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Abstract: Consider a closed subset of a Riemannian manifold, such that all geodesics with end-points in the subset are contained in the subset, and the subset has boundary of codimension one. Is it the case that Riemannian barycentres of probability measures supported by the subset must also lie in the subset? It is shown that this is the case for 2-manifolds but not the case in higher dimensions: a counterexample is constructed which is a conformally-Euclidean 3-manifold, for which geodesics never self-intersect and indeed cannot turn by too much (so small geodesic balls satisfy a geodesic convexity condition), but such that a probability measure concentrated on a single point has a barycentre at another point.

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Introduction:

Riemannian barycentres and Riemannian centres-of-mass have a long history stretching back even to the early history of axiomatic probability theory (Cartan, 1928; Fréchet, 1948). More recent references include Kobayashi & Nomizu (1969), the very detailed geometric comparison estimates established in Karcher (1977), and work relating barycentres (even in a non-Riemannian context) to convexity (Émery & Mokobodzki, 1991). The simplest definition is that x is a barycentre of a probability measure μ on a Riemannian manifold if it is a local minimum of the "energy functional" $\frac{1}{2}\int_{\mathbb{M}} \operatorname{dist}(w,x)^2\,\mu(\mathrm{d}x)$. We extend the definition below to allow for non-uniqueness (and indeed non-minimality) of geodesics, in order better to explore non-uniqueness issues.

Specific applications in probability and statistics include early work by Ziezold on mean shapes of figures defined by datasets of points (Ziezold, 1977, 1989, 1994), later developed by Le (1995) in a geometrical approach to the statistical theory of shape, and also a fully probabilistic approach to the important nonlinear elliptic variational theory of harmonic maps (Kendall, 1990, 1991, 1992a,b; Picard, 1994). Related work (Jost, 1994), deriving from a purely geometric perspective, uses Riemannian barycentres to provide a dramatic simplification of the variational theory which applies to significant infinite-dimensional cases. Finally, Riemannian barycentres have been exploited to provide an intrinsic approach to statistical estimation based on the Fisher-Rao metric (Oller & Corcuera, 1995).

In all these cases it is of interest to establish conditions for which barycentres are unique, or at least constrained to lie on particular subsets. Non-uniqueness is an essentially global phenomenon, arising either because of cut-locus (consider centres of mass on a circle) or curvature (an example of non-uniqueness despite absence of cut-locus is to be found in Kendall, 1992b). Particularly in statistical contexts (for example, of intrinsic estimation constrained by a fixed hypothesis), it is of interest to understand whether or not simple geometric conditions on a subset $\mathcal{C}$ ensure that barycentres of probability measures supported by $\mathcal{C}$ also always lie in $\mathcal{C}$. The simplest possible condition would be, that all barycentres of two-point probability measures supported by $\mathcal{C}$ also always lie in $\mathcal{C}$. This corresponds to the requirement that geodesics which start and end in $\mathcal{C}$ also lie in $\mathcal{C}$; a condition of geodesic convexity. In the following we investigate the extent to which this condition suffices.

The paper is organized as follows. In Section 1 we define the condition above more formally and also give a careful extended definition of Riemannian barycentre as a critical point of a modification of the "energy functional"; care is required because in our situation we should expect geodesics *not* to be uniquely defined by their end-points. Section 2 describes a reduction of the basic problem to the case where the probability measure in question has support on a finite point set, and indeed the number of points (and indeed the number of geodesics involved) is bounded above by $d+1$ where d is the dimension of the underlying manifold. Section 3 establishes that the geodesic convexity condition together with codimension one for the boundary forces the connected components of the complement to be unbounded; this is obtained by a simple non-stochastic argument using Liouville measure and may be of independent interest. We then exploit this in Section 4 to establish a positive result for dimension 2, using the fact that geodesics can separate points in 2-manifolds. On the other hand Section 5 delivers a counterexample in dimension 3, using a conformal metric on $\mathbb{R}^3$ which is flat except on three carefully constructed small "lobes". The construction is so arranged as to provide three geodesics, all emanating from the origin $\mathbf{o}$, which also meet at a point where their tangents sum to zero as vectors (hence allowing a barycentre condition to be satisfied). The main work of the construction is to establish that geodesics in this conformal metric do not intersect themselves if the lobes are sufficiently small, and indeed then cannot turn by too much; this task is facilitated by a simple description of the differential equations governing conformal geodesics which are run at unit speed *in the original metric*. The paper is concluded by Section 6, which discusses related problems mostly to do with Γ-martingales (continuous dynamic analogues of Riemannian barycentres).

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1: Geodesic convexity and barycentres

Here and in the following we suppose $\mathbb{M}$ to be a smooth complete Riemannian manifold (without boundary). Consider the following condition of geodesic convexity for a subset $\mathcal{C}$ of $\mathbb{M}$.

Definition 1.1: **Geodesic convexity condition.** *Let $\mathcal{C} \subseteq \mathbb{M}$ be a subset of a Riemannian manifold $\mathbb{M}$. We say that $\mathcal{C}$ is geodesically convex if the following holds for all geodesics $\gamma : [0,1] \to \mathbb{M}$; whenever γ has both end-points $\gamma(0)$, $\gamma(1)$ lying in $\mathcal{C}$ then the whole image* $\mathrm{Im}(\gamma)$ *is contained in $\mathcal{C}$.*

Remark: It is important to note that γ in the above is *not* constrained to be a local minimum of the length functional: it may be just a critical point.

Definition 1.2: **(Riemannian) barycentre of a probability measure.** *Consider a probability measure μ defined on a Riemannian manifold $\mathbb{M}$. We say that a point $w \in \mathbb{M}$ is a (Riemannian) barycentre of μ if there exists a probability measure $\tilde{\mu}$ on $T_w\mathbb{M}$ which is mapped onto μ by the exponential map, so $\tilde{\mu}((\mathrm{Exp}_w)^{-1}A) = \mu(A)$ for all Borel $A \subseteq \mathbb{M}$) and which satisfies the "criticality condition"*

$$\int_{T_w\mathbb{M}} v\,\tilde{\mu}(\mathrm{d}v) \quad = \quad 0\,. \tag{1.1}$$

Remark: We use the notion of criticality here rather than the more stringent requirement that w be a (local) minimum of the "energy functional". Criticality is more naturally tied to the notion of convexity expressed in Definition 1.1, and disposes very naturally of the complications which arise when cut-locus is present.

Remark: If μ gives measure zero to the cut-locus of w and if in addition $\tilde{\mu}$ is supported within the cut-locus on $T_w\mathbb{M}$ then w is a critical point of the functional

$$\mathcal{Q}(w) \quad = \quad \frac{1}{2}\int_{\mathbb{M}} \mathrm{dist}(w,x)^2\,\mu(\mathrm{d}x)\,. \tag{1.2}$$

In this case

$$\int_{\mathbb{M}} \mathrm{dist}(w,x)\,\mathrm{grad}_1\,\mathrm{dist}(w,x)\,\mu(\mathrm{d}x) \quad = \quad \mathbf{o}\,. \tag{1.3a}$$

and it can be convenient to re-write Eq. (1.3a) informally as

$$\int_{\mathbb{M}} \left[(\mathrm{Exp}_w)^{-1}\,x\right]\mu(\mathrm{d}x) \quad = \quad \mathbf{o}\,. \tag{1.3b}$$

(Note that Exp^{-1} is not well-defined as a point map when there is cut-locus.)

