CASE FOR SUPPORT

PART 1: Previous track record of PI

General experience of the PI in this area

I have worked in the general area of Probability for over 30 years, producing about 100 papers and 6 co-authored or co-edited books. I have served as associate editor for various journals, as inaugural Coordinating Editor for the section *Stochastic Geometry and Statistical Applications* of *Advances of Applied Probability*, and as Chief Editor for *Electronic Communications in Probability.* I was Scientific Secretary for the Bernoulli Society 1996-2000, and Scientific Programme Chair for its Sixth World Congress in 2004, and am currently President-Elect of the Society. I was PI for the 2008 EPSRC Warwick workshop "New scaling limits in probability", initiating a biennial series of UK probability meetings. Since 2006 as PI I have co-directed the EPSRC-funded Academy for PhD Training in Statistics, annually training about 80 Statistics first-year PhD students from across the UK.

Here follows a brief outline of my research activity beyond the specific project proposal, with representative references. I co-authored the main research monograph in the area of stochastic geometry in 1987, currently being revised and extended for a 3rd edition. Other research centered on stochastic differential geometry – the study of Brownian motion and other random processes on geometric manifolds – pioneering a probabilistic approach to the study of the nonlinear elliptic system of harmonic maps [8]. This paper is now achieving citations by applied statisticians working in the statistical theory of shape, as it still contains the best existence/uniqueness result on nonlinear barycentres of probability measures concentrated on small geometric balls. In other work, I developed the so-called Cranston-Kendall reflection coupling, using careful stochastic calculus involving both Itô and Stratonovich forms, stochastic parallel transport, and analysis of cut-locus problems [5]. Kuwada and Sturm have recently applied this to optimal transportation [16].

I also have worked on implementation of stochastic calculus in computer algebra packages, as in [6], and pioneered application of the Propp-Wilson Coupling from the Past (CFTP) simulation algorithm to stochastic geometry [9, 14]. In particular I established a close conceptual link between the resulting generalization of CFTP and geometric ergodicity [10]. In stochastic geometry I have worked with Aldous on efficiency problems for stochastic networks [1]. Recent work with Huiling Le on barycentres has led to a surprising application of optimal transportation to the classical CLT [13].

Specific experience of the PI relevant to this proposal

My work on coupling for nilpotent diffusions began with joint work with Ben Arous and Cranston [2] which developed two simple examples concerning (a) time integrals and (b) stochastic areas. Generalization proved challenging; the time integral case was completely generalized in joint work with my PhD student Price in 2004 [15]. The stochastic area case was generalized to d dimensions in 2007 [11], with initial coupling estimates being derived in 2010 [12]. These results establish the theoretical context for the proposed project, which is about a far-reaching generalization to general nilpotent diffusions. In related work, deriving from application of a coupling to monotonicity for Neumann heat kernels [7], in 2000 I introduced with Burdzy a notion of efficiency for coupling [4], providing a context for evaluating co-adapted nilpotent diffusion couplings. Finally, with Burdzy and Branson, I have explored the use of modern metric geometry in certain other coupling problems [3].

Research environment

The Warwick Statistics Department is located in the modern Zeeman Building, providing an attractive working environment with all facilities required for world-leading research. The Warwick probability group is shared between Statistics and Mathematics (in the same building) and works closely together in running research seminars, and hosting visitors and conferences (the probability research workshop series mentioned above returned to Warwick this year as part of the EPSRC-funded Year of Probability 2011-2012). The Statistics Department also runs a thriving series of workshops on its own account through the EPSRC-funded CRiSM initiative, and owns and runs a small departmental computing cluster (40 fast processors, 80GB of memory) supporting computationally intensive research.

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CASE FOR SUPPORT

Part 2: Description of proposed research "Probabilistic Coupling for Nilpotent Diffusions"

1 Background

The proposed project belongs to the subject of Probability, on the interface of the two research areas of (a) Statistics and Applied Probability and (b) Mathematical Analysis. A major component of the subject is the method of probabilistic coupling: studying a random process by considering the behaviour of two interdependent copies, with dependence carefully chosen to facilitate proofs or algorithms. While this concept dates back to Döblin in the '30s, it has seen most of its development in the last few decades. Much of the initial impetus arose from the study of interacting particle systems, where coupling methods allowed one to overcome severe technical problems. More recently, striking applications include Propp and Wilson's celebrated Coupling from the Past (CFTP) algorithm for stochastic simulation [19], and Thorisson's exploration of conditioning of random point patterns [22], In both these cases the initial advances have given rise to a flourishing literature. A thematic problem for coupling, which has driven much development, has been the question of whether one can design dependence to ensure that the two interdependent copies *couple* at some random time (the *coupling time*). If this can be done, and if good estimates can be derived for the coupling time distribution, then one can deduce significant results about associated differential systems, such as gradient estimates and second eigenvalue estimates; moreover this is often the route of choice for establishing convergence rates for probabilistic algorithms in computer science (for instance, very recent work by Aaron Smith on the Gibbs sampler). Coupling can be used in many other ways (for example to establish approximation, or monotonicity); however the thematic problem, of determining instances in which a coupling time can be finite, has organized and led its development, as pioneered by Lindvall [17]. The literature is now vast and developing rapidly: see Thorisson's monograph [22].

