Divide-and-Conquer Sequential Monte Carlo

Some Properties and Applications

Adam M. Johansen

Collaborators include:
John Aston, Alexandre Bouchard-Côté, Ryan Chan, Francesca Crucinio, Brent Kirkpatrick, Juan Kuntz, Fredrik Lindsten, Christian Næsseth, Murray Pollock, Gareth Roberts and Thomas Schön

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University of Warwick
`a.m.johansen@warwick.ac.uk`
`http://go.warwick.ac.uk/amjohansen/talks/`

Outline

- Sequential Monte Carlo (SMC)
- Divide-and-Conquer SMC

  (D&C-SMC; Lindsten et al. (2017))
- Some Theoretical Properties of D&C-SMC

  (Kuntz et al., 2024)
- Some Illustrative Applications:
  - High-dimensional Filtering

    (Crucinio and Johansen, 2024)
  - Hierarchical Fusion

    (Chan et al., 2023)
- Conclusions

Sequential Monte Carlo

The Essential Problem and SMC Solution

SMC Ingredients:

- Sequence of unnormalized (path space) targets ρ_t on $\mathbf{E}_t = \otimes_{s=0}^{t} E_s$.
- Normalizing constants $Z_t = \rho_t(\mathbf{E}_t)$
- Normalized counterparts $\mu_t = \rho_t / Z_t$.
- Proposals K_t: conditional laws over E_t given $\mathbf{x}_{t-1} \in \mathbf{E}_{t-1}$.
- Importance weights / potentials:

