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by

W.S. Kendall and C.J. Price

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Zeros of Brownian Polynomials

Wilfrid S Kendall* and Catherine J Price†

Abstract

Consider a polynomial-valued stochastic process $P(z) = X_1 z^n + X_2 z^{n-1} + \cdots + X_n z + X_{n+1}$, where the coefficients $X_1, \ldots, X_{n+1}$ are real-valued or complex-valued semimartingales. The time-evolving zeros can be viewed as stochastic processes in their own right: the "root-processes" of P. The root-processes are complex-valued semimartingales except when they collide with each other or explode to infinity, and we describe their statistical behaviour by calculating their stochastic differentials. The main difficulty in analyzing the root-processes lies in determining their behaviour at collisions: this is examined in the case where the coefficients are independent Brownian motions. If the coefficients are independent complex Brownian motions then all roots are distinct save possibly at time zero; almost surely neither collisions nor explosions will occur, and the root-processes are conformal martingales. In the corresponding real case, the root-processes interact in such a way that collisions can only occur between real or complex conjugate pairs. A root-process can only be traced through collisions by making *ad hoc* choices concerning branch-of-root: however pairs of colliding root-processes can be re-parameterized using elementary symmetric functions in a way which continues smoothly across the singularities where collisions occur, allowing us to supply a full description of the behaviour of the root-processes.

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1 Introduction

The theory of random polynomials studies the zeros of polynomials such as $p(z) = a_1 z^n + a_2 z^{n-1} + \cdots + a_n z + a_{n+1}$, where the coefficients a_r are random variables. Properties such as the expected number of real zeros and the joint distribution of the zeros have been investigated for a number of different distributions of the random coefficients [2, 6]. In this paper we consider a corresponding problem in the setting of stochastic processes and study the behaviour of the roots of a polynomial $P(z) = X_1 z^n + X_2 z^{n-1} \cdots + X_n z + X_{n+1}$, where the coefficients X_i are now semimartingales: we say that the random process Z is a *root-process* of the *polynomial-valued semimartingale* $P(z)$ if $P(Z) \equiv 0$ for all time (more formal definitions are

* Statistics, University of Warwick, Coventry CV4 7AL, UK. <w.s.kendall@warwick.ac.uk>
† Lehman Brothers, Tokyo, Japan. <cprice@lehman.com>

to be found in Definitions 3.1 and 3.10 below). In particular, we concentrate on the case where the coefficients are independent Brownian motions. Here are our principal results. In the complex case:

Theorem 1.1 *Let $P(z)$ be the (non-constant) complex Brownian polynomial defined by*

$$P(z) \quad = \quad B_1 z^n + B_2 z^{n-1} \cdots + B_n z + B_{n+1} \,,$$

where the coefficients B_i are independent complex Brownian motions. The generic case (in the sense of Lebesgue measure induced by the coefficients) is for roots to be distinct; moreover the region of multiple roots is polar with respect to the complex Brownian polynomial. If $P(z)$ has no repeated roots at time zero, and if Z is a root-process, then Z is a conformal martingale with stochastic differential defined almost surely for all positive time by

$$\mathrm{d}_I Z \quad = \quad -\sum_{r=1}^{n+1} \frac{Z^{n+1-r}}{P'(Z)} \, \mathrm{d}_I B_r \,. \tag{1}$$

Remark 1.2 *The special case in which roots initially agree is treated in more depth later in the paper, at the end of Section 4.*

The real case is more involved because roots may explode to infinity and also may collide with each other:

Theorem 1.3 *Let $P(z)$ be the (non-constant) real Brownian polynomial defined by*

$$P(z) \quad = \quad B_1 z^n + B_2 z^{n-1} + \cdots + B_n z + B_{n+1} \,.$$

where the coefficients B_i are independent real Brownian motions. Root-processes Z of $P(z)$ cannot be constructed so as to possess the semimartingale property for all time: however a root-process is a semimartingale over time intervals in which it neither collides with other root-processes of $P(z)$ nor explodes, and then its stochastic differential satisfies

$$\mathrm{d}_I Z \;=\; -\sum_{r=1}^{n+1} \frac{Z^{n+1-r}}{P'(Z)} \, \mathrm{d}_I B_r + \sum_{r=1}^{n+1} \frac{Z^{2(n+1-r)}}{P'(Z)^2} \left(\frac{n+1-r}{Z} - \frac{P''(Z)}{2P'(Z)} \right) \mathrm{d}t \,. \tag{2}$$

