMMH Empirical Autocorrelations

A_{11} $\qquad$ A_{41} $\qquad$ A_{55}



In this case:

$$
\begin{aligned}
d &= 5 \\
\mu &= \mathcal{N}\left(\cdot; \mathbf{0}, I_d\right) \\
f(x,\cdot) &= \mathcal{N}\left(\cdot; Ax, I_d\right) \\
\text{and } g(x,\cdot) &= \mathcal{N}\left(\cdot; x, 0.25 I_d\right)
\end{aligned}
$$

$$
A = \begin{pmatrix} 0.9 & 0 & 0 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.6 & 0 & 0 \\ 0.4 & 0.1 & 0.1 & 0.3 & 0 \\ 0.1 & 0.2 & 0.5 & 0.2 & 0 \end{pmatrix},
$$

(unknown lower triangular matrix)

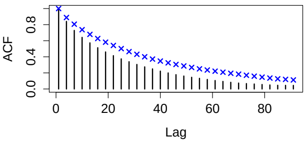
Stochastic Volatility

- A simple stochastic volatility model is defined by:

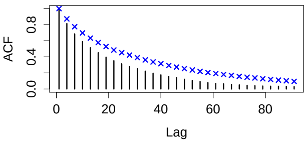
$$\begin{aligned}\mu(\cdot) =& \mathcal{N}(\cdot; 0, \sigma^2/(1-a)^2) \\ f(x,\cdot) =& \mathcal{N}(\cdot; ax, \sigma^2) \\ \text{and } g(x,\cdot) =& \mathcal{N}(\cdot; 0, \beta^2 \exp(x)),\end{aligned}$$

  where $a \in (0,1)$, $\beta > 0$ and $\sigma^2 > 0$ are unknown.
- Considered $T = 945$ observations $y_{1:T}$ corresponding to the mean-corrected daily returns for the GBP/USD exchange rate from 1/10/81 to 28/6/85.