$$w_t = \frac{d\rho_t}{d\rho_{t-1} \otimes K_t}.$$

Goals

- Estimate $Z_1, \ldots, Z_t, \ldots$
- Approximate $\mu_1, \ldots, u_t, \ldots$

Sequential Importance Resampling

```
Propose: for $n \leq N$, draw $\mathbf{X}_0^{n,N}$ independently from $K_0$.
Correct: compute
    $\rho_0^N := \frac{1}{N} \sum_{n=1}^{N} w_0(\mathbf{X}_0^{n,N}) \delta_{\mathbf{X}_0^n},$
    where $w_0 := d\rho_0 / dK_0$, $Z_0^N = \rho_0^N(\mathbf{E}_0)$ and $\mu_0^N := \rho_0^N / Z_0^N$.
for $t = 1, \ldots, T$ do
    Resample: for $n \leq N$, draw $\mathbf{X}_{t-}^{n,N} \sim \mu_{t-1}^N$ independently[1].
    Mutate: for $n \leq N$, draw $X_t^{n,N} \sim K_t(\mathbf{X}_{t-}^{n,N}, dx_t)$ and set $\mathbf{X}_t^{n,N} := (X_t^{n,N}, \mathbf{X}_{t-}^{n,N})$.
    Correct: compute
        $\rho_t^N = \frac{Z_{t-1}^N}{N} \sum_{i=1}^{N} w_t(\mathbf{X}_t^{n,N}) \delta_{\mathbf{X}_t^{n,N}},$
        $Z_t^N = \rho_t^N(\mathbf{E}_t)$ and $\mu_t^N := \rho_t^N / Z_t^N$.
end for
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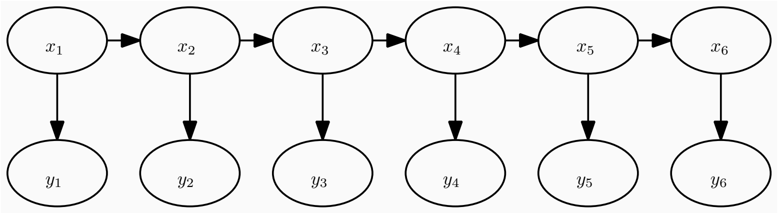
[1] We all know better...

SIR Example: Simple Particle Filters



- Unobserved Markov chain $\{X_n\}$ transition f.
- Observed process $\{Y_n\}$ conditional density g.
- The joint density is available:

$$p(x_{1:n}, y_{1:n}|\theta) = f_1^\theta(x_1)g^\theta(y_1|x_1)\prod_{i=2}^{n} f^\theta(x_i|x_{i-1})g^\theta(y_i|x_i).$$

- Natural SIR target distributions:

$$\mu_n^\theta(x_{1:n}) := p(x_{1:n}|y_{1:n}, \theta) \propto p(x_{1:n}, y_{1:n}|\theta) =: \rho_n^\theta(x_{1:n})$$

$$Z_n^\theta = \int p(x_{1:n}, y_{1:n}|\theta)dx_{1:n} = p(y_{1:n}|\theta)$$

Bootstrap PFs and Similar

- Choosing

$$\mu_n^\theta(x_{1:n}) := p(x_{1:n}|y_{1:n},\theta) \propto p(x_{1:n},y_{1:n}|\theta) =: \rho_n^\theta(x_{1:n})$$
$$Z_n^\theta = \int p(x_{1:n},y_{1:n}|\theta)dx_{1:n} = p(y_{1:n}|\theta)$$

- and $K_p(x_p|x_{1:p-1}) = f^\theta(x_p|x_{p-1})$ yields the bootstrap particle filter of Gordon et al. (1993),
- whereas $K_p(x_p|x_{1:p-1}) = p(x_p|x_{p-1},y_p,\theta)$ yields the "locally optimal" particle filter.
- Note: Many alternative particle filters are SIR algorithms with other targets. Cf. Johansen and Doucet (2008); Doucet and Johansen (2011).

Sequential Monte Carlo Samplers: Another SIR Class

Given a sequence of targets $\bar{\mu}_1, \ldots, \bar{\mu}_n$ on *arbitrary* spaces, Del Moral et al. (2006) extend the space:

$$\begin{aligned}
\mu_n(x_{1:n}) =& \bar{\mu}_n(x_n) \prod_{p=n-1}^{1} L_p(x_{p+1}, x_p) \\
\rho_n(x_{1:n}) =& \bar{\rho}_n(x_n) \prod_{p=n-1}^{1} L_p(x_{p+1}, x_p) \\
Z_n =& \int \rho_n(x_{1:n}) dx_{1:n} \\
=& \int \bar{\rho}_n(x_n) \prod_{p=n-1}^{1} L_p(x_{p+1}, x_p) dx_{1:n} = \int \bar{\rho}_n(x_n) dx_n = \bar{Z}_n
\end{aligned}$$

SIR: Theoretical Justification — Some Of

Under regularity conditions we have:

unbiasedness

$$\mathbb{E}[\widehat{Z}_n^N] = Z_n$$

slln

$$\lim_{N \to \infty} \widehat{\pi}_n^N(\varphi) \overset{\text{a.s.}}{=} \pi_n(\varphi)$$

clt For a normal random variable W_n of appropriate variance:

$$\lim_{N \to \infty} \sqrt{N}[\widehat{\pi}_n^N(\varphi) - \pi_n(\varphi)] \overset{\text{d}}{=} W_n$$

although establishing this requires a little work (cf., e.g. Del Moral (2004).

Auxiliary sequential importance resampling

Ingredients:

- Unnormalized targets ρ_t on $\mathbf{E}_t = \otimes_{s=0}^{t} E_s$.
- Normalized counterparts $\mu_t = \rho_t / Z_t$.
- Normalizing constants $Z_t = \rho_t(\mathbf{E}_t)$
- Sequences of auxiliary targets $\gamma_{t\_}$ and $\gamma_t := \gamma_{t\_} \otimes K_t$.
- Auxiliary normalizing constants $\mathcal{Z}_t = \gamma_t(\mathbf{E}_t)$
- Normalized auxiliary targets $\pi_t = \gamma_t / \mathcal{Z}_t$.
- Proposal kernels K_t: conditional laws over E_t given $\mathbf{E}_{t-1}$.
- Importance weights / potential functions:

$$w_t = \frac{d\gamma_{t\_}}{d\gamma_{t-1}}.$$

Algorithm: iterative importance sampling and resampling targeting auxiliary targets and an extra importance sampling step.

Auxiliary sequential importance resampling