(Note: in the above Z may be complex, over periods in which it is tracking one half of a conjugate pair of roots.) Collisions almost surely only occur between pairs of real root-processes or between complex conjugate root-processes. If (Z_1, Z_2) is a pair of root-processes defined up to and colliding at time τ, then their definition can be extended continuously across the collision time τ (up to necessary ambiguities relating to branch-of-root) by re-expressing the roots in the coordinate system given by elementary symmetric functions:

$$\alpha = Z_1 + Z_2 \quad \textit{and} \quad \gamma = Z_1 Z_2 \,.$$

These processes possess the semimartingale property over stochastic intervals containing only collisions between Z_1 and Z_2. At explosion times we can change coordinates *using $W = 1/Z$ and so re-express the root-process Z in terms of the corresponding root-process $W = 1/Z$ of the Brownian polynomial*

$$P^*(w) \quad = \quad B_1 + B_2 w + \cdots + B_n w^{n-1} + B_{n+1} w^n \,,$$

which will almost surely not explode at such times.

Computation of the stochastic differentials in Eqs. (1) and (2) is a straightforward exercise in Itô calculus for stochastic time intervals over which the polynomial-valued semimartingale has n distinct roots. The main issues of the paper lie in dealing with those times when two root-processes collide or when a root-process explodes. In the case of Theorem 1.1 these issues are easily resolved for positive time: it is shown that after time zero the root-processes almost surely neither collide nor explode. In fact, they turn out to be conformal martingales. However, in the case of real Brownian coefficients, as in Theorem 1.3, both collisions and explosions will occur. To make sense of the explosions we can view the root-processes as semimartingales taking values in complex projective space. The case of collisions is more subtle; there is a genuine ambiguity as to which branch-of-root might be taken by the root-process as it passes through the collision. In the specific case considered in Theorem 1.3 a pair of roots can only collide if both root-processes are real, or if they are a complex conjugate pair. A kind of "dance of the roots" occurs: after collisions, the roots can either continue evolving as real roots (and colliding with other real roots) or move together as a complex conjugate pair, until they meet again on the real axis (which necessarily occurs at their collision time). Note however that the set of collision times is similar to the zero-set of Brownian motion, and in particular has no isolated points, so some care must be exercised in interpreting the phrase "after collision".

The paper is organized in the following way. In Section 2 we review some of the basic definitions and results on complex martingales needed in subsequent sections. The polynomial-valued semimartingales that we will be considering are defined in Section 3, and we study the simple case of the Brownian quadratic, for which the quadratic formula gives an easy way to describe a root-process, and which exposes the phenomena we need to consider in further sections. In Section 4 and Section the statistical behaviour of the root-processes is described in the cases when the coefficients are respectively independent complex and real Brownian motions; in particular, we examine the nature of possible collision points. Finally, in Section 6, we discuss possible generalizations and future work.

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2 Preliminaries about complex local martingales

This section contains the definitions and results about complex martingales which will be required in order to define and study polynomial-valued semimartingales and their root-processes. The definitions given follow those of [8]; most of the results quoted are also taken from this paper.

We assume throughout that all processes are defined on a fixed probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ and adapted to a fixed filtration $\{\mathfrak{F}_t : t \geq 0\}$. We further assume that $\{\mathfrak{F}_t : t \geq 0\}$ is right-continuous and that $\mathfrak{F}_0$ contains all $\mathbb{P}$-null sets.

A complex-valued adapted process Z is said to be a *continuous complex local martingale* if both its real and imaginary components are continuous real local martingales; similarly, Z is a *continuous complex local semimartingale* if its real and imaginary components are continuous real local semimartingales. We define the following spaces of random processes:

$$\begin{aligned}
\mathcal{S} &= \{\text{continuous complex local semimartingales}\} \\
\mathcal{M} &= \{\text{continuous complex local martingales}\} \\
\mathcal{L} &= \{\text{continuous real local martingales}\} \\
\mathcal{A} &= \left\{\begin{array}{l}\text{continuous real valued processes} \\ \text{which are locally of bounded variation}\end{array}\right\}
\end{aligned}$$