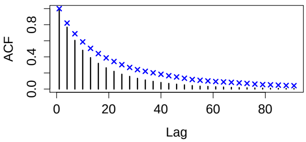
Estimated PMCMC Autocorrelation



a

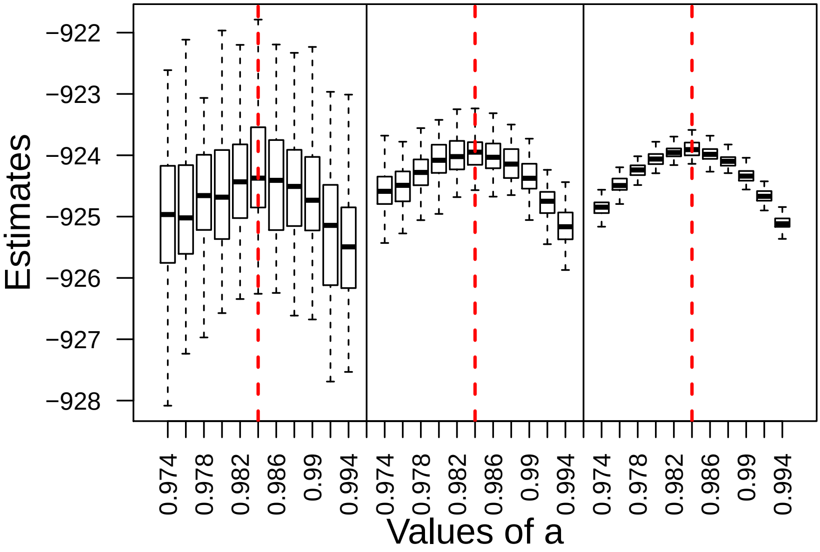


σ

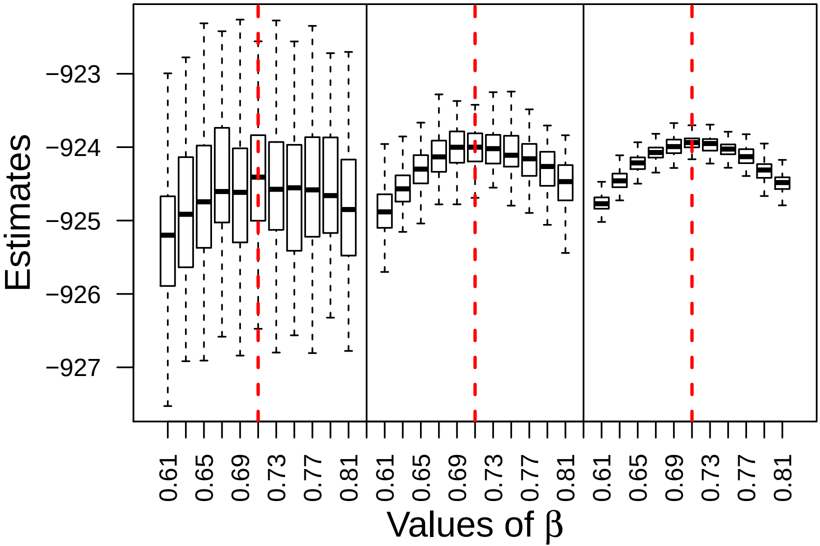


β

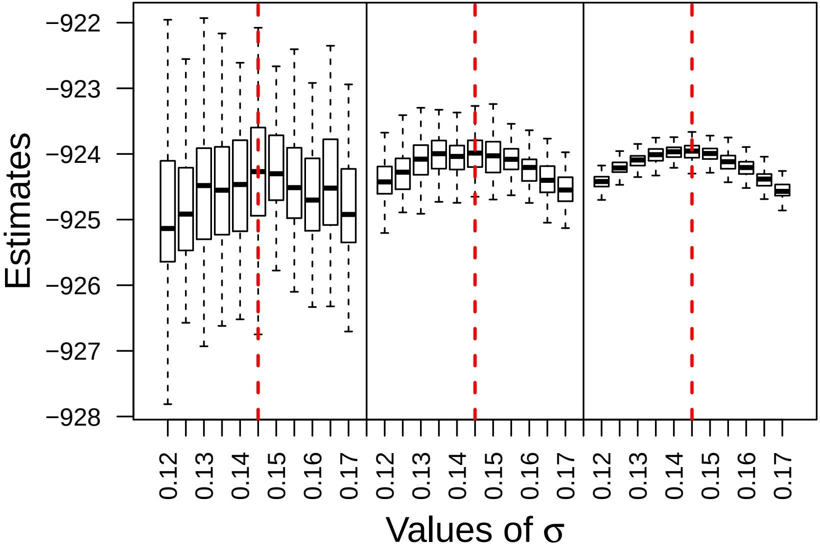
Boostrap $N = 1,000$.
iAPF $N_0 = 100$.
Comparable cost.
150,000 PMCMC iterations.



$Bootstrap : N = 1,000 \ / \ N = 10,000 \ / \ iAPF, N_0 = 100$



$Bootstrap : N = 1,000 \ / \ N = 10,000 \ / \ iAPF, N_0 = 100$



$Bootstrap : N = 1,000 \; / \; N = 10,000 \; / \; iAPF, N_0 = 100$

A More Challenging Stochastic volatility example

- The model is a multivariate stochastic volatility model from Chib et al. [3], with

$$\mu(\cdot) = \mathcal{N}(\cdot; m, U), \quad f(x, \cdot) = \mathcal{N}(\cdot; m + \Phi\left(x - m\right), U),$$

  and $g(x, \cdot) = \mathcal{N}\left(\cdot; 0, \exp\left(\operatorname{diag}\left(x\right)\right)\right)$.
- We set $\Phi = \operatorname{diag}(\phi)$, and U is band-diagonal.
- The dataset is 20 international currencies, in the periods 3/2000–8/2008 (pre-crisis) and 9/2008–2/2016 (post-crisis).
- There are 79 parameters in (m, ϕ, U), and $T = \{102, 90\}$.

- We conducted parameter estimation using particle MCMC.

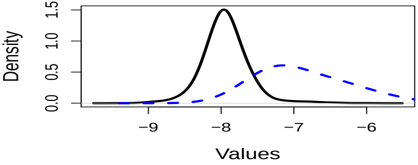
Stochastic volatility: P-MCMC

- The bootstrap particle filter systematically fails to provide reasonable marginal likelihood estimates in a feasible computational time.
- iAPF autocorrelation times sample size adjusted for autocorrelation

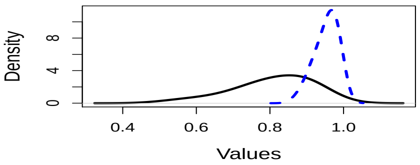
	$m_£$	$\phi_£$	$U_£$	$U_{£,€}$
pre-crisis	408	112	218	116
post-crisis	175	129	197	120

- Average number of particles at final iteration was about 1000.

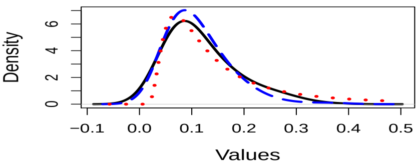
Stochastic volatility: P-MCMC



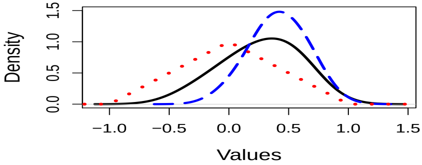
(a) $m_{£}$



(b) ϕ_{pounds}



(c) $U_{£}$



(d) $U_{£,€}$

Figure: Multivariate SV model: density estimates. Pre-crisis chain (solid), post-crisis chain (dashed) and prior density (dotted).