```
Propose: for $n \le N$, draw $\mathbf{X}_0^{n,N}$ independently from $K_0$.
Compute: $\gamma_0^N := N^{-1} \sum_{n=1}^N \delta_{\mathbf{X}_0^{n,N}}$.
for $t = 1, \ldots, T$ do
    Correct: compute $\gamma_{t_-}^N(d\mathbf{x}_{t-1}) := w_{t_-}(\mathbf{x}_{t-1})\gamma_{t-1}^N(d\mathbf{x}_{t-1})$ and $\pi_{t_-}^N := \gamma_{t_-}^N / \gamma_{t_-}^N(\mathbf{E}_{t-1})$.
    Resample: for $n \le N$, draw $\mathbf{X}_{t_-}^{n,N}$ independently from $\pi_{t_-}^N$.
    Mutate: for $n \le N$, draw $X_t^{n,N}$ independently from $K_t(\mathbf{X}_{t_-}^{n,N}, dx_t)$ and set $\mathbf{X}_t^{n,N} := (X_t^{n,N}, \mathbf{X}_{t_-}^{n,N})$.
    Compute: $\gamma_t^N := \frac{\mathcal{Z}_t^N}{N} \sum_{n=1}^N \delta_{\mathbf{X}_t^{n,N}}$ where $\mathcal{Z}_t^N := \gamma_{t_-}^N(\mathbf{E}_{t-1})$.
end for
```

Note: At each step t, one obtains estimates of ρ_t, Z_t, and μ_t.

Auxiliary Particle Filters

In the filtering setting, take:

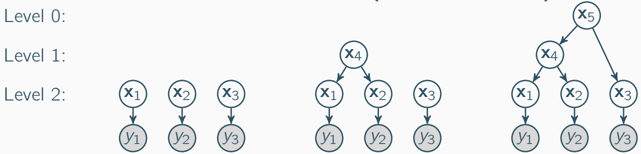
- $\gamma_{t_-}(d\mathbf{x}_{t-1}) = p(\mathbf{x}_{t-1}, \mathbf{y}_{t-1})\hat{p}(y_t|x_{t-1})$
- $\pi_{t_-} = \gamma_{t_-}/\gamma_{t_-}(\mathbf{E}_{t-1})$.

and one recovers the auxiliary particle filter of Pitt and Shephard (1999).

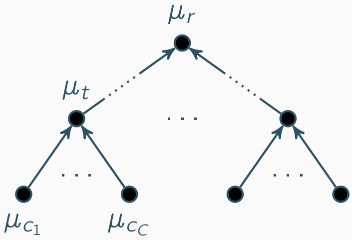
Divide-and-Conquer SMC
see Lindsten et al. (2017)

Divide-and-Conquer

Many models admit natural (or unnatural) decompositions:

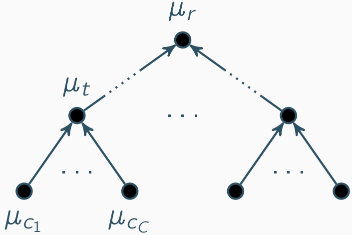


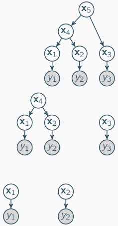
To which we can apply a divide-and-conquer strategy:



A few formalities. . .

- Use a tree, $\mathbb{T}$ of models (with rootward variable inclusion):





- $t \in \mathbb{T}$ denotes a node; $r \in \mathbb{T}$ is the root.
- $\mathcal{C}_t = \{c_1, \ldots, c_C\}$ denotes the children of t.
- E_t is the space of variables included in t but *not* its children.
- $\mathbf{E}_t = E_t \times \otimes_{c \in C(t)} \mathbf{E}_c$ is the space of all variables included in $\mathbb{T}_t$: the subtree rooted at t.
- D&C-SMC can be viewed as a recursion over this tree.

dac_smc(u) **for** u **in** $\mathbb{T}$.