For the subsets of $\mathcal{M}$, $\mathcal{L}$ and $\mathcal{A}$ which consist of processes begun at zero, we write $\mathcal{M}^{(0)}$, $\mathcal{L}^{(0)}$ and $\mathcal{A}^{(0)}$ respectively. A complex valued adapted process $Z = Z(0) + X + iY$ is in the space $\mathcal{M}$ if and only if X and Y are in the space $\mathcal{L}^{(0)}$, and $Z(0) \in \mathfrak{F}_0$. Similarly, Z is in $\mathcal{S}$ if and only X and Y are in $\mathcal{L}^{(0)} \oplus \mathcal{A}^{(0)}$, and $Z(0) \in \mathfrak{F}_0$.

We will use Itô calculus expressed in the notation of stochastic differentials [12]. Formally, the stochastic differential $\mathrm{d}_I X$ of $X \in \mathcal{S}$ is defined to be the equivalence class of X under the equivalence relation $\sim$, where $X \sim Y$ if almost surely $X(t) - X(s) = Y(t) - Y(s)$ for every $0 \leq s \leq t$. If X is of locally bounded variation then the result is a classical differential $\mathrm{d}_I X = \mathrm{d}X$, and in such cases we always write $\mathrm{d}X$ rather than $\mathrm{d}_I X$. We write $\mathrm{d}_I \mathcal{S}$ for the space of stochastic differentials of continuous semimartingales. For $Z \in \mathcal{S}$, consider the unique decomposition $Z = Z(0) + M + V$ into the sum of a local martingale $M \in \mathcal{M}^{(0)}$ and a process of locally bounded variation $V \in \mathcal{A}^{(0)} \oplus i\mathcal{A}^{(0)}$. We write $\mathrm{d}_I Z = \mathrm{d}_I M + \mathrm{d}V$ (note that $\mathrm{d}V$ is a classical differential). The operation of taking stochastic differentials gives a complex linear map $\mathrm{d}_I : \mathcal{S} \to \mathrm{d}_I \mathcal{S}$, to which stochastic integration is a right-inverse. Thus, stochastic differentials are interpreted by integration; for $Z \in \mathcal{S}$ and a locally bounded predictable function f, the semimartingale $\int f \, \mathrm{d}_I Z$ has stochastic differential $f \, \mathrm{d}_I Z$. For the complex semimartingale $Z = Z(0) + X + iY$, where X and Y are in $\mathcal{L}^{(0)}$, we write $\mathrm{d}_I Z = \mathrm{d}_I X + i \, \mathrm{d}_I Y$.

Definition 2.1 *The* increasing process $\langle Z, Z \rangle$ *of* $Z \in \mathcal{M}$ *is defined to be the unique member of* $\mathcal{A}^{(0)}$ *such that* $|Z|^2 - \langle Z, Z \rangle$ *is a real continuous local martingale.*

Definition 2.2 *The* quadratic covariation $\langle Z, W \rangle$ *of* $Z, W \in \mathcal{M}$ *is defined by complex polarization, so that*

$$\begin{aligned}
\langle Z, W \rangle &= \frac{1}{4}\left(\langle Z+W, Z+W \rangle - \langle Z-W, Z-W \rangle\right) \\
&\quad + \frac{i}{4}\left(\langle Z+iW, Z+iW \rangle - \langle Z-iW, Z-iW \rangle\right) . \qquad (3)
\end{aligned}$$

The definition of $\langle Z, W \rangle$ *is extended to* $Z, W \in \mathcal{S}$ *by linearity and the stipulation that the covariation of a semimartingale with a process of locally bounded variation is always zero.*

Remark 2.3 If $Z = X + iY$ and $W = U + iV$ then

$$\langle Z, W \rangle \quad = \quad \langle X, Y \rangle + \langle U, V \rangle - i\langle X, V \rangle + i\langle Y, U \rangle \,. \tag{4}$$

It follows from these definitions that $\langle Z, W \rangle$ is the unique member of $\mathcal{A}^{(0)} \oplus i\mathcal{A}^{(0)}$ such that $Z\overline{W} - \langle Z, W \rangle$ is in $\mathcal{M}$. In the following lemma, representations of $\langle Z, W \rangle$ are given in terms of the real and complex components of Z and W; these are used in the proof of the complex version of Itô's formula.

