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Abstract

We study the probability theory of countable dense random subsets of (uncountably infinite) Polish spaces. It is shown that if such a set is stationary with respect to a transitive (locally compact) group of symmetries then any event which concerns the random set itself (rather than accidental details of its construction) must have probability zero or one. Indeed the result requires only quasi-stationarity (null-events stay null under the group action). In passing, it is noted that the property of being countable does not correspond to a measurable subset of the space of subsets of an uncountably infinite Polish space.

Keywords: HAUSDORFF DIMENSION; HITTING SET; POINT PROCESS; QUASI-STATIONARITY; STATIONARITY; STOCHASTIC GEOMETRY; ZERO-ONE LAW
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1 Introduction

The results derived in this paper make it clear how little can be usefully said about stationary random patterns of points which are both countable and dense. If we restrict ourselves to speaking only about what we can observe from the random pattern itself (rather than any mechanism that might have been used in its construction) then we find that there are strong limits on useful statements, summarized in the zero-one law given in Thm. 4.3 below. It seems worth putting these facts on record, not only because their description has surprised specialists who have not happened to consider the question before, but also because they illustrate some of the limits of sensible enquiry for stochastic phenomena.

Of course in a strict sense the notion of a random countable dense set is far from our firmly bounded and finite perception of reality. However it is important to be clear about the limits of the mathematical abstractions used to discuss our perceptions of such reality. As remarked in [11, page 442], "Thus, while as hewers of wood and drawers of water we can afford for most of the time to ignore these matters, we must not neglect them entirely if we care for the quality of the wood and the purity of the water". Indeed countable dense random sets arise naturally in stochastic geometry (for example, the sets of limiting directions of sensed lines from line processes) and in the theory of Brownian motion (the zeroes of one-dimensional Brownian motion measured in the scale of local time) so it is useful to clarify the situations in which probabilistic statements about such sets can be uninformative.

The paper is organized as follows. In the next section we introduce notation and concepts from stochastic geometry. Section 3 discusses measure-theoretic issues, particularly the fact that the property of being countable does not in general correspond to a measurable event. The main results are to be found in Section 4, where we show that a countable dense random set which is quasi-stationary satisfies a zero-one law, at least under a mild extra condition on the basic space (which is certainly satisfied in the case where the stationarity is taken with respect to the symmetries provided by a Lie group). Finally, Section 5 deals with further questions, in particular showing that there is no concealed phenomenological event concerning such countable dense random sets which has non-trivial probability.

It is a pleasure to acknowledge useful discussions on this topic with my colleagues Saul Jacka and Jon Warren, and with Ilya Molchanov of Glasgow University. I am also very grateful to an anonymous referee who pointed out the centrality to this work of the notion of multifunctions.

2 Definitions and notation

We begin by introducing notation for the space from which the points of the random set are to be drawn. In the following we will consider random patterns of points belonging to a basic space $\mathbb{X}$, which we require to be a Polish space (which is to say, $\mathbb{X}$ is a complete separable metric space), locally compact and of uncountable cardinality. The Borel σ-algebra of $\mathbb{X}$ is denoted by $\mathfrak{B}(\mathbb{X})$.

It will be helpful to bear in mind the specific example of the real line $\mathbb{R}$ with its Borel σ-algebra: we discuss the general case because of its interest in stochastic geometry, but the case of the real line carries all the significant technical issues.

The following standard notation from stochastic geometry will be used throughout this paper. (See [20] for a general introduction to stochastic geometry.) We need a notation to express the fact that the (typically random) set A hits (intersects) the (typically non-random) set B: if A, B are subsets of $\mathbb{X}$ then we write

$$A \Uparrow B$$

(A hits B) to signify that $A \cap B \neq \emptyset$. We denote the event that a random set hits a given target set B by the following:

Definition 2.1 (The hitting-set event): *If $B \subseteq \mathbb{X}$ then*

$$\mathbf{[}B\mathbf{]} \quad = \quad \{A \in \mathfrak{B}(\mathbb{X}) : A \Uparrow B\} \qquad (1)$$

defines the hitting-set event *for B.*

With this notation we can define the *hit-or-miss σ-algebra* (introduced to stochastic geometry by Matheron [15]):

Definition 2.2 : *The* **hit-or-miss σ-algebra** $\mathfrak{H}(\mathbb{X})$ *for* $\mathfrak{B}(\mathbb{X})$ *is generated by the events* $\mathbf{[}B\mathbf{]}$ *as B runs through* $\mathfrak{B}(\mathbb{X})$*:*

$$\mathfrak{H}(\mathbb{X}) \quad = \quad \sigma\{\mathbf{[}B\mathbf{]} : B \in \mathfrak{B}(\mathbb{X})\}\,. \qquad (2)$$

Note that $\mathfrak{H}(\mathbb{X})$ is a collection of subsets of the family $\mathfrak{B}(\mathbb{X})$.

In fact it would suffice to consider B running only through the closed sets of $\mathbb{X}$ (this follows by a capacitability argument).

Notice that in Matheron's original definition the σ-algebra $\mathfrak{H}(\mathbb{X})$ is composed only of subsets of the family of closed subsets of $\mathbb{X}$. However we need to consider random sets which are not closed, and therefore must consider subsets of the family $\mathfrak{B}(\mathbb{X})$ of Borel sets.

We now define a *random set*. In fact there are two different ways in which one might think of a random set: constructively and phenomenologically. From the constructive point of view we define a random set as a subset of "probability space $\times$ basic space";

Definition 2.3 (Constructive definition of a random set): *A* random set *based on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ is an $\mathfrak{F} \otimes \mathfrak{B}(\mathbb{X})$-measurable subset*

$$\Xi \quad \subseteq \quad \Omega \times \mathbb{X}\,.$$

Thus an observed random set is a slice

$$\Xi(\omega) \quad = \quad \{x \in \mathbb{X} \,:\, (\omega, x) \in \Xi\} \quad \in \quad \mathfrak{B}(\mathbb{X})\,,$$

where $\omega \in \Omega$.