```
if $u$ is a leaf (i.e. $u \in \mathbb{T}^{\partial}$) then
    Propose: for $n \leq N$, draw $\mathbf{X}_u^{n,N}$ independently from $K_u$.
    Return: $\gamma_u^N := N^{-1} \sum_{n=1}^N \delta_{\mathbf{X}_u^{n,N}}$.
else
    for $v$ in $\mathcal{C}_u$ do
        Recurse: set $\gamma_v^N :=$ dac_smc($v$).
    end for
    Obtain: $\gamma_{\mathcal{C}_u}^N = \prod_{v \in \mathcal{C}_u} \gamma_v^N$.
    Correct: compute $\gamma_{u-}^N = w_{u-} \cdot \gamma_{\mathcal{C}_u}$ and $\pi_{u-}^N := \gamma_{u-}^N / \gamma_{u-}^N(\mathbf{E}_{\mathcal{C}_u})$.
    Resample: for $n \leq N$, draw $\mathbf{X}_{u-}^{n,N} \sim \pi_{u-}^N$ independently.
    Mutate: for $n \leq N$, draw $X_u^{n,N} \sim K_u(\mathbf{X}_{u-}^{n,N}, dx_u)$ and set $\mathbf{X}_u^{n,N} := (X_u^{n,N}, \mathbf{X}_{u-}^{n,N})$.
    Return: $\gamma_u^N := N^{-1} \mathcal{Z}_u^N \sum_{n=1}^N \delta_{\mathbf{X}_u^{n,N}}$ where $\mathcal{Z}_u^N := \gamma_{u-}^N(\mathbf{E}_{\mathcal{C}_u})$.
end if
```

Theoretical Properties see Kuntz et al. (2024)

Theoretical Properties: Regularity Assumptions i

Assumption (1. Absolute Continuity)

For all u in $\mathbb{T}$ and v in $\mathbb{T}^{\not\partial}$, ρ_u is absolutely continuous w.r.t. $\boldsymbol{\gamma}_u$, $\boldsymbol{\gamma}_{v_-}$ is absolutely continuous w.r.t. $\boldsymbol{\gamma}_{\mathcal{C}_v}$, and the Radon-Nikodym derivatives $w_u := d\rho_u/d\boldsymbol{\gamma}_u$ and $w_{v_-} := d\boldsymbol{\gamma}_{v_-}/d\boldsymbol{\gamma}_{\mathcal{C}_v}$ are positive everywhere.

Assumption (2. Boundedness)

For all u in $\mathbb{T}^{\not\partial}$ and v in $\mathbb{T}$, $w_{u_-} = d\boldsymbol{\gamma}_{u_-}/d\boldsymbol{\gamma}_{\mathcal{C}_u}$ and $w_v = d\rho_v/d\boldsymbol{\gamma}_v$ are bounded: $||w_{u_-}||_\infty < \infty$ and $||w_v||_\infty < \infty$.

Theoretical Properties i

Theorem (L_p Error Bounds (Kuntz et al., 2024, Theorem 5))
If Assumptions 1–2 hold, then, for each $p \geq 1$ and u in $\mathbb{T}$, there exist constants $C_u^\rho, C_u^\mu < \infty$ such that

$$\mathbb{E}\left[\left|\rho_u^N(\varphi) - \rho(\varphi)\right|^p\right]^{\frac{1}{p}} \leq \frac{C_u^\rho ||\varphi||_\infty}{N^{1/2}},$$
$$\mathbb{E}\left[\left|\mu_u^N(\varphi) - \mu_u(\varphi)\right|^p\right]^{\frac{1}{p}} \leq \frac{C_u^\mu ||\varphi||_\infty}{N^{1/2}},$$

for all $N > 0$ and φ in $\mathcal{B}_b(\mathbf{E}_u)$. In particular,

$$\mathbb{E}[|Z_u^N - Z_u|^p]^{1/p} \leq C_u^\rho / N^{1/2}$$

for all $N > 0$.

Theoretical Properties ii

Strong Law of Large Numbers (Kuntz et al., 2024, Theorem 1)
If Assumptions 1–2 are satisfied, u belongs to $\mathbb{T}$, and φ belongs to $\mathcal{B}_b(\mathbf{E}_u)$, then

$$\lim_{N\to\infty} \rho_u^N(\varphi) = \rho_u(\varphi), \quad \lim_{N\to\infty} \mu_u^N(\varphi) = \mu(\varphi), \quad \lim_{N\to\infty} Z_u^N = Z_u,$$

almost surely.

Strong Law of Large Numbers (Kuntz et al., 2024, Theorem 2)
If, in addition to Assumptions 1–2, the spaces $(E_u)_{u\in\mathbb{T}}$ are Polish and $(\mathcal{E}_u)_{u\in\mathbb{T}}$ are the corresponding Borel sigma algebras, then

$$\rho_u^N \rightharpoonup \rho_u, \quad \mu_u^N \rightharpoonup \mu_u, \quad \text{almost surely},$$

for each u in $\mathbb{T}$, where $\rightharpoonup$ denotes weak convergence as $N \to \infty$.

Theoretical Properties iii

Central Limit theorem (Kuntz et al., 2024, Theorem 6)
If Assumptions 1–2 hold, then, as $N \to \infty$,

$$N^{1/2}\left(\rho_u^N(\varphi) - \rho_u(\varphi)\right) \Rightarrow \mathcal{N}(0, \sigma^2_{\rho_u}(\varphi)),$$
$$N^{1/2}\left(\mu_u^N(\varphi) - \mu_u(\varphi)\right) \Rightarrow \mathcal{N}(0, \sigma^2_{\mu_u}(\varphi)),$$

for any given u in $\mathbb{T}$ and φ in $\mathcal{B}_b(\mathbf{E}_u)$, where $\Rightarrow$ denotes convergence in distribution,