In order to avoid consideration of tedious conditions relating to the cut-locus, our definition of barycentre is more generous than this: however our counterexamples can be specialized to the more stringent case by using perturbation arguments. Notice also that if the measure μ has a density then it gives the cut-locus measure zero, so the distance-based "energy" definition can be used. However even in such a case the barycentres defined by Eq. (1.1) are more general than those defined by Eqs. (1.2,1.3a,1.3b), since the second case requires $\tilde{\mu}$ to be supported within the cut-locus.

Remark: The choice of $\tilde{\mu}$ may appear arbitrary and geometrically unnatural to the reader. However it corresponds simply to the choice of a family of geodesics running from w to each of the points in the support of μ; which is certainly a geometrically natural notion. Note in particular that a probability measure supported by a single point z can according to Definition 1.2 have a *different* point $w \neq z$ as barycentre, if there are several different geodesics running from z to w, and if the gradient vectors of these geodesics at z are not all contained in some single open half-space of $T_z\mathbb{M}$.

Remark: The geodesic convexity of a set $\mathcal{C}$ can be re-expressed in terms of barycentres as defined in Definition 1.2: it is equivalent to the requirement that all barycentres of two-point measures supported in $\mathcal{C}$ should also lie in $\mathcal{C}$.

This paper sets out to answer the following question: to what extent may we deduce that if a probability measure μ has support $\operatorname{supp}(\mu)$ contained in a geodesically convex closed set $\mathcal{C}$ then $\mathcal{C}$ contains all barycentres of μ? We shall see that this is the case when $\mathbb{M}$ is two-dimensional (and the boundary $\partial\mathcal{C}$ is of codimension one), but that it does not hold in general.

2: Reduction to case of atomic measures

In this section we show that it suffices to consider atomic probability measures whose support is of cardinality restricted by the dimension.

Proposition 2.1: **Probability measures of finite support suffice.** *Suppose that $\mathcal{C}$ is a closed subset of a Riemannian manifold $\mathbb{M}$ of dimension d, such that if μ is a probability measure on $\mathcal{C}$ concentrated on at most $d+1$ points of $\mathcal{C}$ then all barycentres of μ also lie in $\mathcal{C}$. Then the same is true for general probability measures supported by $\mathcal{C}$.*

Proof: Let μ be a general probability measure supported by $\mathcal{C}$, with barycentre w as defined in Definition 1.2 (we write, $\operatorname{supp}(\mu) \subseteq \mathcal{C}$). Then there is a probability measure $\tilde{\mu}$ on $T_w\mathbb{M}$ such that $\tilde{\mu}((\operatorname{Exp}_w)^{-1}A) = \mu(A)$ for all Borel $A \subseteq \mathbb{M}$, and such that

$$\int_{T_w\mathbb{M}} v\,\tilde{\mu}(\mathrm{d}v) \quad = \quad \mathbf{o}\,.$$

But now $\mathbf{o} \in T_w\mathbb{M}$ lies in the convex hull of the closed support $\operatorname{supp}(\tilde{\mu})$ of $\tilde{\mu}$. So by a vector-space proposition (for example Theorem 3.25 and the following lemma of Rudin, 1973) we know that there must be points ξ_0, ..., ξ_d in the closed support of $\tilde{\mu}$ such that $\mathbf{o} \in \operatorname{conv}\{\xi_0, ..., \xi_d\}$: say $p_0\xi_0 + ... + p_d\xi_d = \mathbf{o}$, or $\int \xi\,\tilde{\nu}(\mathrm{d}\xi) = \mathbf{o}$ where the finitely supported probability measure $\tilde{\nu}$ puts mass p_i on ξ_i. By continuity of the exponential map we know that $x_0 = Exp_w(\xi_0)$, ..., $x_d = Exp_w(\xi_d)$ all lie in $\operatorname{supp}(\mu)$; and by our definition of the x_i and $\tilde{\nu}$ it is immediate that

$$\int_{T_w\mathbb{M}} v\tilde{\nu}(\mathrm{d}v) \quad = \quad \mathbf{o}\,.$$

Since $\operatorname{supp}\mu \subseteq \mathcal{C}$, we can thus define a probability measure ν supported on $\mathcal{C}$ with barycentre w, so that $\nu(A) = \tilde{\nu}((\operatorname{Exp}_w)^{-1}A)$ for all Borel $A \subseteq \mathbb{M}$ and ν is supported on

$$\{x_0 = \operatorname{Exp}_w(\xi_0), ..., x_d = \operatorname{Exp}_w(\xi_d)\} \quad \subseteq \quad \mathcal{C}\,.$$

$\square$

It follows that if μ is a probability measure supported on $\mathcal{C}$ which has a barycentre outside $\mathcal{C}$ then we can find ν supported on $\mathcal{C}$ with barycentre outside $\mathcal{C}$ and with finite support.
Remark: In general we cannot simply define ν by $\nu(x_i) = \tilde{\nu}(\xi_i)$ for all i, since the map $\mathrm{Exp}_w : \{\xi_0, ..., \xi_d\} \to \{x_0, ..., x_d\}$ will generally not be injective.

Corollary 2.2: *In the above the lifted measure $\tilde{\mu}$ can be chosen so as to be supported on a set of cardinality $d+1$.*

Remark: Émery has pointed out to one of us that Proposition 2.1 usefully constrains the scope for counterexamples to possible conjectures relating geodesic convexity to the existence of convex functions, as can be found in Émery & Meyer (1989).

3: Complements of geodesically convex closed sets

In preparation for the positive result of this paper, we show that complements of geodesically convex closed sets cannot have small connected components, at least if the closed sets in question have boundary of codimension one. Of course this is an easy fact in the Euclidean case! (there is only ever one connected component ...) however the global geometry of a general manifold $\mathbb{M}$ makes this result less trivial.

Proposition 3.1: **Unboundedness of connected components of the complement of a geodesically convex closed set.** *Suppose that $\mathbb{M}$ is a Riemannian manifold. Let $\mathcal{C}$ be a closed subset of $\mathbb{M}$, geodesically convex as in Definition 1.1, and suppose that $\mathcal{C}$ has smooth boundary of codimension one. Then every connected component of its complement must be of infinite volume and hence its closure is non-compact.*

Remark: This result would not hold if in Definition 1.1 we used only geodesics which were local minima of the length functional: consider the case of a small hemisphere in a sphere.

Remark: The smooth boundary condition should not be necessary: we impose it in order to avoid involvement in uninformative technicalities about Hausdorff measure.

Proof: Let $S(\mathbb{M})$ be the sphere bundle of unit tangent vectors in $\mathbb{M}$.

Let A be a connected component of the complement $\mathcal{C}^c$, and let $\mathcal{H} \subset S(\partial\mathcal{C}) \subset S(\mathbb{M})$ be the set of unit tangent vectors based in $\partial\mathcal{C}$ but pointing into A:

$$\begin{aligned} \mathcal{H} &= \{v \in S(\mathbb{M}) : \pi v \in \partial\mathcal{C}, \quad \mathrm{Exp}_{\pi v}(tv) \in A \text{ for all small enough } t > 0\} \\ &= \{v \in S(\mathbb{M}) : \pi v \in \mathcal{C}, \quad \mathrm{Exp}_{\pi v}(tv) \in A \text{ for all small enough } t > 0\} . \end{aligned} \tag{3.1}$$

(Here $\pi : S(\mathbb{M}) \to \mathbb{M}$ is induced by the tangent map projection.) Note that $\mathcal{H}$ has open interior, when viewed as a subset of $S(\partial\mathcal{C})$, because $\mathcal{C}$ has smooth boundary.