Lemma 2.4 *Let $Z, W \in \mathcal{M}$ be such that $Z = Z(0) + X + iY$ and $W = W(0) + U + iV$, where $X, Y, U, V \in \mathcal{L}^{(0)}$. Then:*

(i) $\langle Z - Z(0), W - W(0) \rangle = \langle Z, W \rangle$

(ii) $\langle Z, Z \rangle = \langle X, X \rangle + \langle Y, Y \rangle$

(iii) $\langle Z, W \rangle = \langle X, U \rangle + \langle Y, V \rangle + i(\langle Y, U \rangle - \langle X, V \rangle)$

(iv) *The bracket $\langle \cdot, \cdot \rangle$ is sesquilinear and $\overline{\langle Z, W \rangle} = \langle W, Z \rangle$.*

Proof: These are all easy consequences of Definitions 2.1 and 2.2, together with the corresponding properties for real martingales (also see [8]). ☐

Remark 2.5 Note that we then have the following product rule for complex stochastic differentials: $(\,\mathrm{d}_I Z)(\,\mathrm{d}_I W) = \,\mathrm{d}_I \langle Z, \overline{W} \rangle$. Moreover, $\langle Z, \overline{W} \rangle = \langle W, \overline{Z} \rangle$.

Remark 2.6 In defining the quadratic covariation of $Z, W \in \mathcal{M}$ we have followed [8] (and also, for example, [11]). Other authors dealing with complex martingales, such as [21, 27, 29], define quadratic covariation so that it is $(\mathbb{C} \times \mathbb{C})$-linear. It is then the unique member of $\mathcal{A}^{(0)} \oplus i\mathcal{A}^{(0)}$ which, when subtracted from ZW (rather than $Z\overline{W}$), delivers a process which is in $\mathcal{M}$. However, under this alternative definition, the quadratic covariation no longer behaves like a complex inner product, moreover the quadratic covariation of Z with itself is no longer increasing. We have therefore chosen to use the definition which makes $\langle Z, W \rangle$ sesquilinear, rather than the more algebraic alternative. Of course, the map defined on $\mathbb{C} \times \mathbb{C}$ which takes (z, w) to $(z, \overline{w})$ allows one to exchange easily between the two definitions.

Notation

For a function $F : \mathbb{C} \to \mathbb{C}$ we write

$$\frac{\partial F}{\partial z} = \frac{1}{2}\left\{\frac{\partial F}{\partial x} - i\frac{\partial F}{\partial y}\right\} \qquad \text{and} \qquad \frac{\partial F}{\partial \overline{z}} = \frac{1}{2}\left\{\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right\}$$

where $z = x + iy$. The partial derivatives of a function $F : \mathbb{C}^n \to \mathbb{C}$ are defined similarly.

Proposition 2.7 (Itô's formula for complex local semimartingales) *Let $Z_1, \ldots, Z_n$ be continuous complex local semimartingales, and let $F : \mathbb{C}^n \to \mathbb{C}$ be a function which is $\mathcal{C}^2$ in each coordinate, when considered as a function on $\mathbb{R}^{2n}$. Then we have*

$$\begin{aligned} \mathrm{d}_I F(Z_1, \ldots, Z_n) \quad = \quad & \sum_{r=1}^{n} \frac{\partial F}{\partial z_r} \, \mathrm{d}_I Z_r + \sum_{r=1}^{n} \frac{\partial F}{\partial \overline{z}_r} \, \mathrm{d}_I \overline{Z}_r \\ & + \frac{1}{2} \sum_{r=1}^{n} \sum_{s=1}^{n} \left[\frac{\partial^2 F}{\partial z_r \partial z_s} \, \mathrm{d}\langle Z_r, \overline{Z}_s \rangle + \frac{\partial^2 F}{\partial z_r \partial \overline{z}_s} \, \mathrm{d}\langle Z_r, Z_s \rangle \right. \\ & \left. + \frac{\partial^2 F}{\partial \overline{z}_r \partial z_s} \, \mathrm{d}\langle \overline{Z}_r, \overline{Z}_s \rangle + \frac{\partial^2 F}{\partial \overline{z}_r \partial \overline{z}_s} \, \mathrm{d}\langle \overline{Z}_r, Z_s \rangle \right] . \end{aligned} \tag{5}$$