$$\sigma^2_{\rho_u}(\varphi) := \sum_{v \in \mathbb{T}_u} \pi_v([\mathcal{Z}_v \Gamma_{v,u}[w_u \varphi] - \rho_u(\varphi)]^2),$$
$$\sigma^2_{\mu_u}(\varphi) := \sum_{v \in \mathbb{T}_u} \pi_v([\mathcal{Z}_v \Gamma_{v,u}[w_u Z_u^{-1}[\varphi - \mu_u(\varphi)]]]^2).$$

Theoretical Properties iv

More on the CLT
In particular, $N^{1/2}\left(Z_u^N - Z_u\right) \Rightarrow \mathcal{N}(0, \sigma^2_{Z_u})$ as $N \to \infty$ with

$$\sigma^2_{Z_u} := Z_u^2 \sum_{v \in \mathbb{T}_u} \pi_v \left(\left[\frac{d\mu_u^v}{d\pi_v} - 1 \right]^2 \right), \tag{1}$$

where μ_u^v denotes the $\mathbf{E}_v$-marginal of μ_u (i.e. $\mu_u^v(A) := \mu_u(A \times E_{\mathbb{T}_u \setminus \mathbb{T}_v})$ for all A in $\mathcal{E}_v$).

Unbiasedness of NC Estimates (Kuntz et al., 2024, Theorem 3)
If Assumptions 1–2 hold, then for all $u \in \mathbb{T}$:

$$\mathbb{E}\left[\rho_u^N(\varphi)\right] = \rho_u(\varphi), \quad \mathbb{E}\left[Z_u^N\right] = Z_u, \quad \forall N > 0, \;\; \varphi \in \mathcal{B}_b(\mathbf{E}_u).$$

Theoretical Properties v

One Key Ingredient: Multinomial Expansion
Fix any u in $\mathbb{T}^{\not\partial}$ and φ in $\mathcal{B}_b(\mathbf{E}_{\mathcal{C}_u})$. Note that,

$$\gamma^N_{\mathcal{C}_u} - \gamma_{\mathcal{C}_u} = \prod_{v \in \mathcal{C}_u} \left[\gamma^N_v - \gamma_v + \gamma_v\right] - \gamma_{\mathcal{C}_u} = \sum_{\emptyset \neq A \subseteq \mathcal{C}_u} \Delta^N_A \times \gamma^{\not A}_{\mathcal{C}_u}, \quad (2)$$

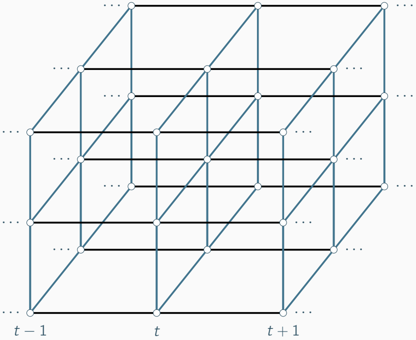
where $\Delta^N_A := \prod_{v \in A} (\gamma^N_v - \gamma_v)$ and $\gamma^{\not A}_{\mathcal{C}_u} := \gamma_{\mathcal{C}_u \setminus A}$ for all $A \subset \mathcal{C}_u$.

Some (Importance) Extensions

1. (Lightweight) Mixture Resampling [with Rejection Sampling]
2. Tempering (Del Moral et al., 2006).
3. Adaptation e.g., Zhou et al. (2016).

Illustrative Application:
High-dimensional Filtering
See Crucinio and Johansen (2024)

DaC-SMC for High-Dimensional Filtering i



Rough idea:

- Decompose space at each time.
- Implement marginal analogue of SMC (over time) — see Kück et al. (2006) and Crucinio and Johansen (2023).

Use, at node u at time t:

$$\text{“}\gamma_{t,u}(z_{t,u}) = g_{t,u}(z_{t,u}; (y_t(i)) : i \in \mathcal{V}_u) \frac{1}{N} \sum_{n=1}^{N} f_{t,u}(z^n_{t-1,\mathfrak{N}}, z_{t,u})\text{”}$$

where $f_{t,u}$ and $g_{t,u}$ approximate appropriate marginal quantities.

DaC-SMC for High-Dimensional Filtering ii

Leaf nodes: IS from with proposal $K_{t,u}$, weights are:

$$w_{t,u}(z_{t,u}, x_{1:t-1,u}) = \frac{g_{t,u}(z_{t,u}, (y_t(i))_{i \in \mathcal{V}_u}) \sum_{n=1}^{N} f_{t,u}(z_{t-1,\mathfrak{R}}^n, z_{t,u})}{\sum_{n=1}^{N} K_{t,u}(z_{t-1,\mathfrak{R}}^n, z_{t,u})}.$$

choosing $K_{t,u} \equiv f_{t,u}$ somewhat simplifies computation.

Intermediate nodes: IS using product of child nodes:

$$m_{t,u}(z_{t,\mathcal{C}_u}) = \frac{g_{t,u}(z_{t,\mathcal{C}_u}, (y_t(i))_{i \in \mathcal{V}_u})}{g_{t,\ell(u)}(z_{t,\ell(u)}, (y_t(i))_{i \in \mathcal{V}_{\ell(u)}}) g_{t,r(u)}(z_{t,r(u)}, (y_t(i))_{i \in \mathcal{V}_{r(u)}})} \times$$
$$\frac{N^{-1} \sum_{n=1}^{N} f_{t,u}(z_{t-1,\mathfrak{R}}^n, z_{t,\mathcal{C}_u})}{N^{-1} \sum_{n=1}^{N} f_{t,\ell(u)}(z_{t-1,\mathfrak{R}}^n, z_{t,\ell(u)}) N^{-1} \sum_{n=1}^{N} f_{t,r(u)}(z_{t-1,\mathfrak{R}}^n, z_{t,r(u)})},$$

with $O(N)$ computation cost... for each of N^2 particle pairs (strategy in paper has $O(N^{5/2})$ cost overall).

D&C Filtering: Spatial Example i

Lattice $V = \{1, \dots, d\}^2$. We take $d = 8$ and $d = 16$.

Dynamics $X_t(v) = X_{t-1}(v) + U_t(v)$, where $U_t(v) \overset{\text{i.i.d}}{\sim} N(0, \sigma_x^2)$.