Note further that the geodesically convex condition means that Eq. (3.1) is equivalent to

$$\mathcal{H} = \{v \in S(\mathbb{M}) : \pi v \in \mathcal{C}, \quad \mathrm{Exp}_{\pi v}(tv) \in A \text{ for all } t > 0\} . \tag{3.2}$$

Consider the geodesic flow map $f : \mathcal{H} \times (0+, \infty) \to S(\mathbb{M})$ determined by $f(v,t) = (\mathrm{Exp}_{\pi v})_* (tv)$. The above shows that $\mathrm{Im}(f) \subset S(A)$. Furthermore f must be injective, since failure of injectivity would deliver a periodic closed geodesic which could not be reached from $\mathcal{H}$ by f.

Now select B to be an open subset of $\mathcal{H} \subseteq S(\partial\mathcal{C})$ such that $\{f(v,t) : v \in B, t \in (0,\varepsilon)\} \subset S(A)$. *Topologically* the set $\{f(v,t) : v \in B, t \in ((n-1)\varepsilon, n\varepsilon)\}$ can be viewed as a product subset of $S(\mathcal{C}) \times \mathbb{R}$, using geodesic convexity and properties of the geodesic flow. Since the open sets B and $((n-1)\varepsilon, n\varepsilon)$ are both non-empty, and the codimension of $S(\mathcal{C})$ is one, it follows that the product is a non-empty open subset in $S(M)$. Thus if m is Liouville measure on $S(\mathbb{M})$ then

$$m\{f(v,t) : v \in B, t \in ((n-1)\varepsilon, n\varepsilon)\}$$

is positive. Moreover by the invariance of Liouville measure under the geodesic flow it does not depend on n. Finally, since f is injective it follows that $m(S(A)) = \infty$ and hence that A is of infinite volume, as required. □

Remark: The proof above extends to the case of $\mathcal{C}$ a singleton set in $\mathbb{M}$ when $\mathbb{M}$ is of dimension two, if the geodesic convexity condition is strengthened. We then require not only that geodesics can hit the singleton set $\mathcal{C}$ once only, but also that geodesics which do not hit $\mathcal{C}$ cannot come arbitrarily close to it. This is equivalent to requiring geodesic convexity for small geodesic balls centred on the singleton point of $\mathcal{C}$. It is thus a local version of the Γ-geodesic Liouville property ($\mathbf{D}'$) described in Kendall (1991).

Remark: For the proof to apply to closed sets $\mathcal{C}$ of general codimension we must impose a kind of *strict* geodesic convexity: the set $\mathcal{C}_\epsilon$ must be geodesically convex for all sufficently small ϵ, where

$$C_\epsilon \quad = \quad \bigcup_{x\in\mathcal{C}} \text{ball}(x,\epsilon)\,.$$

The proof then works by application to each of the geodesically convex C_ϵ.

4: The two-dimensional case: a positive result

The question posed by this paper has a positive answer in dimension 2, essentially because geodesics can separate points in 2-manifolds.

Theorem 4.1: **Barycentres and geodesically convex closed sets in 2-manifolds.** *Suppose that $\mathbb{M}$ is a two-dimensional Riemannian manifold. Let $\mathcal{C}\subseteq\mathbb{M}$ be geodesically convex as in Definition 1.1, and suppose that $\mathcal{C}$ has smooth boundary of codimension one. Let μ be a probability measure supported in $\mathcal{C}$. Then every barycentre of μ is contained in $\mathcal{C}$.*

Proof: First observe that $\mathcal{C}$ is path-wise connected, as a direct consequence of being geodesically convex in the sense of Definition 1.1. (Riemannian geometry shows that for x, $y\in\mathcal{C}$ there is at least one geodesic of minimal length connecting x to y in $\mathbb{M}$, and by geodesic convexity such geodesics must lie entirely in $\mathcal{C}$.)

Minimality of the geodesics involved is not required for barycentres defined according to Definition 1.2. Consequently we can take universal covers, so that there is no loss of generality in supposing that $\mathbb{M}$ is simply connected, which is to say that $\mathbb{M}$ is a topological plane.

We now argue by contradiction. Suppose that μ is a probability measure supported on $\mathcal{C}$ with a barycentre w lying outside $\mathcal{C}$. By Proposition 2.1 we may suppose that μ is actually supported on a three-point set $\{p,q,r\}\subset\mathcal{C}$. We need to refer to the actual geodesics realizing the barycentre condition: if w is a barycentre according to Definition 1.2 then there are points $\tilde{p}$, $\tilde{q}$, $\tilde{r}\in T_w\mathbb{M}$ and weights α_p, α_q, α_r ($\alpha_p+\alpha_q+\alpha_r=1$ and $\alpha_p\geq 0$, $\alpha_q\geq 0$, $\alpha_r\geq 0$) such that

(i) $\text{Exp}_w\,\tilde{p}=p$, $\text{Exp}_w\,\tilde{q}=q$, $\text{Exp}_w\,\tilde{r}=r$;
(ii) $\alpha_p\tilde{p}+\alpha_q\tilde{q}+\alpha_r\tilde{r}=\mathbf{o}$ in $T_w\mathbb{M}$.

We define geodesics $\gamma_p(t)=\text{Exp}_w(t\tilde{p})$, $\gamma_q(t)=\text{Exp}_w(t\tilde{q})$, $\gamma_r(t)=\text{Exp}_w(t\tilde{r})$, for $t\in[0,1]$, so

$$\sum_p \alpha_p\gamma_p'(0) \quad = \quad \alpha_p\tilde{p}+\alpha_q\tilde{q}+\alpha_r\tilde{r} \quad = \quad \mathbf{o}\,. \tag{4.1}$$

We may suppose that $\tilde{p}$, $\tilde{q}$, $\tilde{r}$ all have positive weights, since otherwise the two geodesics with non-zero weight combine to produce a single geodesic with end-points in $\mathcal{C}$ and passing through $w\notin\mathcal{C}$. This would violate the geodesic convexity of $\mathcal{C}$.

Note however that it is possible for some or all of the end-points p, q, r to coincide: it is the intervening geodesics which must be distinct.

We may re-adjust the positive weights given to the various geodesic directions $\tilde{p}$, $\tilde{q}$, $\tilde{r}$ and alter their end-points and lengths so as to arrange that p, q, r belong to the boundary $\partial\mathcal{C}$ and the geodesics $\gamma_p(t)$, $\gamma_q(t)$, $\gamma_r(t)$ lie outside $\mathcal{C}$ for $t\in[0,1)$. This means that we may suppose that $\mathcal{C}^c$ is connected, for otherwise we may add to $\mathcal{C}$ all the connected components of $\mathcal{C}^c$ apart from the one containing w, without destroying either the closed-ness or the geodesic convexity (hence connected-ness) of $\mathcal{C}$. We therefore have simplified the topological setting: $\mathbb{M}$ is a plane with non-Euclidean metric; $\mathcal{C}$ is connected; $\mathcal{C}^c$ is connected and its closure is not compact; the points p, q, r lie on the boundary $\partial\mathcal{C}$. Since both $\mathcal{C}$ and $\mathcal{C}^c$ are connected, and C^c has non-compact closure, it follows by planar duality that $\mathcal{C}$ is simply-connected.