One line of enquiry has focused on coupling for Brownian motion and associated random processes on manifolds, including the Riemannian reflection coupling (see [11] for a review): Von Renesse has recently generalized the notion to suitable metric space contexts. Note also the important relationship to Bubley and Dyer's notion of path-coupling for discrete spaces. A strong motivation here is that elliptic partial differential equations carry a natural Riemannian structure; so coupling of Brownian motion on manifolds immediately enables probabilistic approaches to gradient estimates *etc*, in work of Cranston, Wang, and others. Typically one considers *co-adapted* couplings (in which one couples only the step-by-step evolutions of the two processes, essentially using a greedy algorithm). There can be a price for this restriction [5]; co-adapted coupling will be slower than *optimal couplings* [7]. However the resulting couplings are much more amenable to calculation and generalization, particularly because they allow the use of stochastic calculus. The tension between co-adapted and optimal couplings has led recently to promising links with optimal transportation [16].

With collaborators, I have been exploring a far-reaching extension of the thematic coupling problem, asking to what extent it might be possible simultaneously co-adaptively to couple not only elliptic diffusions but also path functionals of the diffusions. The two simplest examples are (a) scalar Brownian motion and its time integral, (b) planar Brownian motion and its Lévy stochastic area; coupling can be achieved for these examples [3]. The difficulty is that one can only control the dependence between the diffusive behaviours of the two processes; thus obtaining only very indirect control of the joint evolution of the path functionals of the two copies. In effect, one has to solve a degenerate and nonstandard reachability problem in stochastic control. Alternative non-co-adapted couplings have been used with success [8, 9] but require technical results on densities. More recent work on this challenging problem [13–15] has shown that the co-adaptive approach works for finite sets of iterated time integrals, and for all possible stochastic areas of n-dimensional Brownian motion, and initial estimates on coupling rates have been obtained. These results provide strong evidence for a general conjecture [13]:

Conjecture: *One can co-adaptively couple copies of any "nilpotent diffusion", namely a process driven by Brownian motion using vectorfields forming a nilpotent Lie algebra.*

This corresponds to being able to couple Brownian motion together with a finite (but arbitrarily large) set of iterated time integrals and iterated stochastic integrals. There are general control-theoretic reasons why one should expect this conjecture to mark a limit to this kind of generalization, at least insofar as stochastic integrals are concerned. Further evidence for this conjecture arises from noting that nilpotent diffusions are hypoelliptic, and thus have probability densities, from which one can certainly devise non-co-adapted coupling schemes as in [9]. The conjecture is challenging for a number of reasons. Firstly, a lesson from [3, 13] is that it is important somehow to relate differences of iterated stochastic areas to invariant quantities, and to learn how to compute with these in invariant terms. Secondly one must deal systematically with Lie algebras of high step size k (where $k = 2$ corresponds to conventional stochastic areas, and higher steps arise from iterated integration).

The purpose of this project is to determine the validity of the conjecture, and to explore the consequences both in terms of applications to mathematical analysis and also within probability and statistics. Validity of the conjecture (which is thought to be the case) would dramatically extend the range of probabilistic coupling, and would open up a wide variety of possibilities for applications in stochastic modelling.

2 National Importance

Primarily this proposal fits into the Mathematical Sciences theme as a foundational topic in Probability. Probability is currently an extremely active area of international research, and (as stated in IRM 2011) the UK probability community is vibrant and strongly engaged at the highest international standards of activity, while the subject itself is now pervasive throughout much of science and engineering. The work described here will maintain the UK's strong reputation for innovation and depth in this important research area.

The aim of the project is considerably to enlarge the range of applicability of *probabilistic coupling*, a central tool in modern probability. The results will both relate strongly to theoretical work in Analysis and also open up new possibilities in stochastic modelling in Applied Probability. The advances projected here will firmly establish coupling as an effective tool not just in elliptic but also in hypoelliptic situations. In due course this may be expected to impact strongly on analysis of all sorts of dynamical systems involving noise. Such systems are pervasive in many areas of national importance: for example in modelling of global meteorology and ocean dynamics. Advances in coupling may also be expected in the longer term to play a very direct role in furthering understanding of data assimilation algorithms, where the technique provides a natural tool to measure inferential differences arising from unknown initial variations.