But what do we actually observe when given a random set? If all reference to the actual construction is to be avoided then it is reasonable to confine ourselves to the σ-algebra generated by the various hitting events $[\![B]\!]$ for $B \in \mathfrak{B}(\mathbb{X})$ (we address this issue further in Thm. 5.1 below). These events are indeed measurable for a countable random set, by the argument sketched after Defn. 2.4 below. So it is natural, following Matheron [15], to consider a notion of a random set involving rather less information, for which we stipulate *only* the measurability of these hitting events. This turns out to be exactly the notion of a *Borel multifunction*: a set-valued function Ξ satisfying the following.

Definition 2.4 (Phenomenological definition of a random set): *Consider a probability space* $(\Omega, \mathfrak{F}, \mathbb{P})$. *The map*

$$\Xi : \Omega \quad \to \quad \mathfrak{B}(\mathbb{X})$$

is a Borel multifunction *if it is measurable as a map from* $(\Omega, \mathfrak{F})$ *to* $(\mathfrak{B}(\mathbb{X}), \mathfrak{H}(\mathbb{X}))$, *which is to say that for all Borel subsets* B *of* $\mathbb{X}$ *we have*

$$\{\omega \,:\, \Xi(\omega) \Uparrow B\} \quad \in \quad \mathfrak{F}\,.$$

Wagner [21, 22] (see also [9]) surveys the very extensive literature on multifunction properties and results.

Note that the above definitions still make sense if the basic space $\mathbb{X}$ fails to be a Polish space. The main requirement is for singleton point sets to be Borel, so that $[\![\{x\}]\!] \in \mathfrak{H}(\mathbb{X})$, thus ensuring that $\Xi^{-1}([\![\{x\}]\!])$ belongs to $\mathfrak{F}$. This is important if the random countable set

$$\Xi \quad = \quad \{\xi_1, \xi_2, \ldots\}\;,$$

(produced using $\mathbb{X}$-valued random variables ξ_1, ξ_2, $\ldots$) is to yield a Borel multifunction in the sense of Defn. 2.4.

The contrast between random sets and multifunctions reflects a fundamental tension between synthetic and analytic approaches to probability. The synthetic approach defines random quantities by construction, as in Defn. 2.3. (Indeed from a historical perspective this is the original way in which random sets were defined: see [17, 18].) The analytic approach describes random quantities using their probability distributions on suitable σ-algebras, just as in the use of the hit-or-miss σ-algebra $\mathfrak{H}(\mathbb{X})$ in the phenomenological Defn. 2.4. If the random set Ξ is almost surely closed then Defns. 2.3 and 2.4 are equivalent: this is a consequence of Himmelberg's theorem [6, Thm. 3.5].

The work of Aldous and Barlow [1], on countable dense random subsets of the real line, is close in subject matter to the material treated here. However Aldous and Barlow treat dense *Poisson* point processes using the synthetic approach, considering the dense Poisson point process as defined using a projection from some higher-dimensional space, and using the higher-dimensional information to formulate their results. In the following we are guided by the analytic approach; we will aim for results concerning random sets of the form given in Defn. 2.3 but based on observation of the random set itself, without making reference to whatever stochastic mechanism has been used for its construction. Thus our results concern random sets defined using Defn. 2.3 but viewed in terms of Defn. 2.4, using the hitting σ-algebra $\mathfrak{H}(\mathbb{X})$.

Finally in this section we introduce notions of stationarity. All the essential technicalities in the following arise in the case of the real line $\mathbb{X} = \mathbb{R}$ with symmetry group given by translations; we would then abbreviate $\mathfrak{B} = \mathfrak{B}(\mathbb{R})$, $\mathfrak{H} = \mathfrak{H}(\mathbb{R})$. and let $T_x : \mathbb{R} \to \mathbb{R}$

be the translation $T_x(y) = x + y$. However we give a general treatment here because the results of this paper are of interest in stochastic geometry, where both the basic space and the symmetry group can be considerably more general.

Recall the following basic facts about invariant measures, to be found for example in [12] or in part in [5]. If G is a locally compact topological group then it carries a left-invariant Borel measure μ, the *Haar measure*, which is unique up to a multiplicative constant. In general this measure is not right-invariant, but there is a *modular function*

$$\Delta : G \quad \to \quad (0, \infty) \tag{3}$$

such that for $g \in G$ the right-translated measure $R_g\mu$ satisfies the identity $R_g\mu = \Delta(g)\mu$. In fact Δ is a continuous homomorphism [12, page 117].

Suppose that G acts transitively on a locally compact space $\mathbb{X}$, with stability group the closed subgroup $H \leq G$, so that we can view $\mathbb{X}$ as

$$\mathbb{X} \quad = \quad G/H\,. \tag{4}$$

Then $\mathbb{X} = G/H$ carries an invariant measure (unique up to a multiplicative constant) if and only if the modular functions Δ and δ of G and H are related by

$$\Delta(h) \quad = \quad \delta(h) \qquad \text{for all } h \in H\,. \tag{5}$$

Moreover, given a Borel measurable selection $s : G/H \to G$ (so that $s(gH)H = gH$), there is a natural construction of the invariant measure ν which will be useful to us later. (The existence of a Borel measurable selection follows here from a theorem due to Mackey: see [21, Theorem 11.6].) Suppose that Eq. (5) holds and fix $W \subseteq H$ to be a relatively compact open subset of H. If Z is a Borel subset of G/H then

$$\nu(Z) \quad = \quad \text{constant} \times \mu\left\{s(z) \times w \;:\; z \in Z, w \in W\right\} \tag{6}$$

is an invariant measure on G/H. (In the case when the stability group H is compact the construction can be simplified by using $W = H$.)

We define extensions of the G-action $g : \mathfrak{B}(\mathbb{X}) \to \mathfrak{B}(\mathbb{X})$ and $g : \mathfrak{H}(\mathbb{X}) \to \mathfrak{H}(\mathbb{X})$ in the canonical way.