Suppose now that $\mathcal{C}$ is not compact. As $\mathcal{C}$ is simply-connected, the (smooth) boundary $\partial\mathcal{C}$ can be viewed as a one-dimensional non-closed curve (if $\mathcal{C}$ is not the closure of its interior – which happens if it is the image of a geodesic – then this remains true if we replace $\partial\mathcal{C}$ by a non-topological potential-theoretic boundary using equivalence classes of geodesic rays rooted in $\partial\mathcal{C}$ but otherwise lying wholly in $\mathcal{C}^c$). Consequently we can use the ordering of the boundary $\partial\mathcal{C}$ to argue that two of p, q, r are connected to each other by a curve in $\mathcal{C}^c$ which does not intersect γ_p, γ_q, γ_r, and which together with $\mathcal{C}$ separates w from infinity. Suppose these two points are p and q.

There is exactly one infinite component of $\mathcal{C}^c \setminus \bigcup_p \mathrm{Im}(\gamma_p)$, say D, and D is bounded by fragments of the geodesics $\mathrm{Im}(\gamma_p) \cap \mathcal{C}^c$, $\mathrm{Im}(\gamma_q) \cap \mathcal{C}^c$, $\mathrm{Im}(\gamma_r) \cap \mathcal{C}^c$, and part of ∂C. Moreover the angles of intersection of the geodesic fragments are all convex when viewed from D. Furthermore the geodesic fragment portion of ∂D must lie between p and q and separate them in ∂D.

Consider the class of piecewise-smooth continuous curves running from p to q, which never cross any of γ_p, γ_q, γ_r, and which separate w from infinity in $\mathcal{C}^c$ and which can be expressed as chains of at most N geodesic fragments each of length at most $\delta > 0$. Then N, δ can be chosen so that

(a) such curves exist (simply ensure that the total length bound $N\delta$ is sufficiently large);

(b) geodesic segments in the region $\mathrm{ball}(p, N\delta)$ are minimal if they are of length at most 2δ (simply choose δ small enough while keeping the total length $N\delta$ fixed).

The family of all such curves can be viewed as a closed subset of the compact topological product space $\mathrm{ball}(p, N\delta)^N$, and therefore its closure contains at least one curve $\tilde{\gamma}$ of minimal length. Since $\mathrm{dist}(w, \{p, q\}) > 0$ it follows that $\tilde{\gamma}$ is of non-zero length; furthermore it is not contained wholly in $\mathcal{C}$ (since it separates w from infinity in $\mathcal{C}^c$); finally it must be a geodesic because it is possible to shorten the geodesic chain wherever the join between geodesics is not differentiable, and this does not lead to a curve crossing γ_p, γ_q, γ_r, since the exposed intersections of these geodesics are convex from infinity (except at p, q, where anyway the curve $\tilde{\gamma}$ is fixed).

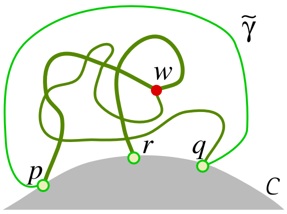

Figure 1: Illustration of $\tilde{\gamma}$, p, q, r.

Hence $\tilde{\gamma}$ is a geodesic from p to q which is not contained wholly in $\mathcal{C}$, contradicting the geodesic convexity of $\mathcal{C}$.

If $\mathcal{C}$ is compact then we argue as above, but using the universal cover of $\mathbb{M} \setminus \{y\}$ for some $y \in \mathcal{C}$. Again, if $\mathcal{C}$ is not the closure of its interior then we need to take care with the definition of $\partial\mathcal{C}$. □

If $\mathcal{C}$ is a singleton then its boundary is clearly not of codimension one. However the proof carries through if we can show that a geodesic which does not intersect $\mathcal{C}$ does not come arbitrarily close to $\mathcal{C}$, which is all that we will require in the counterexample in the following section.

5: Counterexamples

Theorem 4.1 cannot extend to higher dimensions. A four-dimensional counterexample, using a set of codimension two, can be based on the two-dimensional manifold $\mathcal{P}$ known as the "propeller" (Kendall, 1992b); geodesics in $\mathcal{P}$ are uniquely determined by their end-points and yet it supports a three-point probability measure with at least three distinct barycentres w_1, w_2, w_3. Émery has pointed out to us that $\mathcal{P} \times \mathcal{P}$ immediately supplies a counterexample. Geodesics in $\mathcal{P} \times \mathcal{P}$ are uniquely determined by their end-points (a property inherited directly from $\mathcal{P}$). Now consider the copy of $\mathcal{P}$ embedded as a totally geodesic sub-manifold as the diagonal $\mathcal{D}$ in $\mathcal{P} \times \mathcal{P}$. Being totally geodesic, this is geodesically convex as required. On the other hand, the three-point probability measure used as counterexample in Kendall (1992b), when embedded in $\mathcal{P} \times \mathcal{P}$ *via* $\mathcal{D}$, has barycentres which do not lie in $\mathcal{D}$ (use coordinate barycentre w_1 for the first coordinate, w_2 for the second, and Lemma 4 of Émery & Mokobodzki, 1991).

In the remainder of this paper we demonstrate a *three*-dimensional counterexample to the higher-dimensional version of Theorem 4.1: geodesics are free of self-intersections and indeed have uniformly bounded total angle of turning, but there is a probability measure concentrating on a single point $\mathbf{o}$ which has another point as barycentre. The limit on total turning angle means that this counterexample extends to small geodesic balls centered on $\mathbf{o}$. This counterexample is obtained from Euclidean 3-space by a conformal change of metric which leaves the Euclidean metric unchanged except in three small lobes. Its analysis depends on the behaviour of geodesics in conformally transformed metrics: we therefore commence with a subsection which summarizes this behaviour.

5.1: Geodesics and conformal changes of metric

We describe these results in complete generality, though we shall only require the case when the reference metric is that for Euclidean 3-space. In the following γ', γ'', *etc*, and $\operatorname{grad}\phi$, refer to the original Levi-Civita connection and Riemannian metric.

Let $\mathbb{M}$ be a Riemannian manifold with Riemannian metric f. Let $\phi : \mathbb{M} \to \mathbb{R}$ be a smooth function and consider the conformal metric

$$g \quad = \quad \exp(2\phi) \times f\,.$$

Apply the standard variational arguments to the g-length of a path $\gamma_u : [0,1] \to \mathbb{M}$, with u ranging over $(-\varepsilon, \varepsilon)$ and with end-points held fixed. We suppose that at $u = 0$ the path γ_0 is of constant f-speed K. The g-length is

$$L[\gamma_u] \quad = \quad \int_0^1 e^\phi \sqrt{f(\gamma_u'(s), \gamma_u'(s))}\,\mathrm{d}s \quad = \quad \int_0^1 e^\phi \sqrt{f(T,T)}\,\mathrm{d}s \tag{5.1}$$

where $T = \gamma_u'$ is the tangent vector of γ_u (the constant f-speed condition implies $|T|_f = \sqrt{f(T,T)} = K$ at $u = 0$). Let U be the vector-field supplied by the u-variation. Then

$$\frac{\mathrm{d}}{\mathrm{d}u} L[\gamma_u] \quad = \quad \int_0^1 e^\phi \sqrt{f(T,T)}(U\phi)\,\mathrm{d}s + \int_0^1 e^\phi \frac{f(T, \nabla_U T)}{\sqrt{f(T,T)}}\,\mathrm{d}s \tag{5.2}$$

and integration by parts (using $U = 0$ at $s = 0$, $s = 1$), together with $\nabla_U T = \nabla_T U$ (a consequence of zero torsion for the Levi-Civita connection ∇), shows

$$\left[\frac{\mathrm{d}}{\mathrm{d}u} L[\gamma_u]\right]_{u=0} \quad = \quad K \int_0^1 e^\phi (U\phi)\,\mathrm{d}s - K^{-1} \int_0^1 e^\phi f(\nabla_T T, U)\,\mathrm{d}s - K^{-1} \int_0^1 e^\phi f(T,U)(T\phi)\,\mathrm{d}s\,. \tag{5.3}$$

Consequently if γ_0 is a critical path for the g-length then

$$\nabla_T T \quad = \quad f(T,T)\operatorname{grad}\phi - (T\phi)T \tag{5.4}$$

(recall that $K = |T|_f = \sqrt{f(T,T)}$).