I do not know of any directly related research in UK or overseas. There is some peripheral overlap with the work of Connor (supported by EPSRC) on the relationship between co-adapted and optimal couplings, while I refer below to some work on the associated theme of analytical gradient estimates, which is likely to provide important stimulus. However this specific research topic is essentially an original line of enquiry.

3 Aims and Objectives

The fundamental research hypothesis for this project is that one can devise explicit implementations of co-adapted coupling for all nilpotent diffusions and associated dynamical systems, and the aim is to devise such implementations and to investigate their properties and applications. The specific objectives are as follows:

1. To construct such couplings, first in the relatively straightforward cases of step $k = 3, 4$ (thus extending the known case [13] of $k = 2$) and also in instances where the corresponding Lie brackets are solely multilinear; and then to use recent advances in the theory of bases of Lie algebras [6] to address the general case of finite k.

2. To investigate the rate at which coupling can occur, following preliminary work in [14].
3. To explore links with related mathematical areas: geometry for nilpotent groups, coupling in infinite dimensions, relationships with stochastic flow theory.
4. To develop applications, for example in optimal transportation, simulation methodology, stochastic dynamical systems, and rough path theory.

4 Work Programme and Methodology

We now discuss details of the four sections of the programme of research.

1. *Resolution of the Conjecture by constructing co-adapted couplings for iterated Brownian stochastic areas.* As evidenced in [3, 13], it is crucial to work with invariant analogues of differences between stochastic areas. In the case of iterated stochastic areas corresponding to right-normed iterated Lie brackets, these analogues are derived from the group law of the corresponding free nilpotent group, and can be expressed recursively; moreover I have now worked out the corresponding stochastic calculus (drifts, volatilities) in recursive form using these invariant differences. These calculations form the basis for future progress.

   (a) To begin with, the cases of steps 3 and 4 now appear to be amenable to calculation, because the corresponding Lyndon basis of Lie brackets can be transformed easily into a basis composed of right-normed iterated Lie brackets. Moreover it is possible similarly to exploit the fact that the space of multilinear Lie brackets has a simple basis [21, §5.6.2]. The plan is to establish coupling using a variation of the 2-step method of [13]; however rather than using traces one will use the recursive analysis to control the nonlinear drift formulae and deal with complications caused by volatility by using scaling bounds, in the manner of [15];

   (b) The case of general finite step k is complicated by the need to find a general algorithm to re-express the Lyndon basis in terms of right-normed iterated Lie brackets. Such an algorithm has now been provided by Chibrikov [6] using rewriting rules; so the work here will be to show that the Chibrikov basis, or a suitable adaptation, subject to the operation "drift of stochastic differential", leads to a sufficiently wide range of possible drifts.

   (c) The above methods are directed at the case of iterated stochastic areas involving only Brownian coordinates. The methods of [15] deal with the case of iterated stochastic areas involving a single instance of a Brownian coordinate and a multiplicity of instances of the time variable. The final step towards a positive resolution of the conjecture is to allow for general admixtures of time and Brownian coordinates; initial considerations suggest that this should fall to a careful modification of the above approach.

   There is of course a possibility of the conjecture not holding in complete generality, notwithstanding the above evidence; were this so then it would be of considerable interest as exhibiting an intrinsic limit on the range of co-adapted coupling. This would lead to an extra set of questions concerning the extent of this intrinsic limit, while the research questions discussed below would still remain important in the range of cases for which co-adapted coupling occurs.

2. *Investigation of rate at which coupling can occur.*

   (a) Following the preliminary work of [14], estimates on rates of coupling should follow by careful choice of a modified successful coupling strategy so as to facilitate either exact expression of the coupling distribution, or estimates.

   (b) In parallel with this, the detailed nature of coupling strategies will be explored using numerical computations (compare the work of Jansons and Metcalfe [10]) and extensive simulations.

(c) There is now a substantial body of work on analytical gradient estimates for various nilpotent diffusions (Bakry, Baudoin, Wang and others). Comparison of analytical estimates with coupling rate bounds will be used to locate the nilpotent couplings in the contexts of efficiency of coupling [5] and optimal coupling [7].

3. *Exploration of links with related areas.*

   (a) The treatment outlined above is essentially Lie-algebraic. By analogy with the reflection coupling for Riemannian manifolds, the question then arises of supplying geometric interpretations of the successful couplings, using Carnot-Caratheodory distances or related distances on nilpotent Lie groups. Experience with intrinsic metrics in another context [4], as well as calculations facilitated by computer algebra, suggests that the obvious direct approach cannot succeed: some adaptation either of metric or approach will be required. Tools developed in the analytical approach will be of value here [2, 20].