Proof: The proposition is proved by writing the complex semimartingales in their real and imaginary components, and then using Itô's formula for functions on $\mathbb{R}^{2n}$, together with Lemma 2.4. To speed up the proof, to note that the second-order part of the result (once $\partial^2 F / \partial z_r \partial z_s$ *etc* are expanded) is a sum of terms such as

$$\frac{\partial^2 F}{\partial x_r \partial y_s} \times \langle Z_r, \overline{Z}_s \rangle ,$$

using various permutations of presence and absence of complex conjugates and sign changes. Orthogonality relations can then be used to deduce that all terms vanish save those appearing in Eq. (5) above. This result is to be found in [21]. □

Definition 2.8 *The complex martingale $Z = Z(0) + X + iY \in \mathcal{M}$ is said to be* conformal *if $\langle X, X \rangle = \langle Y, Y \rangle$ and $\langle X, Y \rangle = 0$.*

From Lemma 2.4 it is equivalent to require that $\langle Z, \overline{Z} \rangle = 0$. Moreover Z is conformal if and only if its conjugate $\overline{Z}$ is conformal. Thus complex Brownian motion is clearly a conformal martingale. Proposition 2.9 below shows that in some sense it is a canonical example.

Proposition 2.9 *If Z is a conformal martingale, then (enlarging the probability space if required) there exists a Brownian motion W such that $Z_t = W(\langle X, X \rangle)$ for $Z = X + iY$.*

Proof: See for example [27, p. 178, Theorem 2.4]. □

Proposition 2.10 *If Z is a conformal martingale such that $\mathbb{P}\left[\, Z(0) = 0 \,\right] = 0$, then*

$$\mathbb{P}\left[\, Z_t = 0 \textit{ for some } t \geq 0 \,\right] = 0$$

In other words, if a conformal martingale does not start at the origin then it will almost surely never hit the origin.

Proof: The lemma can be proved by using the time-change argument from Proposition 2.9 to convert Z into a complex Brownian motion, for which the corresponding result is well known. A full proof is given in [8, Theorem 5.6]. □

We conclude this section by stating two forms of Itô's formula: for holomorphic functions of complex semimartingales (together with an easy consequence in the case of conformal martingales), and for a function $f : \mathbb{R}^n \times \mathbb{C} \to \mathbb{C}$. These will be used to calculate the stochastic differentials of the root-processes; essentially they are special cases of Proposition 2.7.

Proposition 2.11 (Itô's formula for holomorphic functions) *Let $F : \mathbb{C}^n \to \mathbb{C}$ be holomorphic. Then for $Z_1, \dots, Z_n \in \mathcal{M}$,*

$$\mathrm{d}_I F(Z_1, \dots, Z_n) \quad = \quad \sum_{r=1}^{n} \frac{\partial F}{\partial z_r} \mathrm{d}_I Z_r + \frac{1}{2} \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{\partial^2 F}{\partial z_r \partial z_s} \mathrm{d}\langle Z_r, \overline{Z}_s \rangle \tag{6}$$

Corollary 2.12 *Let $Z_1, \dots, Z_n$ be conformal martingales such that $\langle Z_r, \overline{Z}_s \rangle = 0$ for all r, s (such Z_r may be thought of as having the same orientation). Then $F(Z_1, \dots, Z_n)$ is a conformal martingale whenever $F : \mathbb{C}^n \to \mathbb{C}$ is holomorphic. In particular, $F(B_1, \dots, B_n)$ is a conformal martingale for the n-dimensional complex Brownian motion $B = (B_1, \dots, B_n)$.*

Proposition 2.13 (Itô's formula for $F : \mathbb{R}^n \times \mathbb{C} \to \mathbb{C}$) *Let $F : \mathbb{R}^n \times \mathbb{C} \to \mathbb{C}$ be a function that is $\mathcal{C}^2$ in each coordinate when viewed as a function on $\mathbb{R}^{n+2} = \mathbb{R}^n \oplus \mathbb{C}$, and holomorphic in the complex coordinate. Then the following formula holds for $X_1, \dots, X_n \in \mathcal{L} + \mathcal{A}$ and $Z \in \mathcal{S}$:*

$$\begin{aligned} \mathrm{d}_I F(X_1, \dots, X_n, Z) \quad &= \\ \sum_{r=1}^{n} \frac{\partial F}{\partial x_r} \mathrm{d}_I X_r + \frac{\partial F}{\partial z} \mathrm{d}_I Z &+ \frac{1}{2} \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{\partial^2 F}{\partial x_r \partial x_s} \mathrm{d}\langle X_r, X_s \rangle \\ + \sum_{r=1}^{n} \frac{\partial^2 F}{\partial x_r \partial z} \mathrm{d}\langle X_r, \overline{Z} \rangle &+ \frac{1}{2} \frac{\partial^2 F}{\partial z^2} \mathrm{d}\langle Z, \overline{Z} \rangle \end{aligned} \tag{7}$$