In the following we shall assume that the basic space $\mathbb{X}$ can be written as $\mathbb{X} = G/H$ with symmetry group G, such that the modularity relation Eq. (5) applies, with resulting invariant measure ν on $\mathbb{X} = G/H$. Examples include:

1. the real line $\mathbb{R}$ with symmetry group given by translations;
2. Euclidean space $\mathbb{R}^d$ with symmetry group given by rigid motions;
3. the Euclidean sphere S^{d-1} (for $d > 1$) with symmetry group the special orthogonal group SO(d);
4. the space of unimodular lattices in $\mathbb{R}^d$ with symmetry group given by the special linear group SL$(d, \mathbb{R})$.

The last example differs in that the stability subgroup $H =$ SL$(d, \mathbb{N})$ is non-compact. It has arisen in stochastic geometry in the study of line processes; see the letter from Kingman presented by Kallenberg in [10].

Remark 2.5 : *It is* not *the case that invariant measures can be found for all transitive homogeneous spaces with locally compact group symmetry. Consider for example Euclidean space with symmetry group given by the group of affine transformations. Rather clearly this cannot have an invariant measure, since the symmetry group can be used to transform compact sets into strictly smaller compact sets!*

We now define the notion of a stationary random subset of $\mathbb{X}$. It is convenient also to introduce the weaker notion of "quasi-stationarity", corresponding to quasi-invariance of measures as discussed for example by Mackey [14, page 134], since this arises naturally in the hypotheses of our main results.

Definition 2.6 (Stationarity for random sets): *Let Ξ be a random subset of $\mathbb{X}$ based on $(\Omega, \mathfrak{F}, \mathbb{P})$, and let G act transitively as above. We say*

(i) Ξ *is* G-stationary *if, for all $E \in \mathfrak{H}(\mathbb{X})$, the probability $\mathbb{P}\left[\, g\Xi \in E \,\right]$ doesn't depend on $g \in G$;*

(ii) Ξ *is* quasi-G-stationary *if, for all $E \in \mathfrak{H}(\mathbb{X})$ and for all $g \in G$, the probability $\mathbb{P}\left[\, g\Xi \in E \,\right]$ vanishes if and only if $\mathbb{P}\left[\, \Xi \in E \,\right] = 0$.*

In the following we shall generally omit the adjectival "G-", as we will only consider stationarity and quasi-stationarity with respect to a single fixed symmetry group G.

3 Countable random sets

In order to discuss countable dense random sets in an analytic approach, we first need to consider how to specify when a random set is countable. We insisted on the basic space $\mathbb{X}$ being uncountable exactly in order to avoid trivialities here.

It would be convenient if we could show that the set U of all countable sets belongs to $\mathfrak{H}(\mathbb{X})$. However it is rather easy to show that (except in the trivial case of countable $\mathbb{X}$) the set of all countable random sets is not measurable with respect to the hit-or-miss σ-algebra, as shown in the following theorem. (Note that the proof of the theorem, rather unsurprisingly, depends on the Axiom of Countable Choice. It is closely related to the ideas of Vitali's classic example of a non-measurable set as described, for example, in [4, §2.7.17, page 141]. Hoffman-Jørgensen [8, Vol. 2, P.12, pages 502ff] gives a probabilistically-orientated discussion of logical axioms.)

Theorem 3.1 (Non-measurability of the countable property): *The hit-or-miss σ-algebra $\mathfrak{H}(\mathbb{X})$ (for uncountable basic space $\mathbb{X}$) does not contain the set of all countable sets.*

Proof: Suppose $E \in \mathfrak{H}(\mathbb{X})$. By standard measure-theoretic arguments E belongs to a countably-generated sub-σ-algebra

$$\sigma\left\{ [\![B_1]\!], [\![B_2]\!], \ldots \right\}$$

for some sequence of Borel sets B_1, B_2, ... from $\mathfrak{B}(\mathbb{X})$. Without loss of generality we may suppose all of the B_i are non-empty. By the Axiom of Countable Choice we may choose $x_i \in B_i$ for each $i = 1, 2, \ldots$, and consider the countable set

$$C \quad = \quad \{x_1, x_2, \ldots\} \, .$$

By construction $C \Uparrow B_i$ for all i. However it is also trivially true that $\mathbb{X} \Uparrow B_i$ for all i.

If $\mathbb{X} \in E$ then (since $\mathbb{X}$ is uncountable) E cannot be the set of all countable sets in $\mathbb{X}$.

On the other hand, if $\mathbb{X} \not\in E$ we can argue that $C \not\in E$. For by construction of C we know $C \Uparrow B_i$ agrees with $\mathbb{X} \Uparrow B_i$ for all i. A routine measure-theoretic argument then shows that $\mathbb{X}$ and C must agree in membership or non-membership of all sets in

$$\sigma\left\{ [\![B_1]\!], [\![B_2]\!], \ldots \right\} \, .$$

Therefore either C, $\mathbb{X}$ both belong to E or they both do not belong. So it follows that if $\mathbb{X} \not\in E$ then $C \not\in E$.

Thus, whether or not $\mathbb{X} \in E$, we deduce E is not the set of all countable sets in $\mathbb{X}$, and the result follows. □

In practice the Axiom of Countable Choice is not required, as in any practical presentation of $\mathbb{X}$ there will be some means of selecting x_i from B_i.

Of course there are probability measures $\mathbb{P}$ on $(\mathfrak{B}(\mathbb{X}), \mathfrak{H}(\mathbb{X}))$ giving outer measure 1 to the set of all countable sets (for example, any Poisson point process with intensity measure a diffuse σ-finite measure μ on $\mathbb{X}$). The phenomenon above is comparable with the non-Borel-measurability of the set of continuous paths in general path-space, which does not preclude the construction of Brownian motion as a random continuous path.