   (b) In infinite dimensions one is typically concerned with "coupling at time infinity", for fundamental reasons related to the Cameron-Martin space approach to Gaussian measures on Banach spaces; I have written notes which relate an asymptotic form of Brownian coupling in Banach spaces to the existence of Schauder bases or appropriate generalizations, based on Borell inequality techniques. Extending this to stochastic areas (at least in step 2) should essentially be a matter of relating the coupling to appropriate norms for alternating tensors; similar questions for higher finite step k will be more challenging.

   (c) Relationships with stochastic flow theory arise when one seeks to couple not just two instances of a diffusion, but a range of instances beginning at different starting points. The corresponding flows are *coalescing* flows (studied in the elliptic case by Arnaudon, LeJan, Li, Raimond and others); by analogy with the simulation technique of CFTP, interest focuses on the question of how long one must wait for complete coalescence. The numerical investigations of Section 2 of the work will be crucial for gaining a strategic appreciation of how to address this.

4. *Development of applications.* Details in this final section depend heavily on how work in previous sections develops.

   (a) Riemannian reflection coupling is already linked with optimal transportation [16]. Generalization of this link to the case of nilpotent diffusions depends on careful examination of the simplest cases to determine sharp bounds on coupling probabilities.

   (b) Simulation methodology: rates of convergence to stochastic equilibrium will follow directly from the work in Section 2. A known conceptual link of geometric ergodicity with a generalization of CFTP [12] will motivate an examination of simulation implications of the link to stochastic flows.

   (c) A key step in Bailleul's investigation [1] of the Poisson boundary of a relativistic diffusion falls easily to coupling of diffusions of Kolmogorov type. Similar potential-theoretic applications should occur in other stochastic dynamical systems, to be addressed by more general nilpotent diffusions.

   (d) Once the Conjecture is resolved, contact needs to be made between nilpotent diffusion coupling and rough path theory [18], which makes essential use of step 2 stochastic areas to develop a deterministic integration theory for rough paths. There should be a significant interaction between the two theories, directed by and enhancing applications of rough paths for Brownian motion problems.

5 Academic Impact

The development of co-adapted coupling for nilpotent diffusions will primarily impact probabilists, for whom coupling is a fundamental tool. The expected affirmative solution of the Conjecture will considerably extend the range of coupling from elliptic to hypoelliptic situations, and will also substantially influence our understanding of the evolution of stochastic dynamical systems. The work will also be very relevant to mathematical analysts working on dynamical and hypoelliptic systems, and should be of considerable interest to algebraists working on Lie algebras, and to control theorists. The primary routes by which this work will be made accessible are of course publication in international journals and presentations at academic conferences. Early access will be provided by means of preprints in `arXiv` and Warwick research repositories, particularly involving crossposting to analysis and control groups with the `arXiv`. I will also follow my current practice of providing illustrative animation simulations on my web-page when appropriate. In the past this has been highly successful in disseminating graphic and succinct overviews of my research (for example, go.warwick.ac.uk/wsk/abstracts/dead).

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DIAGRAMMATIC WORK PLAN

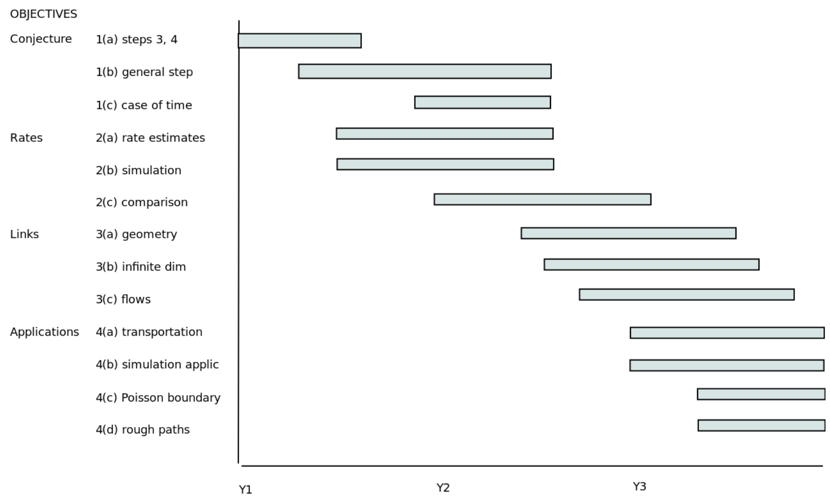

Figure 1: The PI and RA will work closely together on all aspects of the project. Results will be written up regularly throughout the project.