Specializing to Euclidean space, we see that if we change the metric conformally to be $\exp(2\phi)$ times the Euclidean metric then a new geodesic γ, when run at unit speed *in the old Euclidean metric*, satisfies the simple second-order differential equation

$$\gamma'' \quad = \quad [\operatorname{grad}\phi - \langle\gamma', \operatorname{grad}\phi\rangle\gamma'] \,. \tag{5.5}$$

So a conformal geodesic behaves as if the gradient of ϕ is subjecting it to a turning moment but not altering its speed. This simple description is fundamental to the construction of the required counterexample.

5.2: Three-dimensional counterexample

We will construct a conformal metric for $\mathbb{R}^3$, three-fold symmetric under rotations through $2\pi/3$ about the z-axis, parametrized by ε, such that for small ε we have

(a) all conformal geodesics are simple, and indeed have total angle of turning bounded above away from π;

(b) there are three conformal geodesics, permuted by the three-fold symmetry, begun at $\mathbf{o} = (0,0,0)$, and meeting at $\mathbf{e} = (0,0,1)$, such that their tangent vectors are parallel to the x:y-plane at $\mathbf{e}$ and hence (by three-fold symmetry) $\mathbf{e}$ is a barycentre for the degenerate probability measure concentrated at $\mathbf{o}$.

It follows that $\mathbf{e}$ is a barycentre of a measure concentrated on $\mathcal{C} = \{\mathbf{o}\}$. Furthermore $\mathcal{C}$ is geodesically convex in the sense of Definition 1.1, and small geodesic balls centered on the point which forms $\mathcal{C}$ are also geodesically convex. See Fig. 2 for an illustration of this construction.

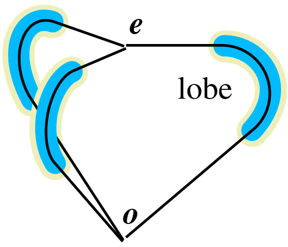

Figure 2: Three-dimensional counterexample

We first describe how to construct a conformal metric on $\mathbb{R}^3$ which has as one of its geodesics a specified smooth curved (unit-speed) segment $\alpha : [-1,2] \to \mathbb{R}^3$ which is free of self-intersections. We suppose that α is straight-line motion on $[-1,0]$, $[1,2]$.

Notice that by differentiation of $|\alpha'|^2 = 1$ we see that

$$\langle\alpha'', \alpha'\rangle \quad = \quad \langle\alpha'', T\rangle \quad = \quad 0\,.$$

Let $T(t) = \alpha'(t)$ be the tangent vector-field for α, and let $U(t)$, $V(t)$ be unit vector-fields along α such that $U(t)$, $V(t)$, $T(t)$ form an orthonormal basis for each $t \in [-1,2]$.

Finally let $n : \mathbb{R}^2 \to [0,\infty)$ be a smooth radially symmetric functions, with $n(u,v)$ decreasing in u^2+v^2, and such that $n(0) = 1$, $n(u,v) = 0$ if $u^2+v^2 \geq 1$. Let n_1, n_2 denote the two partial derivatives of the function n.

For each sufficiently small $\varepsilon > 0$ we define $\psi_\varepsilon : \mathbb{R}^3 \to \mathbb{R}$ by

$$\psi_\varepsilon\left(\alpha(t) + uU(t) + vV(t)\right) \quad = \quad \varepsilon\, n(u/\varepsilon, v/\varepsilon)\,. \tag{5.6}$$

(We need ε small enough so that $\alpha(t) + uU(t) + vV(t) = x$ is solved uniquely for $u^2+v^2 \leq \varepsilon^2$.) Observe that

$$\begin{aligned} \psi_\varepsilon(\alpha(t)) \quad &= \quad \varepsilon \\ \operatorname{grad}\psi_\varepsilon(\alpha(t)) \quad &= \quad 0 \end{aligned} \tag{5.7}$$

for all t, while we can solve the equations for $\operatorname{grad}\psi_\varepsilon$ in general,

$$\begin{aligned}\langle \operatorname{grad}\psi_\varepsilon\left(\alpha(t)+uU(t)+vV(t)\right), T(t)+uU'(t)+vV'(t)\rangle &= 0\\ \langle \operatorname{grad}\psi_\varepsilon\left(\alpha(t)+uU(t)+vV(t)\right), U(t)\rangle &= n_1(u/\varepsilon, v/\varepsilon)\\ \langle \operatorname{grad}\psi_\varepsilon\left(\alpha(t)+uU(t)+vV(t)\right), V(t)\rangle &= n_2(u/\varepsilon, v/\varepsilon)\end{aligned} \tag{5.8}$$

to establish that $\operatorname{grad}\psi_\varepsilon$ is bounded in norm uniformly in ε.

We must now modify ψ_ε to produce a function ϕ_ε with the following properties:

(i) its gradient is bounded uniformly in ε,
(ii) it is supported on the ε-dilation of the set $\{\alpha(t) : t\in[-\varepsilon, 1+\varepsilon]\}$,
(iii) it delivers the required turning moment on α.

Set

$$\theta_\varepsilon\left(\alpha(t)+uU(t)+vV(t)\right) = 1+\langle (u/\varepsilon)U+(v/\varepsilon)V, \alpha''(t)\rangle\,, \tag{5.9}$$

and choose $h:\mathbb{R}\to[0,\infty)$ to be a smooth monotonic increasing function such that $h=0$ on $(-\infty,-1]$ and $h=1$ on $[1,\infty)$. Then we define

$$\phi_\varepsilon\left(\alpha(t)+uU(t)+vV(t)\right) = \psi_\varepsilon\left(\alpha(t)+uU(t)+vV(t)\right)\times\theta_\varepsilon\left(\alpha(t)+uU(t)+vV(t)\right)\times h(t/\varepsilon)\times h((1-t)/\varepsilon)\,. \tag{5.10}$$

Boundedness of the gradient follows from boundedness of the gradient of ψ_ε and the fact that $\psi_\varepsilon(\alpha(t))=\varepsilon$; the support property follows from the definition of n and h; finally we can compute the *normal component* (to α) of the gradient of ϕ_ε at $\alpha(t)$ by using the fact that $\operatorname{grad}\psi_\varepsilon(\alpha(t))=0$, and $\alpha''(t)=0$ unless $t\in[0,1]$: differentiating $\phi_\varepsilon(\alpha(t)+uU(t)+vV(t))$ with respect to u, v,

$$\langle \operatorname{grad}\phi_\varepsilon(\alpha(t)), U(t)\rangle = \begin{cases}\langle U(t),\alpha''(t)\rangle & \text{if } t\in[0,1],\\ 0 & \text{if } t\le 0 \text{ or } t\ge 1,\end{cases} \tag{5.11}$$

and similarly if U is replaced by V. Hence

$$(\operatorname{grad}\phi_\varepsilon)\,|_{\mathrm{Im}(\alpha)} = (\alpha''+\langle\operatorname{grad}\phi_\varepsilon, T\rangle T)\,|_{\mathrm{Im}(\alpha)} \tag{5.12}$$

as required.