We digress briefly to sketch an argument to show that the property of being of Hausdorff dimension at most α is also not measurable with respect to the hit-or-miss σ-algebra. Suppose the basic space $\mathbb{X}$ can be viewed as a complete separable metric space of Hausdorff dimension β for $\beta > \alpha$, furnished with the Borel σ-algebra. For convenience we require a certain dimensional uniformity: namely, that within each non-empty metric ball B_i we may construct a subset H_i of Hausdorff dimension α. Once again, any $E \in \mathfrak{H}(\mathbb{X})$ lies in a sub-σ-algebra which is countably generated. Moreover we may suppose that the countable sequence of subsets of $\mathbb{X}$ used to generate the sub-σ-algebra is a sequence of non-empty metric balls $B_1, B_2, \ldots$.

Now construct a set H by taking the countable union of subsets $H_i \subset B_i$ of Hausdorff dimension α. Then H also has Hausdorff dimension α. Arguing as in Theorem 3.1, we deduce that H and $\mathbb{X}$ belong or do not belong to E together, and accordingly that $E \in \mathfrak{H}(\mathbb{X})$ cannot be the set of all subsets of Hausdorff measure at least α.

The appeal to the Axiom of Countable Choice is circumvented here by our (very reasonable) assumption that we can explicitly *construct* $H_i \subset B_i$.

The failure of measurability of the countability property, as described by Thm. 3.1, opens up an ambiguity. What do we *mean* by saying a random set is countable? There are two possible alternative definitions. Let $\Xi : (\Omega, \mathfrak{F}, \mathbb{P}) \to (\mathfrak{B}(\mathbb{X}), \mathfrak{H}(\mathbb{X}))$ be a random set as defined in Defn. 2.3.

Definition 3.2 (Weak countability): *We say* Ξ *is* weakly countable *if the image* $\operatorname{Im}(\Xi)$ *is almost surely contained in the (non-measurable) set of countable sets.*

The alternative is a constructive approach. In the literature there can be found two main ways of constructing countable random sets:

(1) as $\Xi = \{\xi_1, \xi_2, \ldots\}$ using an explicit sequence ξ_1, ξ_2, $\ldots$ of $\mathbb{X}$-valued random variables;

(2) as the projection of an appropriate locally finite point process on $\mathbb{X} \times \mathbb{Y}$, where $\mathbb{Y}$ is another locally compact Polish space.

Random countable dense sets can arise naturally in other ways (for example, the starting times for Brownian excursions measured in the scale of local time at zero), but typically can be recast easily in one of the two above forms. Moreover *all* countable random sets of the form given in Defn. 2.3 can be represented in the sequential form (1) above: we formulate this below in Thm. 3.4. This motivates the alternative definition:

Definition 3.3 (Constructive countability): *We say* Ξ *is* constructively countable *if there exists a sequence (possibly of finite length, and this length if finite may itself be random)*

$$\xi_1, \xi_2, \ldots : \Omega \quad \to \quad \mathbb{X}$$

of random variables ξ_1, ξ_2, $\ldots$ *such that*

$$\mathbb{P}\left[\Xi = \{\xi_1, \xi_2, \ldots\}\right] \quad = \quad 1\,. \tag{7}$$

The notions of weak and constructive countability are in fact equivalent, using an argument due to Lusin [13, pages 237ff] (see also [21, Cor. 10.2]) which can be obtained by a straightforward argument using transfinite induction and the celebrated *section theorem* of measure theory:

Theorem 3.4 (The equivalence of weak and constructive countability): *The random set Ξ based on the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ is weakly countable if and only if it is constructively countable.*

In general this equivalence runs into difficulties for Borel multifunctions, as can be seen from [7, Example 5]. Consider Ω to be the space of all countable nonempty sets σ of real numbers, and take $\mathfrak{F} = \mathfrak{H}(\mathbb{R})$. If F is the multifunction given by $F(\sigma) = \sigma$ then it is trivially a Borel multifunction, but (following from [2, Cor. 2]) there is no Borel measurable function f defined on Ω such that $f(\sigma) \in \sigma$ for all σ. Thus this particular Borel multifunction is not derived from a random set, and delivers weak countability without constructive countability.

As our results concern random sets viewed using the hitting σ-algebra $\mathfrak{H}(\mathbb{X})$, we shall no longer distinguish between weak and constructive countability, but shall instead simply speak of *countable* random sets.

4 A zero-one law for dense countable random sets

We consider a basic space $\mathbb{X}$ which has a transitive symmetry group G (a locally compact topological group with left-invariant Haar measure μ) and which has invariant measure ν. To obtain understanding in a simple case the reader may find it convenient to consider the real line case $\mathbb{X} = \mathbb{R}$, together with the group of translations, in what follows.

We first prove a partial result for the completely general case.

Theorem 4.1 : *Let Ξ be a quasi-stationary countable random subset of $\mathbb{X}$ based on the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. For $A \in \mathfrak{B}(\mathbb{X})$ (so A is a Borel subset of $\mathbb{X}$) the invariant measure $\nu(A)$ is non-zero if and only if $\mathbb{P}[\,\Xi \Uparrow A\,] > 0$.*

Proof: Suppose that $\mathbb{P}[\,\Xi \Uparrow A\,] > 0$. Then $\mathbb{P}[\,g\Xi \Uparrow A\,] > 0$ for all $g \in G$, by quasi-stationarity. Using a representation $\Xi = \{\xi_1, \xi_2, \ldots\}$ of the countable set Ξ, and subadditivity of probability,

$$\sum_{i=1}^{\infty} \mathbb{P}[\,g\xi_i \in A\,] \quad > \quad 0$$

for all $g \in G$. For convenience, set

$$f_i(g) \quad = \quad \mathbb{P}[\,g\xi_i \in A\,]$$

so that

$$\sum_{i=1}^{\infty} f_i(g) > 0 \qquad \text{for all } g \in G.$$

It follows that for at least one i we have that

$$D_i \quad = \quad \{g : f_i(g) > 0\}$$

is a set of positive μ-measure.