Ongoing work

- We consider

$$d\overline{X}_s = a\left(\overline{X}_s\right)ds + b\left(\overline{X}_s\right)dW_s, \qquad 0 \le s \le 1$$

  with standard Brownian motion W and the condition $\overline{X}_0 = \overline{x}_0$.
- We are interested in (approximately)

1. Simulating diffusion bridges, conditioning on the event $\left\{\overline{X}_0 = \overline{x}_0, \overline{X}_1 = \overline{x}_1\right\}$.
2. Evaluation of transition densities, e.g. $p\left(\overline{x}_0, \overline{x}_1\right)$.

- We employ an Euler–Maruyama approximation defined by $X_1 = \overline{x}_0$ and

$$X_t \sim \mathcal{N}\left(X_{t-1} + a\left(X_{t-1}\right)h, b^2\left(X_{t-1}\right)h\right),$$

  for $t \in \{2, \ldots, T\}$, with $T = 1/h$ so $X_T \approx \bar{X}_{1-h}$.

Model for a particle filter

- Euler–Maruyama approximation: $X_1 = \overline{x}_0$ and

$$X_t \sim \mathcal{N}\left(X_{t-1} + a\left(X_{t-1}\right)h, b^2\left(X_{t-1}\right)h\right),$$

  for $t \in \{2, \ldots, T\}$, and $T = 1/h$ so $X_T \approx \bar{X}_{1-h}$.
- If we want

$$p(\bar{x}_0, \bar{x}_1) \approx Z = \int_{\mathsf{X}^T} \mu_1\left(x_1\right) g_1\left(x_1\right) \prod_{t=2}^{T} f\left(x_{t-1}, x_t\right) g_t\left(x_t\right) dx_{1:T}$$

  we take $g_1 \equiv \cdots \equiv g_{T-1} \equiv 1$ and

$$g_T(\cdot) = \mathcal{N}\left(\overline{x}_1; x_T + a\left(x_T\right)h, b^2\left(x_T\right)h\right).$$

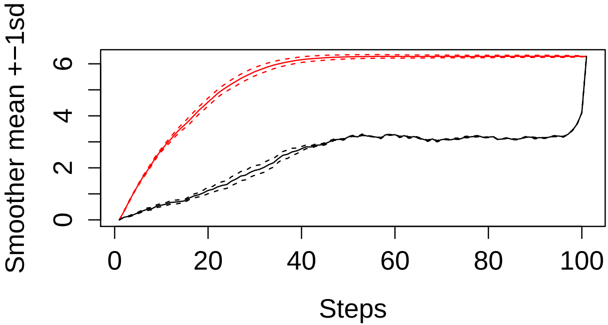
- All the information comes at the end, if we run a standard particle filter.

Example

- We take

$$d\overline{X}_s = 50s \cdot \sin\left(\overline{X}_s\right) ds + 2dW_s,$$

$\bar{x}_0 = 0$ and $\bar{x}_1 = 2\pi$. [iAPF (red), BPF (black), $h = 1/100$]



- sin is negative on $(\pi, 2\pi) \Rightarrow$ more likely for the diffusion to approach 2π from above than below.

Conclusions

- To fully realise the potential of PMCMC we should exploit its flexibility.
- Even very simple variants on the standard particle filter can significantly improve performance.
- The iAPF can improve performance substantially in some settings.
- Extending the extent of its applicability / characterising it theoretically is ongoing work.
- In principle any *function approximation* scheme can be employed: provided that f_t^{ψ} can be sampled from, and g_t^{ψ} evaluated pointwise.
- Other [standard and less standard] ideas including blocking and tempering can also be readily employed (cf. [6]).

References

[1] C. Andrieu and G. O. Roberts. The pseudo-marginal approach for efficient Monte Carlo computations. *Annals of Statistics*, 37(2):697–725, 2009.

[2] C. Andrieu, A. Doucet, and R. Holenstein. Particle Markov chain Monte Carlo. *Journal of the Royal Statistical Society B*, 72(3):269–342, 2010.

[3] S. Chib, Y. Omori, and M. Asai. Multivariate stochastic volatility. In T. G. Andersen, R. A. Davis, J.-P. Kreiss, and T. V. Mikosch, editors, *Handbook of Financial Time Series*, pages 365–400. Springer, 2009.

[4] N. J. Gordon, S. J. Salmond, and A. F. M. Smith. Novel approach to nonlinear/non-Gaussian Bayesian state estimation. *IEE Proceedings-F*, 140(2): 107–113, 1993.

[5] P. Guarniero, A. M. Johansen, and A. Lee. The iterated auxiliary particle filter. ArXiv mathematics e-print 1511.06286, ArXiv Mathematics e-prints, 2015.

[6] A. M. Johansen. On blocks, tempering and particle MCMC for systems identification. In Y. Zhao, editor, *Proceedings of 17th IFAC Symposium on System Identification*, pages 969–974, Beijing, China., 2015. IFAC.

[7] J. Owen, D. J. Wilkinson, and C. S. Gillespie. Likelihood free inference for markov processes: a comparison. *Statistical Applications in Genetics and Molecular Biology*, 2015. In press.

[8] N. Whiteley and A. Lee. Twisted particle filters. *Annals of Statistics*, 42(1): 115–141, 2014.