Together with Eq. (5.5) this establishes that the conformal metric

$$\exp(2\phi_\varepsilon)\times\ \text{Euclidean}$$

has α as a geodesic curve, and alters the Euclidean metric only in an ε-neighbourhood of the curved part of α, and yet is controlled in the sense that $\operatorname{grad}\phi_\varepsilon$ is bounded in norm uniformly in ε.

The counterexample $\mathcal{K}_\varepsilon$ is constructed as follows. Choose a once-differentiable curve α_0 in the x:z-plane, starting at $\mathbf{o}$ and moving for a while at a constant angle $\frac{\pi}{4}$ to the z-axis, then curving round through $\frac{3\pi}{4}$ radians of angle in a manner which is convex when viewed from the origin $\mathbf{o}$ of the x:z-plane and then moving back to the z-axis in parallel with the x-axis (see Fig. 3). We further require that the curved portion of α_0 approximates $3/8$ of a unit-radius circle, in the following sense: for fixed $\kappa>0$ and fixed x_0 in the x:z-plane, the curved portion of α_0 has tangent less than distance κ from the tangent for the corresponding $3/8$ of the unit circle centred at x_0.

Use α_0, and its two copies α_1, α_2 under $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ rotation about the z-axis, to build $\phi_{\varepsilon,0}$, $\phi_{\varepsilon,1}$, $\phi_{\varepsilon,2}$ as indicated in above, and as illustrated in Fig. 2. Set

$$\phi_{\varepsilon,+} = \phi_{\varepsilon,0}+\phi_{\varepsilon,1}+\phi_{\varepsilon,2}\,.$$

The resulting conformal metric certainly makes α_0, α_1, α_2 into geodesics and so by construction we see that $\mathbf{e}$ is a barycentre for the degenerate probability measure at $\mathbf{o}$, assigning probability $1/3$ to each of the three geodesics. In order to produce a counterexample to the higher-dimensional version of Theorem 4.1, it only

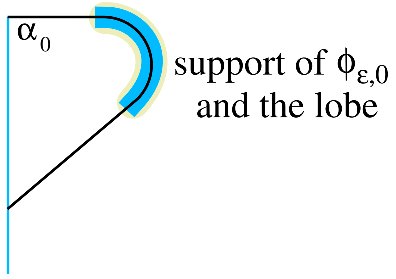

Figure 3: The curve α_0, the support of $\phi_{\varepsilon,0}$, and the lobe.

remains to show that, for sufficiently small ε, all geodesics are simple curves in this conformal metric and cannot turn by too much.

The proof of this amounts to a number of steps which we describe using lemmas and corollaries. We need some terminology. If γ is a (conformal) geodesic in $\mathcal{K}_\varepsilon$ then we say it has *collisions* corresponding to the connected components of

$$\{t \in \mathbb{R} \ : \ \phi_\varepsilon(\gamma(t)) > 0\} .$$

The *lobes* of $\mathcal{K}_\varepsilon$ contain the three connected components of the support of ϕ_ε; for convenience they are slight expansions of the support components, being the ε'-neighbourhoods of the curved portions of α_0, α_1, α_2, for $\varepsilon' = \sqrt{2}\varepsilon$.

Let U_i, V_i be the vector-fields used together with α_i' to form an orthonormal basis in the definition of $\phi_{\varepsilon,i}$ near the curved portion of α_i.

The *angle on entry* at a collision is defined as follows. Suppose that, in the above notation, entry to a collision is at $(\alpha_i(t), U_i(t), V_i(t)) \in \mathbb{R}^3$. The (Euclidean!) angle between the tangent vector of γ on entry and the tangent vector to the appropriate α_i at the point $\alpha_i(t)$ (the perpendicular projection onto α_i).

There is of course also an *angle of exit*, defined analogously, which will in general differ from the angle of entry for the same collision.

Let C_ϕ be the bound on the norm of $\operatorname{grad}\phi_\varepsilon$ (recall that this bound is uniform in ε).

Lemma 5.1: *For all $\beta \in (0, \pi/2)$, for all sufficiently small $\delta > 0$, there is $\varepsilon_* = \varepsilon_*(\delta, \beta)$ such that for all $\varepsilon < \varepsilon_*$ we have the following: Suppose that γ is a geodesic (with respect to the conformal metric) such that $\gamma(0)$ is in the lobe corresponding to α_i. Let $\gamma(0) = \alpha_i(t) + uU_i(t) + vV_i(t)$ with $u^2 + v^2 \leq (\varepsilon')^2 = 2\varepsilon^2$. If $\gamma'(0)$ makes an angle with $\alpha_i'(t)$ of at least β then $\gamma(\delta)$ is not in any of the lobes, and $\gamma'(u)$ differs from $\gamma'(0)$ in Euclidean norm by at most $C_\phi \cdot \delta$ for $u \in [0, \delta]$.*

Proof: Suppose that $\gamma(0)$ is in any lobe. Then from Eq. (5.5) and the uniform bound on $\operatorname{grad}\phi_\varepsilon$ we can deduce the following bounds on Euclidean norms:

$$\begin{aligned} \|\gamma'(u) - \gamma'(0)\| &\leq C_\phi \cdot u \text{ for } u \in [0, \delta] \\ \|\gamma(\delta) - \gamma(0) - \delta\gamma'(0)\| &\leq C_\phi \cdot \tfrac{\delta^2}{2} . \end{aligned} \tag{5.13}$$

Thus the bound on the difference between $\gamma'(u)$ and $\gamma'(0)$ follows directly, and the lemma is proved completely if we can also show that $\gamma(0) + \delta\gamma'(0)$ (representing the *Euclidean* geodesic with the same initial conditions) is at distance at least $C_\phi \frac{\delta^2}{2}$ from any lobe.

Because the lobes are separated, and we need the result only for sufficiently small ε, it suffices to prove the result only in so far as it concerns the lobe corresponding to α_i. Moreover it suffices to establish conditions such that if $\gamma(0)$ lies *on* the curve α_i then $\gamma(0) + \delta\,\gamma'(0)$ is at least distance $\varepsilon' + C_\phi\frac{\delta^2}{2}$ from α_i. This is because when $\gamma(0) = \alpha_i(t) + uU_i(t) + vV_i(t)$ satisfies the conditions of the Lemma then $\gamma(0)$ is at most ε' from $\alpha_i(t)$, which (for ε small enough for $\gamma(0)$ to avoid focal points) is the closest point on α_i to this $\gamma(0)$.

Since α_i is smooth and does not intersect itself, it suffices to show that, for small enough δ and $\varepsilon \leq \varepsilon_*(\delta,\beta)$, the point $\gamma(0) + \delta\,\gamma'(0)$ is at least distance $2\varepsilon' + C_\phi \frac{\delta^2}{2}$ from the tangent line running through $\gamma(0) = \alpha_i(t)$ with direction vector $\alpha_i'(t)$. But this follows by trigonometry from the condition on the angle between $\gamma'(0)$ and $\alpha_i'(t)$, so long as $\beta > 0$ and $\varepsilon_*(\delta,\beta)$ is sufficiently small. (We use $2\varepsilon'$ here rather than ε' to allow for the deviation of α_i from its tangent line in Euclidean space.) □

Corollary 5.2: *For all sufficiently small $\varepsilon > 0$, if a geodesic γ collides with two lobes in succession then these are the only collisions that occur, and moreover the angle between $\gamma'(-\infty)$ and $\gamma'(\infty)$ tends to zero with ε.*

Proof: This follows from Lemma 5.1 and the three-dimensional Euclidean geometry of the construction of $\mathcal{K}_\varepsilon$: the geodesic segment leading from one lobe to another cannot have exit, respectively entry, angles too close to zero, and so by Lemma 5.1 the difference between entry and exit angles for each of the two lobes tends to zero with ε. □

From Corollary 5.2 it follows that we need consider only those geodesics which hit just one lobe: without loss of generality we may suppose this to be the lobe of α_0. There are three cases to consider:

(i) Geodesics which, at some stage in a collision, have tangent directions pointing substantially away from the x:z-plane; see Lemma 5.3 below.
(ii) Geodesics which, at some stage in a collision, have tangent directions pointing substantially away from the corresponding α_0 tangent *when resolved onto the x:z-plane*; see Lemma 5.4 below.
(iii) Geodesics which at all stages during collisions have tangent directions approximately aligned with the corresponding α_0 direction; see the argument following Lemma 5.4 below.