Now this means the following: there is a relatively compact open set $O \subset G$ such that $D_i \cap O$ is of positive μ-measure and therefore

$$\mathbb{P}[U\xi_i \in A] \quad = \quad \frac{1}{\mu(O)} \int_O f_i(g)\mu(\mathrm{d}g) \quad > \quad 0$$

where U is a G-valued random variable μ-uniformly distributed over the open set O and independent of the construction of Ξ. We deduce that A is charged by the law of $U\xi_i$. But $U\xi_i$ has a probability density with respect to the invariant measure ν. Therefore A is of positive invariant measure: $\nu(A) > 0$.

Suppose on the other hand that A is of positive ν-measure. Then it follows that for all large enough relatively compact open subsets O of G, and for U as above, then

$$\mathbb{P}[U\Xi \Uparrow A] \quad \geq \quad \mathbb{P}[U\xi_1 \in A] \quad > \quad 0\,.$$

It follows that

$$\frac{1}{\mu(O)} \int_O \mathbb{P}[g\Xi \Uparrow A]\,\mu(\mathrm{d}g) \quad > \quad 0$$

and therefore, for some (in fact $\mu|_O$-almost all) $g \in G$,

$$\mathbb{P}[g\Xi \Uparrow A] \quad > \quad 0\,.$$

Therefore $\mathbb{P}[g\Xi \Uparrow A] > 0$ for some $g \in G$, and so (by quasi-stationarity) for all g including the case $\mathbb{P}[\Xi \Uparrow A] > 0$ when g equals the identity. □

The main result of this paper is to show that this can be strengthened to a zero-one law for quasi-stationary countable *dense* random sets. It suffices to show that if A is of positive ν-measure then $\mathbb{P}[\Xi \Uparrow A] = 1$. We require two extra regularity conditions in order to make the proof work. Firstly, the topology for the basic space must be generated by open sets with boundaries which are negligible with respect to invariant measure, and secondly there must be local continuous selections lifting the basic space to the symmetry group. Both are satisfied in the case when the symmetry group G is a Lie group, since then it is a consequence of the closed subgroup theorem [23, Thm. 3.42] that the closed subgroup H is also Lie, G/H is a manifold, and locally the set-up is Euclidean with H lying in G as an imbedded sub-manifold. We therefore impose the requirement that G is Lie for the next lemma and theorem.

We begin with a preparatory lemma formulating and proving a version of Lebesgue's density theorem.

Lemma 4.2 : *Suppose that the Polish metric space $\mathbb{X}$ is a transitive homogeneous G-space, with symmetry group a Lie group G, and stability group a closed subgroup $H < G$. Suppose further that $\mathbb{X}$ possesses a G-invariant measure ν. Suppose that a Borel subset $A \subseteq \mathbb{X}$ has positive ν-measure. Then we can find a point $a \in A$ and a decreasing sequence $\{N_n : n \geq 1\}$ of relatively compact open sets whose boundaries are of null ν-measure, such that*

(a) $a \in N_n$;

(b) $N_n \downarrow \{a\}$;

(c) $\rho_n = \nu(N_n \cap A)/\nu(N_n) \to 1$ *as* $n \to \infty$.

Proof: Since $\nu(A) > 0$, there is a point $a \in A$ of full density by Lebesgue's density theorem. For the case of $\mathbb{X} = \mathbb{R}$ this is a standard result in real analysis [19, Theorem 8.8]: for every $n = 1, 2, \ldots$ we can find an interval (c_n, d_n) such that

(a') $a \in (c_n, d_n)$;

(b') $d_n - c_n = 2^{-n}$ and $(c_{n+1}, d_{n+1}) \subseteq (c_n, d_n)$;

(c') $\rho_n = \text{Leb}((c_n, d_n) \cap A)/2^{-n} \to 1$ as $n \to \infty$.

It also applies in the general setting of the G-space $\mathbb{X}$ when G is a Lie group; a general proof can be obtained using discrete-time martingale theory as follows. Without loss of generality suppose that $0 < \nu(A) < \infty$. Because the homogeneous space $\mathbb{X}$ is derived from a Lie group it is a smooth manifold, and therefore we can construct a filtration $\{\mathcal{C}_n : n \geq 1\}$ of locally finite coverings $\mathcal{C}_n$ of $\mathbb{X}$ using relatively compact open sets with boundaries of ν-measure zero. This filtration in turn allows us to construct a sequence of functions $f_n : \mathbb{X} \to [0, \infty)$ as follows: the covering $\mathcal{C}_n$ produces an algebra $\mathcal{A}(\mathcal{C}_n)$ of subsets of $\mathbb{X}$ and if $x \in \mathbb{X}$ belongs to the atom H of $\mathcal{A}(\mathcal{C}_n)$ then

$$f_n(x) \quad = \quad \frac{\nu(A \cap H)}{\nu(H)} \,.$$

(We set the ratio equal to zero if $\nu(H) = 0$.)

Let U be ν-uniformly distributed over A. Then the sequence $\{f_n(U) : n \geq 1\}$ defines a bounded martingale (by properties of discrete conditional expectation) which converges almost surely to 1 (as can be seen by approximating the indicator of A using indicators of elements of $\mathcal{A}(\mathcal{C}_n)$). Hence there is a point $a \in A$ of full density (in fact, the set of such a is of positive ν-measure): for every $n = 1, 2, \ldots$ we can find a relatively compact open set N_n of positive ν-measure such that

(a) $a \in N_n$;

(b) $N_n \downarrow \{a\}$;

(c) $\rho_n = \nu(N_n \cap A)/\nu(N_n) \to 1$ as $n \to \infty$,

as required. In fact if H_n is the atom of $\mathcal{A}(\mathcal{C}_n)$ containing the random variable U then we can set $N_n = \text{int}(H_n)$, arguing that with probability one all the H_n will be open and of positive measure and their boundaries (being of measure zero) will not contain U. □

Theorem 4.3 (A zero-one law for quasi-stationary dense random sets): *Suppose that the Polish metric space $\mathbb{X}$ is a transitive homogeneous G-space, with symmetry group a Lie group G, and stability group a closed subgroup $H < G$. Suppose further that $\mathbb{X}$ possesses a G-invariant measure ν. Let Ξ be a quasi-stationary countable dense random subset of $\mathbb{X}$ based on the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. Then for any Borel $A \in \mathfrak{B}(\mathbb{X})$ it is the case that $\mathbb{P}[\,\Xi \Uparrow A\,] = 1$ if and only if A is of non-zero ν-measure. Moreover for* any *$E \in \mathfrak{H}(\mathbb{X})$ we have*

$$\mathbb{P}[\,\Xi \in E\,] \quad = \quad 0 \textit{ or } 1 \tag{8}$$

depending only on $E \in \mathfrak{H}(\mathbb{X})$ and the nature of the symmetry afforded by G via *the invariant measure ν, but* not *on the specific random set Ξ.*

Proof: Thm. 4.1 proves half of this result, so it suffices to consider the case when a set $A \subseteq \mathbb{X}$ is given which is of positive ν-measure.