Observation $Y_t = X_t + V_t$; V_t to be multivariate t-distributed with $\nu = 10$ d.o.f., and precision structure $\Sigma_{vj}^{-1} = \tau^{D(j,v)}$ if $D(j, v) \leq r_y$ and 0 otherwise. D denotes graph distance.

Data Simulated with $\sigma_x^2 = 1$, $\tau = -0.25$, $r_y = 1$ and $t = 10$.

D&C Filtering: Spatial Example ii

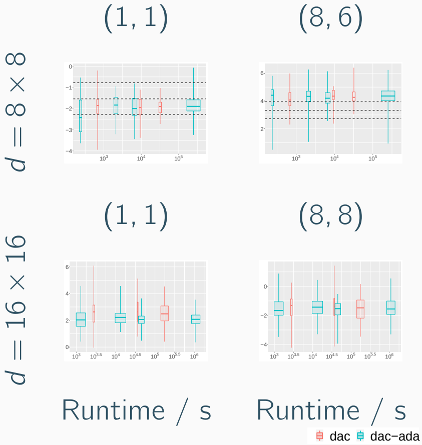


Figure 1: Filtering mean estimates for two nodes for a 8×8 and a 16×16 lattice at time $t = 10$. 50 repetitions for $N = 100, 500, 1000$ and 5000. The reference lines for the 8×8 grid show the average value of the filtering mean estimate and the interquartile range obtained with 50 repetitions of a bootstrap PF with $N = 10^5$ particles.

Illustrative Application:
Hierarchical Monte Carlo Fusion
See Chan et al. (2023)

Monte Carlo Fusion i

Objective: combine approximations of "subposteriors":

$$f(\mathbf{x}) \propto \prod_{c \in \mathcal{C}} f_c(\mathbf{x}), \tag{3}$$

Proposition (Dai et al. (2019))

If p_c is f_c^2-invariant on $\mathbb{R}^d$ then the density proportional to

$$g_{\mathcal{C}}\left(\vec{\mathbf{x}}^{(\mathcal{C})}, \mathbf{y}^{(\mathcal{C})}\right) := \prod_{c \in \mathcal{C}} \left[f_c^2\left(\mathbf{x}^{(c)}\right) \cdot p_c\left(\mathbf{y}^{(\mathcal{C})} \middle| \mathbf{x}^{(c)}\right) \cdot \frac{1}{f_c\left(\mathbf{y}^{(\mathcal{C})}\right)} \right], \tag{4}$$

admits marginal density $f^{(\mathcal{C})} \propto \prod_{c \in \mathcal{C}} f_c$ over $\mathbf{y}^{(\mathcal{C})} \in \mathbb{R}^d$.

Monte Carlo Fusion ii

This can be exploited by taking a proposal distribution proportional to:

$$h_{\mathcal{C}}\left(\vec{\mathbf{x}}^{(\mathcal{C})}, \mathbf{y}^{(\mathcal{C})}\right) := \prod_{c \in \mathcal{C}} f_c\left(\mathbf{x}^{(c)}\right) \cdot \exp\left\{-\frac{(\mathbf{y}^{(\mathcal{C})} - \tilde{\mathbf{x}}^{(\mathcal{C})})^\intercal \mathbf{\Lambda}_{\mathcal{C}}^{-1} (\mathbf{y}^{(\mathcal{C})} - \tilde{\mathbf{x}}^{(\mathcal{C})})}{2T}\right\}$$

where

$$\tilde{\mathbf{x}}^{(\mathcal{C})} := \left(\sum_{c \in \mathcal{C}} \mathbf{\Lambda}_c^{-1}\right)^{-1} \left(\sum_{c \in \mathcal{C}} \mathbf{\Lambda}_c^{-1} \mathbf{x}^{(c)}\right), \qquad \mathbf{\Lambda}_{\mathcal{C}}^{-1} := \sum_{c \in \mathcal{C}} \mathbf{\Lambda}_c^{-1}.$$

Monte Carlo Fusion iii

Proposition

If $p_c\left(\mathbf{y}^{(\mathcal{C})}|\mathbf{x}^{(c)}\right)$ *is the transition density of a suitable Langevin diffusion*

$$\begin{aligned}
\frac{g_{\mathcal{C}}(\vec{\mathbf{x}}^{(\mathcal{C})}, \mathbf{y}^{(\mathcal{C})})}{h_{\mathcal{C}}(\vec{\mathbf{x}}^{(\mathcal{C})}, \mathbf{y}^{(\mathcal{C})})} \propto & \rho_0(\vec{\mathbf{x}}^{(\mathcal{C})}) \cdot \rho_1(\vec{\mathbf{x}}^{(\mathcal{C})}, \mathbf{y}^{(\mathcal{C})}), \\
\rho_0(\vec{\mathbf{x}}^{(\mathcal{C})}) :=& \exp\left\{-\sum_{c\in\mathcal{C}} \frac{(\tilde{\mathbf{x}}^{(\mathcal{C})} - \mathbf{x}^{(c)})^{\intercal}\mathbf{\Lambda}_c^{-1}(\tilde{\mathbf{x}}^{(\mathcal{C})} - \mathbf{x}^{(c)})}{2T}\right\}, \\
\rho_1(\vec{\mathbf{x}}^{(\mathcal{C})}, \mathbf{y}^{(\mathcal{C})}) :=& \prod_{c\in\mathcal{C}} \mathbb{E}_{\mathbb{W}_{\Lambda_c}}\left[\exp\left\{-\int_0^T \phi_c\left(\mathbf{X}_t^{(c)}\right) dt\right\}\right], \\
\phi_c(\mathbf{x}) :=& \frac{1}{2}\left(\nabla \log f_c(\mathbf{x})^{\intercal}\mathbf{\Lambda}_c \nabla \log f_c(\mathbf{x}) + Tr(\mathbf{\Lambda}_c \nabla^2 \log f_c(\mathbf{x}))\right),
\end{aligned}$$

where $Tr(\cdot)$ *denotes the trace of a matrix, and* $\mathbb{W}_{\Lambda_c}$ *denotes the law of a Brownian bridge* $\{\mathbf{X}_t^{(c)}, t \in [0, T]\}$ *with* $\mathbf{X}_0^{(c)} := \mathbf{x}^{(c)}$, $\mathbf{X}_T^{(c)} := \mathbf{y}^{(\mathcal{C})}$ *and covariance matrix* $\mathbf{\Lambda}_c$*.*

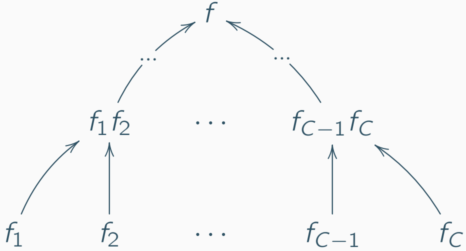
Merging Subposteriors