3 Root-processes for polynomial-valued semimartingales

In Section 1 we gave an informal definition of a root-process of a polynomial-valued semimartingale $P(z)$ as a stochastic process Z solving $P(Z) = 0$. In this section we proceed more formally, beginning by defining polynomial-valued semimartingales.

Definition 3.1 *Let $X_1, \dots, X_{n+1} \in \mathcal{S}$ be continuous local semimartingales such that X_1 is not identically equal to zero and $n > 0$. Then, the polynomial valued stochastic process $P(z) = \{P(z, t) : t \geq 0\}$ defined by*

$$P(z,t) \quad = \quad X_1(t) z^n + X_2(t) z^{n-1} + \dots + X_n(t) z + X_{n+1}(t) \qquad \text{for all } t \geq 0\,, \tag{8}$$

is said to be a (local semimartingale) polynomial process of degree n *(more briefly,* polynomial-valued semimartingale*), in the formal variable z, with coefficients $X_1, \dots,$ X_{n+1}.*

In the case where the coefficients $X_1, \dots, X_{n+1}$ are independent real (or complex) Brownian motions, the polynomial process is called a real *(respectively* complex*)*

Brownian polynomial. *The* $n+1$ *(real, respectively complex) dimensional Brownian motion* $X = (X_1, \dots, X_{n+1})$ *formed by the coefficients is called the* Brownian motion associated with $P(z)$.

Remark 3.2 Constant polynomials (case $n = 1$) are excluded since their root-processes either do not exist or else behave in a degenerate way.

Remark 3.3 A polynomial process can be thought of as being a function of the semimartingale coefficients as well as the variable z. Thus, for the polynomial process $P(z)$ in Definition 3.1 above, we can write $P = F(X_1, \dots, X_{n+1}, z)$ where F is the function defined by $F(x_1, \dots, x_{n+1}, z) = x_1 z^n + \cdots + x_n z + x_{n+1}$, for any $(x_1, \dots, x_{n+1}, z)$. This representation of the polynomial process will be used in order to calculate the stochastic differentials of its root-process.

Remark 3.4 There are many possible definitions of a "polynomial process", of varying levels of generality and rigour. The idea here of a polynomial process of degree n is analogous to that of the random polynomials of degree n mentioned in Section 1. For comparison we consider the definition of random polynomials given in [2]. Here random polynomials are considered to be the special cases of random power series which have only finitely many non-zero terms in the expansion. With this definition, random polynomials are allowed to have random (but almost surely finite) degree; a definition of what is meant by the degree of the polynomial is also given. Alternative definitions of random polynomials are also discussed. For example, the polynomials can be thought of as random analytic functions. This allows notions from probabilistic functional analysis to be used. It is easy to see how these definitions could be adapted to generalize the notion of random processes of polynomials. However, the characterization of a polynomial process above is sufficient for our present purposes.

3.1 Real Brownian Quadratics

Before defining the root-processes of a polynomial process, we consider the special case of a real Brownian quadratic. This example is important for several reasons. To begin with, the Brownian quadratic is easy to study: the quadratic formula gives an obvious way of describing the root-processes. This then suggests the properties expected of the root-processes in the higher degree case, for which the roots cannot in general be expressed explicitly in terms of the coefficients. In the following, we consider the problems of existence and uniqueness presented by the times when the roots of the real Brownian quadratic explode or collide. The methods of resolving these problems carry over to cases of higher degree. In particular, we will see later that locally the root-processes of real Brownian polynomials processes only interact in pairs; the description of the root-processes here allows us to keep track of the pairs of root-processes over collision points.