We deal with these in the two lemmas and theorem below.

Lemma 5.3: *For all $\beta \in (0, \pi/2)$, for all sufficiently small $\delta > 0$, there is $\varepsilon_* = \varepsilon_*(\delta)$ such that for all $\varepsilon < \varepsilon_*$ we have the following: Suppose that γ is a geodesic (with respect to the conformal metric) such that $\gamma(0)$ is in the lobe corresponding to α_0 and this is the only lobe with which it collides. If $\gamma'(0)$ makes an angle with the y-axis of at most $\pi/2 - \beta$ then collisions involve only $\gamma|_{[-\delta,\delta]}$, and for all t the difference between $\gamma'(0)$ and $\gamma'(t)$ in Euclidean norm is at most $C_\phi \cdot \delta$.*

Proof: We argue as follows. By reversibility of the dynamics of the geodesic equations for γ, we need consider only $\gamma|_{[0,\infty)}$. From Lemma 5.1 we can choose δ, ε_* to ensure that if $\varepsilon < \varepsilon_*$ then $\gamma(\delta)$ is not in the lobe, and $\gamma'(u)$ differs from $\gamma'(0)$ in Euclidean norm by at most $C_\phi \cdot \delta$. Then the y coordinate of $\gamma(\delta)$ is at least

$$\sin(\beta - 2\arcsin(C_\phi \cdot \delta/2)) \cdot \delta$$

in absolute value, and $\gamma'(\delta)$ makes an angle with the y-axis of at least

$$\pi/2 - \beta + 2\arcsin(C_\phi \cdot \delta/2)\,.$$

So long as δ is small enough for this second quantity to be less than $\pi/2$, and the ε_* bound is tightened to ensure that

$$\sin(\beta - 2\arcsin(C_\phi \cdot \delta/2)) \cdot \delta \quad \geq \quad \varepsilon_*\,,$$

we can be assured that $\gamma|_{[\delta,\infty)}$ cannot experience any collisions (recall we require that γ has collisions only with the α_0 lobe, which lies entirely in an ε_* neighbourhood of the x:z-plane). This proves the lemma. □

From Lemma 5.3 it follows that we need consider only those geodesics which hit just one lobe and lie approximately in the plane defined by the lobe. We now show that if such geodesics make a sufficiently non-tangential collision then either on the entrance or the exit side there are no further collisions.

Lemma 5.4: *For all $\beta \in (0, \pi/2)$, for all sufficiently small $\delta > 0$, there is $\varepsilon_* = \varepsilon_*(\delta)$ such that for all $\varepsilon < \varepsilon_*$ we have the following: Suppose that γ is a geodesic (with respect to the conformal metric) such that $\gamma(0)$ is in the lobe corresponding to α_0 and this is the only lobe with which it collides. Suppose that $\gamma'(0)$ makes an angle with α_0 which when resolved on the x:z-plane is at least β. Then at least one of $\gamma|_{[\delta,\infty)}$, $\gamma|_{(-\infty,-\delta]}$ is involved in no collisions, and on that side γ' differs from $\gamma'(0)$ by at most $C_\phi \cdot \delta$ in Euclidean norm.*

Furthermore, if $\beta + 2\arcsin(C_\phi \cdot \delta/2) < \pi/2$ then the first collision for the other of $\gamma|_{[\delta,\infty)}$, $\gamma|_{(-\infty,-\delta]}$ must have entry angle less than $\pi/2$.

Proof: We arrange for δ to be chosen small enough so that $\beta - 2\arcsin(C_\phi \cdot \delta/2)$ is positive.

From Lemma 5.1 we know that $\gamma(\delta)$ and $\gamma(-\delta)$ both lie outside the lobe, and both $\gamma'(\delta)$ and $\gamma'(-\delta)$ differ in Euclidean distance from $\gamma'(0)$ by at most $C_\phi \cdot \delta$. When resolved onto the x:y-plane, one of $\gamma(\delta)$ and $\gamma(-\delta)$ must lie on the non-convex side of α_0.

Since the collision-free part of γ is actually straight-line motion, $\gamma'(u)$ is constant on it and therefore differs from $\gamma'(0)$ in Euclidean norm by at most $C_\phi \cdot \delta$ for $u \in [0, \delta]$. Therefore the corresponding $\gamma|_{[\delta,\infty)}$, $\gamma|_{(-\infty,\delta]}$ must be free of collisions. (Here we use the choice of δ at the beginning of the proof.)

The final remark follows directly from the bounds on $\gamma'(\delta)$ and $\gamma'(-\delta)$, and convexity. □

From Lemma 5.4, those geodesics which hit just one lobe and lie approximately in the plane defined by the lobe can hit the lobe in a substantially non-tangential manner in at most two places. Such a substantially non-tangential collision must be closely approximated on at least one side by a geodesic ray with no collisions at all. On the other side, there are two possibilities. If the next collision is substantially non-tangential then the arguments of Lemma 5.4 apply again to show there can be no further collisions, and indeed γ' differs from $\gamma'(0)$ in Euclidean distance by at most $2C_\phi \cdot \delta$.

On the other hand, if the next collision is approximately tangential then the last part of Lemma 5.4 assures us that the geodesic must still be running along α_0 in the same sense as in the original collision. There will then be a sequence of approximately tangential collisions until either the geodesic exits the lobe, or an exit is made with exit angle exceeding β. The exit angle cannot exceed $\beta + 2\arcsin(C_\phi \cdot \delta/2)$, since by Lemma 5.1 we know exit will commence as soon as the geodesic angle exceeds β. The same arguments apply in reverse time to the initial entrance angle, and this together with the geometry of α_0 assures us that the maximum difference in angle between start and finish for a geodesic intersecting a single lobe is

$$3\pi/4 + 2\beta + 4\arcsin(C_\phi \cdot \delta/2)\,. \tag{5.14}$$

Thus such a geodesic cannot intersect itself so long as β, δ, and thus the ε bound ε^*, are chosen to be sufficiently small.

This deals with all the possibilities; so there can be no self-intersecting geodesics in $\mathcal{K}_\varepsilon$. We sum up the result in a theorem.