It is convenient to fix a relatively compact open subset $O \subset G$, and to replace Ξ by $\Xi' = V\Xi$, where V is a G-valued random variable uniformly distributed over O. It is immediate that Ξ' is also a countable dense quasi-stationary random set. As we will explain in detail below, if $\mathbb{P}[\,\Xi' \Uparrow A\,] = 1$ then quasi-stationarity implies that $\mathbb{P}[\,\Xi \Uparrow A\,] = 1$.

Fix a constructively countable representation

$$\Xi' \quad = \quad \{\xi_1, \xi_2, \ldots\}$$

of the countable dense random set Ξ'.

Now apply Lemma 4.2 to obtain a density point $a \in \mathbb{X}$ for the set A in question, with corresponding decreasing sequence of open sets $\{N_n : n \geq 1\}$.

Let $Q \subseteq H$ be a relatively compact open neighbourhood of the identity in the subgroup H. Using our discussion of invariant measure for homogeneous spaces in Section 2, and particularly Eq. (6) together with the existence (following from the remarks before Lemma 4.2, since G is Lie and H is a closed subgroup) of a *locally continuous* selection $s : G/H \to G$ which is chosen to be continuous in a neighbourhood of the density point a, we can construct sets $M_n = s(N_n) \times Q \subseteq G$ such that;

- M_n is decreasing in n, and all M_n contain $s(a)$;
- M_n is open for all large enough n (specifically, n such that s is continuous when restricted to N_n);
- if the G-valued random variable U_n is μ-uniform on M_n (and independent of Ξ'), then

$$\mathbb{P}\left[\, U_n H \in A \,\right] \quad = \quad \nu[N_n \cap A] \quad \to \quad 1$$

as $n \to \infty$.

Without loss of generality we may discard the initial part of the sequence $\{M_n : n \geq 1\}$ and suppose that in fact M_n is open for all n.

Choose a decreasing sequence $\{W_n : n \geq 1\}$ of relatively compact open subsets of G which are neighbourhoods of the identity and such that $\operatorname{diam}(W_n) \to 0$.

Using the fact that the countable random set Ξ' is dense in $\mathbb{X} = G/H$, we can find $g_n \in W_n \subseteq G$ (depending implicitly on Ξ') such that $g_n s(a) H \in \Xi'$.

Let $f_n(\cdot)$ be the density of U_n on G with respect to the Haar measure μ, and let $L_{g_n} f_n(\cdot)$ be the density of $g_n U_n$. (The notation $L_{g_n} f_n$ is standard for the left-translate of the function f_n by the group element g_n.) Their supports M_n and $g_n M_n$ are contained in the relatively compact set $W_1 \times M_1$. Furthermore both $f_n(\cdot)$ and $L_{g_n} f_n(\cdot)$ take just two values: either zero or $1/\mu(U_n) = 1/\mu(g_n U_n)$ (recall that μ is left-invariant). Finally the supports are both open sets with boundaries of null μ-measure, and the fact that g_n converges to the identity on G therefore implies

$$f_n - L_{g_n} f_n \quad \to \quad 0 \qquad \mu\text{-almost everywhere}\,.$$

We can therefore deduce from the dominated convergence theorem that

$$\frac{1}{\mu(M_n)} \int_{g \in G : gH \in A} (f_n - L_{g_n} f_n)\, \mathrm{d}\mu \;=\; \mathbb{P}\left[\, U_n H \in A \,\right] - \mathbb{P}\left[\, g_n U_n H \in A \,\right] \;\to\; 0\,.$$

It follows from $\mathbb{P}\left[\, U_n H \in A \,\right] = \nu[N_n \cap A] \to 1$ that

$$1 \quad \geq \quad \mathbb{P}\left[\, \Xi' \Uparrow A \,\right] \quad \geq \quad \mathbb{P}\left[\, g_n U_n H \in A \,\right] \quad \to \quad 1\,.$$

Now $\mathbb{P}\left[\, \Xi' \Uparrow A \,\right] = \int_O \mathbb{P}\left[\, g\Xi \Uparrow A \,\right] \mu(\mathrm{d}g)/\mu(O)$ and therefore $\mathbb{P}\left[\, g\Xi \Uparrow A \,\right] = 1$ for μ-almost all $g \in O$, and indeed for μ-almost all $g \in G$, since O was arbitrary. Therefore by quasi-stationarity we deduce

$$\mathbb{P}\left[\, \Xi \Uparrow A \,\right] \quad = \quad 1 \qquad \text{if } \nu(A) > 0\,. \tag{9}$$

The final conclusion (the zero-one law for $U \in \mathfrak{H}(\mathbb{X})$, not depending on the specific construction of Ξ) follows by standard measure theory arguments once we recall the definition of $\mathfrak{H}(\mathbb{X})$; that it is generated by the hitting events $[\Xi \Uparrow A]$ which are zero or one depending on whether the invariant measure of A is zero or positive. □

As an immediate consequence of this result we obtain some apparently paradoxical facts:

Remark 4.4 : *The zero-one law for the quasi-stationary countable dense random set* Ξ *implies that for all* $\mathfrak{H}(\mathbb{X})$*-measurable random variables* Y *we have* $\mathbb{P}[Y \in \Xi] = 0$*. So there are no Borel selections of points in* Ξ*, and the randomness in* Ξ *is "hidden", if all we can use is hit-or-miss information.*