Example 3.5 *Let* B_1*,*B_2 *and* B_3 *be real, independent Brownian motions and let* $Q(z)$ *be the Brownian polynomial*

$$Q(z) \quad = \quad B_1 z^2 + B_2 z + B_3 \,.$$

The quadratic $Q(z)$ has two (not necessarily distinct) roots Z^+ and Z^-, given by

$$Z^{\pm} \quad = \quad \frac{-B_2 \pm \sqrt{(B_2)^2 - 4B_1 B_3}}{2B_1} \,. \tag{9}$$

Here we write $\sqrt{-1} = i$ (the ambiguity between i and $-i$ is typical for the issue of selecting roots of polynomials).

We can define root-processes Z^+ and Z^- as roots of $Q(z)$ given by the formula Eq. (9) over stopping-time intervals such that both the leading coefficient B_1 and also the discriminant $D = (B_2)^2 - 4B_1B_3$ are non-vanishing. Away from the origin, the square-root function is $\mathcal{C}^2$. Thus, whenever both D and B_1 are non-zero, we can write Z^+ and Z^- as $\mathcal{C}^2$-functions of the Brownian motion $B = (B_1, B_2, B_3)$ and hence, by Itô's formula (Proposition 2.7), they are semimartingales on stochastic intervals over which neither D nor B_1 vanish. Unfortunately, in this real case we cannot ignore the times when either $B_1 = 0$ or $D = 0$. The following properties of polynomials will be used to study Z^+ and Z^- at these times:

Lemma 3.6 *If $p(z) = a_1z^n + a_2z^{n-1} + \cdots + a_nz + a_{n+1}$ is a polynomial of degree n defined on the complex plane, then:*

(i) *The roots of $p(z)$ are continuous functions of the coefficients (when the roots are viewed as belonging to the complex projective plane discussed in Subsection 3.3: otherwise we must exclude the case $a_1 = 0$).*

(ii) *The roots of $p(z)$ are contained (in $\mathbb{C}$) in the circle*

$$\left\{ z : |z| < 1 + \max_{r=2,\ldots,n+1} \left| \frac{a_r}{a_1} \right| \right\}$$

(iii) *If the coefficient a_1 is made suitably small in relation to at least one other of the coefficients then at least one zero of $p(z)$ may be made to lie outside an arbitrarily large circle in $\mathbb{C}$.*

Proof: For the proof of these statements, we refer to [22, Theorem (1,4), Theorem (27,1), Section 31]. Similar results are also proved in [9]. The method is essentially a winding-number argument, using re-parameterizations on the complex projective plane. □

3.2 Monic Brownian quadratics and polynomials

From Lemma 3.6, or indeed Eq. (9), we see that as $B_1 \to 0$ while B_2 and B_3 are held fixed so one root must become arbitrarily large whilst the other must approach $-B_3/B_2$. Thus, in order to define the roots of $Q(z)$ as continuous stochastic processes, we must make sense of the roots being infinite at certain times. We do this in the next section by viewing the root-processes as stochastic processes on complex projective space.

The complications associated with the explosions of the roots can be avoided by considering *monic polynomial processes*. A monic polynomial over $\mathbb{C}$ of degree n always has n roots and so it is intuitively clear that the roots of a monic polynomial process will not explode: this intuition is justified later by Lemma 3.6.

Definition 3.7 *Let $P(z)$ be a polynomial process such that the leading coefficient is identically almost surely equal to 1. Then $P(z)$ is defined by*

$$P(z) \quad = \quad z^n + X_1z^{n-1} + \cdots + X_{n-1}z + X_n \,,$$

for some $X_1, \dots, X_n \in \mathcal{S}$. In this case, $P(z)$ is said to be a monic (local semimartingale) polynomial process of degree n, *in the formal variable z.*

A monic polynomial process $P(z)$ as above is a monic Brownian polynomial *if the coefficients $X_1, \dots, X_n$ are independent Brownian motions. The n-dimensional Brownian motion $X = (X_1, \dots, X_n)$ is again called the* Brownian motion associated with $P(z)$.