Theorem 5.5: *For all small enough ε, the conformally Euclidean 3-manifold $\mathcal{K}_\varepsilon$ described above has no self-intersecting geodesics but has a geodesically convex closed subset $\mathcal{C}$ (in fact $\mathcal{C} = \{\mathbf{o}\}$) which supports a probability measure with a barycentre lying outside $\mathcal{C}$.*

Remark: It should be noted that we have *not* shown that the "energy functional" $\mathcal{Q}$ for the degenerate probability measure at $\mathbf{o}$ has a local minimum away from $\mathbf{o}$; we have only shown that the "criticality" condition Eq. (1.1) holds. However it is a simple matter to modify the function ϕ_ε controlling the conformal metric locally to α_i so as to ensure that the metric is locally flat for most of the path of the geodesics α_i, using the fact that

$$\mathrm{d}s^2 \quad = \quad \frac{\mathrm{d}r^2 + r^2\mathrm{d}\theta^2}{r^2} \tag{5.15}$$

delivers a flat metric in the coordinates r and θ. By this means one can ensure that the critical point is in fact a local minimum of a functional similar to $\mathcal{Q}$, but computing the distance along specified (possibly non-minimal) geodesics. Notice however that the propeller product $\mathcal{P} \times \mathcal{P}$ described at the start of Section 5 provides a 4-dimensional codimension two counterexample which immediately delivers a local minimum of $\mathcal{Q}$ itself.

6: Conclusion

In this paper we have shown how to construct a three-dimensional counterexample (Theorem 5.5) to the following assertion: that a geodesically convex closed set with boundary of codimension one can support no barycentres outside itself. As well as being of interest for its own sake as a part of Riemannian geometry, this is of significance both in stochastic differential geometry, where barycentres are strongly related to the concept of Γ-martingales (Émery & Mokobodzki, 1991; Kendall, 1990, 1991, 1992a,b; Picard, 1994) and in geometrically based statistical inference, where barycentres are used as "intrinsic mean values" (Oller & Corcuera, 1995) and uniqueness conditions for barycentres are of basic importance.

Its non-triviality is underlined by the result that the assertion *is* true in two dimensions (Theorem 4.1). In fact the proof of the two-dimensional result generalizes to hold for iterated barycentres: it would be interesting to formulate and prove a result for a suitable class of Γ-martingales in 2-manifolds. However a simple example shows that care must be taken in formulating the corresponding Γ-martingale result:

Example 6.1: *Consider the Euclidean plane $\mathbb{R}^2$. Clearly the singleton set $\mathcal{C} = \{\mathbf{o}\}$ is geodesically convex. However the random process $\{(B_{t\wedge\tau}, 0) : t \geq 0\}$, where B is one-dimensional Brownian motion begun at 1 and τ is the stopping time when B first hits 0, provides an example of a Γ-martingale which ends up inside the geodesically convex set $\mathcal{C}$ even though it begins (at $(1,0)$) outside.*

In essence, the problem is that Γ-martingales are really *local* martingales. Of course one can get round this by restricting attention to bounded subsets (this is automatic in the work of Kendall, 1991, 1992a), or by installing a condition of H^p type (Darling, 1996; Thalmaier, 1996). However we do not pursue this here.

Finally, we have noticed above that the property $(\mathbf{D}')$ of Kendall (1991, 1992a) (the "geodesic Liouville property": there are no non-trivial geodesics with coincident end-points, moreover a geodesic with almost coincident end-points is almost trivial) is a slight strengthening of the requirement that singleton sets be geodesically convex. The results of this paper thus provide a *3-dimensional* counterexample to a conjecture by Émery, that $(\mathbf{D}')$ for a compact manifold with boundary implies a further condition $(\mathbf{A}')$; singleton sets can be realized as zero-sets of bounded non-negative convex functions. For suppose that $y \in \mathcal{K}_\varepsilon$ is barycentre to the degenerate probability measure concentrated on $\mathbf{o}$. If $\Phi : \text{ball}(\mathbf{o}, R) \to [0,1]$ is convex, with $\Phi(\mathbf{o}) = 0$ and $y \in \text{ball}(\mathbf{o}, R)$, then the standard Jensen-type inequality for barycentres shows that $\Phi(y) \leq \Phi(\mathbf{o}) = 0$, so $\Phi(y) = 0$ also. It would be interesting to pursue the question of whether Émery's conjecture *does* hold in dimension 2 (for compact manifolds with boundary): this is related to the Γ-martingale question above using the ideas discussed in Kendall (1991, 1992a).

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References

Cartan, H. (1928), *Leçons sur la géometrie des espaces de Riemann*, Gauthiers-Villars, Paris.

Darling, R.W.R. (1996), "Martingales on non-compact manifolds: maximal inequalities and prescribed limits," *Ann. Inst. H. Poincaré. Probab. Statist.* **32**, 431–454.

Émery, M. & P.A. Meyer (1989), *Stochastic Calculus in Manifolds, with an Appendix by P.A. Meyer*, Springer-Verlag, New York–Heidelberg–Berlin.

Émery, M. & G. Mokobodzki (1991), "Sur le barycentre d'une probabilité dans une variéte," in *Sém. Probab. XXV*, Lect. Notes in Math. **1485**, Springer-Verlag, New York–Heidelberg–Berlin, 220–233.

Fréchet, M. (1948), "Les éléments aléatoires de nature quelconque dans un espace distancié," *Ann. Inst. Henri Poincaré* **10**, 215–310.

Jost, J. (1994), "Equilibrium maps between metric spaces," *Calc. Var.* **2**, 173–204.

Karcher, H. (1977), "Riemannian centre of mass and mollifier smoothing," *Comm. Pure Appl. Math.* **30**, 509–541.

Kendall, W.S. (1990), "Probability, convexity, and harmonic maps with small image I: Uniqueness and fine existence," *Proc. London Math. Soc. (3)* **61**, 371–406.

Kendall, W.S. (1991), "Convex geometry and nonconfluent Γ–martingales I: Tightness and strict convexity," in *Stochastic Analysis, Proceedings, LMS Durham Symposium, 11th – 21st July 1990*, M.T. Barlow & N.H. Bingham, eds., 163–178.

Kendall, W.S. (1992a), "Convex Geometry and nonconfluent Γ-martingales II: Well–posedness and Γ-martingale convergence," *Stochastics and Stochastic Reports* **38**, 135–147.

Kendall, W.S. (1992b), "The Propeller: A counterexample to a conjectured criterion for the existence of certain convex functions," *J. London Math. Soc. (2)* **46**, 364–374.

Kobayashi, S. & K. Nomizu (1969), *Foundations of Differential Geometry volume II*, Interscience, New York.

Le, H.L. (1995), "Mean size-and-shapes and mean shapes: a geometric point of view," *Adv. in Appl. Probab.* **27**, 44–55.

Oller, J.M. & J.M. Corcuera (1995), "Intrinsic analysis of the statistical estimation," *Ann. Statist.* **23**, 1562–1581.

Picard, J. (1994), "Barycentres et martingales sur une variété," *Ann. Inst. H. Poincaré. Probab. Statist.* **30**, 647–702.

Rudin, W. (1973), *Functional Analysis*, McGraw-Hill, New York.

Thalmaier, A. (1996), "Brownian motion and the formation of singularities in the heat flow for harmonic maps," *Probab. Theor. Relat. Fields* **105**, 335–367.

Ziezold, H. (1977), "On expected figures and a strong law of large numbers for random elements in quasi-metric spaces," in *Trans. Seventh Prague Conference on Information Theory, Statistical Decision Functions, Random Processes* **A**, Academia, Prague, 591–602.

Ziezold, H. (1989), "On expected figures in the plane," in *Geobild '89*, A. Hübler, W. Nagel, B.D. Ripley & G. Werner, eds., Akademie-Verlag, Berlin, 105–110.

Ziezold, H. (1994), "Mean figures and mean shapes applied to biological figure and shape distributions in the plane," *Biometrical J.* **36**, 491–510.