Remark 4.5 : *If one considers the stationary countable dense random set* $\Xi = \mathbb{Q} + U$*, where* $\mathbb{Q}$ *is the set of rationals and* U *is Uniform*$(0, 1)$*, then this translates into the known fact that one cannot produce a* $\mathfrak{H}$*-measurable selection for the fibration of* $\mathbb{R}$ *induced by its subgroup* $\mathbb{Q}$*. This is related to Vitali's example of a non-Lebesgue-measurable set mentioned above before Thm. 3.1.*

Remark 4.6 : *As a further consequence, we can deduce that* any *two stationary countable dense random sets* Ξ_1*,* Ξ_2 *on* $\mathbb{R}$ *are independent, in the sense that*

$$\mathbb{P}[\Xi_1 \in U_1, \Xi_2 \in U_2] \quad = \quad \mathbb{P}[\Xi_1 \in U_1] \times \mathbb{P}[\Xi_2 \in U_2]$$

whenever $U_1, U_2 \in \mathfrak{H}$*. This follows for the trivial reason that all sets of probability zero or one are independent of each other.*

5 Further questions and comments on the literature

One might wonder whether there is some more complicated σ-algebra for $\mathfrak{B}$ than $\mathfrak{H}$, which supports non-trivial events for stationary countable dense random sets but does not use randomness not somehow available from observation of the random set in question. This can be formulated as follows. Let $\Xi \subset E \times \mathbb{X}$ be a countable dense random set such that

$$\{\xi_1, \xi_2, \dots\}$$

is a representation as a constructively countable random set. The particular representation is arbitrary: so we introduce the notion of *set-equivalence* of two sequences of $\mathbb{X}$-valued random variables by

$$\begin{aligned} \underline{\xi} = \{\xi_1, \xi_2, \dots\} \quad &\sim \quad \underline{\eta} = \{\eta_1, \eta_2, \dots\} \\ \text{if and only if} & \\ \{\xi_1(\omega), \xi_2(\omega), \dots\} \quad &= \quad \{\eta_1(\omega), \eta_2(\omega), \dots\} \text{ for almost all } \omega\,. \end{aligned}$$

Here the equality is taken in the sense of equality of the two sets (*ie:* order of enumeration is irrelevant).

We set

$$\mathfrak{G} \quad = \quad \bigcap_{\underline{\eta}:\underline{\eta}\sim\underline{\xi}} \sigma\{\eta_1, \eta_2, \dots\} \tag{10}$$

so that events in $\mathfrak{G}$ are exactly those which can be formulated under arbitrary (including random!) reorderings of the sequence used to derive constructive countability. Thus $\mathfrak{G}$ encompasses the widest possible class of events which might be viewed as concerning the random set Ξ itself, rather than the means of its construction.

Theorem 5.1 : *It is the case that* $\mathfrak{G}$ *is contained in any completion of the* σ*-algebra* $\mathfrak{H}(\mathbb{X})$*.*

Proof: This is actually a problem about *fibered spaces*. Let $\mathbb{S}$ be the space of all $\mathbb{X}$-valued sequences, with the usual σ-algebra $\mathfrak{S}$, generated by evaluation maps. Then we have a sequence of measurable projections

$$\pi : (\mathbb{S}, \mathfrak{S}) \to (\mathbb{S}, \mathfrak{G}) \to (\mathfrak{B}(\mathbb{X}), \mathfrak{H}(\mathbb{X})) \tag{11}$$

representing $\mathbb{S}$ as a fibration over the space $\mathfrak{B}(\mathbb{X})$ of (in fact countable) subsets of $\mathbb{X}$.

The matter resolves to the study of sets $E \subseteq \mathfrak{B}(\mathbb{X})$ such that E is the *projection* of some $\mathfrak{G}$-measurable set $\tilde{E} \in \mathbb{S}$, so

$$E \quad = \quad \left\{ \pi(\alpha) \,:\, \alpha \in \tilde{E} \right\} .$$

Note that $\tilde{E}$ will also be $\mathfrak{S}$-measurable. Hence (since the space $\mathbb{S}$ of $\mathbb{X}$-valued sequences can be metrized so as to be Polish with σ-algebra $\mathfrak{S}$) we see the projection set E is analytic ([3, Theorem III.13]). Now the issue can be recognized as one which concerns analytic sets and Choquet's famous capacitability theorem ([3, III.33]). In general projection sets E need *not* be measurable in the sense of belonging to $\mathfrak{H}(\mathbb{X})$: however they will be measurable with respect to any completion of $\mathfrak{H}(\mathbb{X})$. □

Clearly this settles the matter as far as zero-one laws are concerned.

Ilya Molchanov has asked the question, whether Thm. 4.3 above can be generalized to apply to a stationary random subset of $\mathbb{R}$ which is everywhere dense and yet which has probability zero of containing any specified point. The answer is no, as may be seen from the following examples.

In the first example we use a theorem of Mattila [16, Theorem 3.2] on intersections of fractal sets. This implies that if H, $\tilde{H} \subset [0,1]^2$ are sets of finite positive Hausdorff $5/3$-dimensional measure then the set of rigid motions $\mathfrak{r}$ such that $\mathfrak{r}H \Uparrow \tilde{H}$ has positive Haar measure. Note that the conditions of this theorem force us to consider at least $\mathbb{R}^2$ rather than $\mathbb{R}$!

Example 5.2 : *The random set $\Xi \subset \mathbb{R}^2$ is constructed as follows. Let $\mathfrak{r}$ be a random rigid motion, obtained by first uniformly randomly translating the origin $\mathbf{o}$ over the unit square $[0,1]^2$ and then applying a uniform random rotation. Let $\mathbb{Z}^2 \subset \mathbb{Q}^2 = \mathbb{Q} \times \mathbb{Q}$ denote the integer lattice. Let H, $\tilde{H} \subset [0,1/3]^2$ be sets of finite positive Hausdorff $5/3$-dimensional measure, so that the probability of $\mathfrak{r}H$ intersecting $\tilde{H}$ lies strictly between 0 and 1. Consider the* uncountable *dense random set*

$$\Xi \quad = \quad \mathfrak{r}\left(\mathbb{Q}^2 \cup \bigcup_{z \in \mathbb{Z}^2} (z + H) \right) .$$

This stationary (and isotropic) dense random set generates non-trivial probabilities for the hit-or-miss σ-algebra $\mathfrak{H}(\mathbb{R}^2)$.