```
general.fusion($\mathcal{C}, \{\{\mathbf{x}_{0,i}^{(c)}, w_i^{(c)}\}_{i=1}^{M}, \mathbf{\Lambda}_c\}_{c\in\mathcal{C}}, N, T$)
Input: Samples $\{\mathbf{x}_{0,i}^{(c)}, w_i^{(c)}\}_{i=1}^{M}$ for $c \in \mathcal{C}$, matrices, $\{\mathbf{\Lambda}_c : c \in \mathcal{C}\}$,
particle count, $N$, and time horizon, $T > 0$.

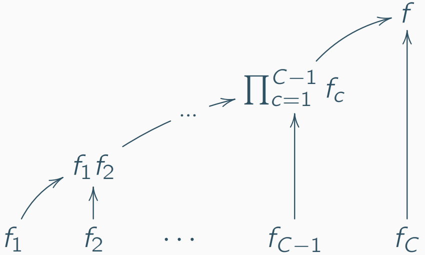
  1. Partial proposal: Compose samples $\{\vec{\mathbf{x}}_{0,j}^{(\mathcal{C})}, \vec{w}_j\}_{j=1}^{M}$ where
     $\vec{w}_j := \left(\prod_{c\in\mathcal{C}} w_j^{(c)}\right) \cdot \rho_0(\vec{\mathbf{x}}_{0,j}^{(\mathcal{C})})$ for $j \in \{1, \dots, M\}$.
  2. For $i$ in 1 to $N$,
     2.1 $\vec{\mathbf{x}}_{0,i}^{(\mathcal{C})}$: Sample $I \sim \text{categorical}(\vec{w}_{1:M})$ and set $\vec{\mathbf{x}}_{0,i}^{(\mathcal{C})} := \vec{\mathbf{x}}_{0,I}^{(\mathcal{C})}$.
     2.2 Complete proposal: Simulate $\mathbf{y}_i^{(\mathcal{C})} \sim \mathcal{N}_d(\tilde{\mathbf{x}}_i^{(\mathcal{C})}, T\mathbf{\Lambda}_{\mathcal{C}})$.
     2.3 $\tilde{\rho}_{1,i}^{(\mathcal{C})}$: Compute importance weight $\tilde{\rho}_{1,i}^{(\mathcal{C})} := \tilde{\rho}_1^{(b)}(\vec{\mathbf{x}}_{0,i}^{(\mathcal{C})}, \mathbf{y}_i^{(\mathcal{C})})$.
  3. For $i$ in 1 to $N$ compute $w_i^{(\mathcal{C})} = \tilde{\rho}_{1,i}^{(\mathcal{C})} / \sum_{k=1}^{N} \tilde{\rho}_{1,k}^{(\mathcal{C})}$.

Output: $\left\{\vec{\mathbf{x}}_{0,i}^{(\mathcal{C})}, \mathbf{y}_i^{(\mathcal{C})}, w_i^{(\mathcal{C})}\right\}_{i=1}^{N}$.
```

Some Decompositions Leading to an Algorithm



A balanced-binary tree.



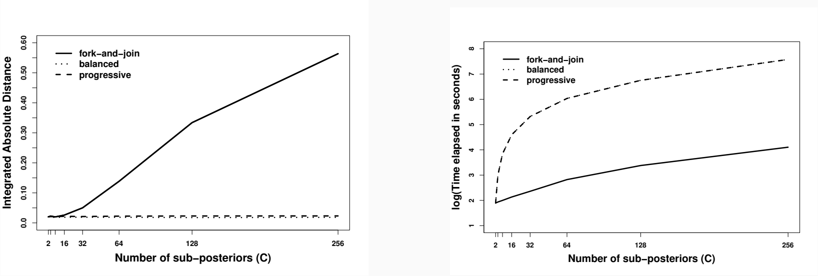
A progressive tree.

`d&c.fusion`(v, N, T)

Given: Sub-posteriors, $\{f_u\}_{u \in \mathrm{Leaf}(\mathbb{T})}$, and preconditioning matrices $\{\boldsymbol{\Lambda}_u\}_{u \in \mathbb{T}}$.
Input: Node in tree, v, the number of particles N, and time horizon $T > 0$.

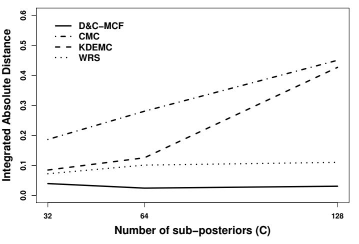
```
1. For $u \in \mathrm{Ch}(v)$,
    1.1 $\left\{\mathbf{x}_i^{(u)}, \mathbf{y}_i^{(u)}, w_i^{(u)}\right\}_{i=1}^{N} \leftarrow$ d&c.fusion$(u, N, T)$.
2. If $v \in \mathrm{Leaf}(\mathbb{T})$,
    2.1 For $i = 1, \dots, N$, sample $\mathbf{y}_i^{(v)} \sim f_v(\mathbf{y})$.
    2.2 Output: $\{\emptyset, \mathbf{y}_i^{(v)}, \frac{1}{N}\}_{i=1}^{N}$.
3. If $v \notin \mathrm{Leaf}(\mathbb{T})$,
    3.1 Output: Call
        general.fusion$(\mathrm{Ch}(v), \{\{\mathbf{y}_i^{(u)}, w_i^{(u)}\}_{i=1}^{N}, \boldsymbol{\Lambda}_u\}_{u \in \mathrm{Ch}(v)}, N, T)$.
```

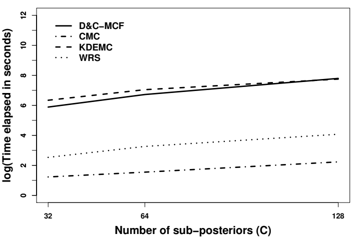
An Illustration of the Impact of the D&C Approach



Illustrative comparison of the effect of using different hierarchies, with $f \propto \prod_{c=1}^{C} f_c$, where $f_c \sim \mathcal{N}(0, C)$ for $c = 1, \ldots, C$ (averaged over 50 runs).

Some Results for a Logistic Regression Example





[CMC=Consensus Monte Carlo; KDEMC=kernel density averaging approach of Neiswanger et al. (2014); WRS=Weierstrass Rejection Sampler]

* The *'Default of credit card clients'* data set available from https://archive.ics.uci.edu/ml/datasets. The data set comprised $m = 30000$ records of **response:** whether a default had occurred and binary covariates **Gender** and **Education**.

Conclusions

- SMC $\approx$ SIR
- D&C-SMC $\approx$ SIR + Coalescence
- Distributed implementation is often straightforward
- D&C strategy can improve even serial performance
- D&C-SMC inherits many theoretical guarantees from SMC
- Some questions remain unanswered, e.g.:
    - How can we construct (near) optimal tree-decompositions?
- Some other recent applications include:
    - Parallel (in time) Smoothing (Ding and Gandy, 2018; Corenflos et al., 2022)

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