To see this, consider the event $\Xi \Uparrow \tilde{H}$. This has probability lying strictly between 0 and 1. For

$$\mathbb{P}\left[\Xi \Uparrow \tilde{H} \right] \quad = \quad \mathbb{P}\left[\mathfrak{r}\left(\bigcup_{z \in \mathbb{Z}^2} (z + H) \right) \Uparrow \tilde{H} \right] ,$$

since the countable part $\mathfrak{r}(\mathbb{Q}^2)$ of Ξ misses $\tilde{H}$ almost surely, by Thm. 4.1. But $\mathfrak{r}(z+H) \cap \tilde{H}$ is empty when $\mathfrak{r}$ is a random rigid motion produced as above, and z is a member of $\mathbb{Z}^2 \setminus (\text{ball}(\mathbf{o}, \sqrt{2}/3) \oplus [-4/3, 0]^2)$, since H and $\tilde{H}$ are both contained in $[0,1/3]^2$ and hence $\mathfrak{r}(z + H)$ is disjoint from $\tilde{H}$ for such z. Consequently the Mattila result shows

$$\mathbb{P}\left[\Xi \Uparrow \tilde{H} \right] \quad = \quad \mathbb{P}\left[\left(\bigcup_{z \in \mathbb{Z}^2 \cap (\text{ball}(\mathbf{o}, \sqrt{2}/3) \oplus [-4/3, 0]^2)} \mathfrak{r}(z + H) \right) \Uparrow \tilde{H} \right] \quad \in \quad (0,1) .$$

So there is no zero-one law for probabilities of hitting events in this case.

The above is a two-dimensional example: here is a less explicit example in just one dimension, which in addition is ergodic and mixing:

Example 5.3 : *The random set $\Xi \subset \mathbb{R}$ is constructed as follows. Split $\mathbb{R}$ into domains according to a unit intensity Poisson process, such that in each domain independently Ξ is*

- either *the intersection with the domain of a random translate of the rationals $\mathbb{Q}$,*
- or *the intersection with the domain of a random translate of $\mathbb{Q} \oplus \mathcal{Z}$, the rationals Minkowski-added to the zero set $\mathcal{Z}$ of a linear Brownian motion.*

In each case the random translate vector should be chosen uniformly distributed over $[0, 1]$ independently of the Poisson process and of other translations, and the Brownian motions should be independent of the Poisson processes and the translations.

There are bounded sets of measure zero which have a positive chance of being hit by linear Brownian motion (for example the zero set of a Bessel(ν) process for $\nu \in (0, 1)$, continued by reflection at the origin): such sets will have a positive but not certain chance of being hit by the Ξ whose construction is indicated above.

Our final example is motivated by problems from the theory of Brownian local times, which lead to pairs of random countable dense sets which exhibit dependence phenomena appearing to contradict the observation in Remark 4.6 above. Saul Jacka and Jon Warren proposed the following simplified example. (More complicated examples involve non-independent Brownian motions which share some but not all of their respective zero sets.)

Example 5.4 : *Consider three independent Poisson point processes X, Y, Z living on the infinite rectangle $[0, 1] \times \mathbb{R}$. Suppose their intensity measures are given by* Leb $\times$ Leb, Leb $\times \nu$, *and* Leb $\times$ Leb *respectively, where* Leb *is Lebesgue measure and ν is some measure on $\mathbb{R}$, possibly finite, possibly σ-finite, possibly null. Imagine the points of X, Y, Z being coloured red, white, blue respectively. Now construct the new point processes $\Xi_1 = \pi(X \cup Y)$, $\Xi_2 = \pi(Y \cup Z)$ where $\pi : [0, 1] \times \mathbb{R} \to [0, 1]$ is projection onto the first coordinate.*

Clearly, depending on the total mass of ν, we have three different situations depending on whether there are no white points at all, infinitely many, or only finitely many. So the dependence/independence structure of Ξ_1, Ξ_2 is non-trivial. However this example really concerns a *marked* point process based on the points of $\pi(X \cup Y \cup Z)$, where the points are marked red, white, or blue according to their originating point process. Consequently more information is being used here concerning Ξ_1, Ξ_2 than would be available from their hitting σ-algebras alone. It follows that there is no contradiction between this example and Remark 4.6.

We have already noted the work of Aldous and Barlow [1], which investigates non-trivial structure of countable dense random sets; this is possible because these random sets are viewed constructively (because they are related to filtrations of σ-algebras specified *a priori*) rather than phenomenologically. Note that the difference here is the *a priori* specification of the filtration, rather than the one-dimensional nature of the basic space: as Zuyev [24] has demonstrated, it is possible to make effective use of similar notions for point processes on multidimensional spaces. For example it is straightforward to follow Zuyev's ideas to generalize a pretty characterization due to Aldous and Barlow [1, Theorem 4(c)]: if a countable dense subset $\Xi \subseteq \mathbb{R}^d$, viewed as a random set defined on a probability space with filtration localizing to compact subsets of $\mathbb{R}^d$, satisfies

$$\{\omega \in E \ : \ \Xi(\omega) \cap C(\omega)\} \quad = \quad \{\omega \in E \ : \ \text{Leb}(C(\omega)) = 0\} \tag{12}$$

for every previsible set C (see [24] for a definition) then Ξ is the projection of a locally finite Poisson point process on $\mathbb{R}^d \times (0, \infty)$. The proof uses [1, Theorem 4(c)] applied to predictable sets following one-parameter increasing families of compact subsets of $\mathbb{R}^d$.

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