Example 3.8 *Let B_1 and B_2 be independent real Brownian motions and let $\tilde{Q}(z)$ be the monic Brownian quadratic defined by*

$$\tilde{Q}(z) \quad = \quad z^2 + B_1 z + B_2\,, \qquad\qquad \textit{for all } z \in \mathbb{C}.$$

We obtain from the quadratic formula that root-processes $Z^{\pm}$ must satisfy

$$\tilde{Z}^{\pm} \quad = \quad \frac{1}{2}\left(-B_1 \pm \sqrt{(B_1)^2 - 4B_2}\right)\,. \tag{10}$$

It is clear that construction of $\tilde{Z}^+$ and $\tilde{Z}^-$ requires us to make *ad-hoc* decisions about choice of branch-of-root whenever the discriminant $(B_1)^2 - 4B_2$ vanishes. If these choices are made then Eq. (10) shows that root-processes always exist and are finite in the monic quadratic case. As in Example 3.5, Itô's formula from Proposition 2.7 shows that $\tilde{Z}^+$ and $\tilde{Z}^-$ are both semimartingales over stochastic intervals during which the roots can be expressed as C^2-functions of the Brownian coefficients: that is, whenever $(B_1)^2 \neq 4B_2$. Thus the times at which $\tilde{Z}^+$ and $\tilde{Z}^-$ are not well defined as semimartingales correspond to those times when $\tilde{Z}^+$ and $\tilde{Z}^-$ coincide, or equivalently, when the planar Brownian motion (B_1, B_2) hits the curve $y = x^2/4$. Due to the neighbourhood recurrence of planar Brownian motion, this is almost sure to happen. At collision times, the function defining $\tilde{Z}^{\pm}$ is no longer smooth, and indeed it is no longer possible to define root-processes as semimartingales; here is an argument which is a variant of the proof of Yor [32] for the case of the square-root of the absolute value of Brownian motion, but extended to allow us to deal later with the case of roots of higher-degree Brownian polynomials.

First consider $\tilde{D} = (B_1)^2 - 4B_2$, and notice

(a) $\texttt{drift}\ \mathrm{d}_I \tilde{D} = \mathrm{d}t$ is absolutely continuous, when considered as a measure;

(b) $\mathbb{P}\left[\tilde{D}(t) = 0\right] = 0$ for all positive t (easily seen since B_1, B_2 are independent);

(c) if $\mathrm{d}L = \mathrm{d}_I|\tilde{D}| - \mathrm{sgn}(\tilde{D})\,\mathrm{d}_I\tilde{D}$ then the increasing process L is non-constant over every interval in which $\tilde{D}$ changes sign.

For this last fact, note that we can use a Girsanov change-of-measure to express $\tilde{D}$ as a local martingale with respect to the new probability measure; moreover

$$(\,\mathrm{d}_I\tilde{D})^2 \quad = \quad 4((B_1)^2 + 1)\,\mathrm{d}t\,.$$

Working in a new time-scale $\mathrm{d}\tau = 4((B_1)^2 + 1)\,\mathrm{d}t$, we can now view $\tilde{D}$ as a Brownian motion, and L as its local time. The last fact is then a simple martingale and stopping-time argument, following [32].

Now it is a consequence of the following lemma that any process Z satisfying $(2Z + B_1)^2 = \tilde{D}$ must fail to be a semimartingale in any stochastic interval in which $\tilde{D}$ changes sign.

Lemma 3.9 *Suppose Y, D are processes adapted to a common filtration such that D is a semimartingale, and is adapted to a sub-filtration generated by an n-dimensional Brownian motion B, has absolutely continuous drift $\texttt{drift}\ \mathrm{d}_I D = G\,\mathrm{d}t$, and furthermore satisfies*

$$\mathbb{P}\left[\, D(t) = 0\, \right] \quad = \quad 0$$

for all positive t. If

$$Y^2 \quad = \quad |D|$$

then $\mathrm{d}L$ must vanish on all stopping-time intervals for which Y is a semimartingale, where

$$\mathrm{d}L \quad = \quad \mathrm{d}_I|D| - \mathrm{sgn}(D)\,\mathrm{d}_I D\,.$$

Proof: If a semimartingale Y satisfies $Y^2 = |D|$ and has Doob-Meyer decomposition

$$\mathrm{d}_I Y \quad = \quad \mathrm{d}_I M + \mathrm{d}V$$

then the Itô-Tanaka formula shows

$$2Y\,\mathrm{d}_I M + 2Y\,\mathrm{d}V + (\,\mathrm{d}_I M)^2 \quad = \quad \mathrm{sgn}(D)\,\mathrm{d}_I D + \mathrm{d}L\,. \qquad (11)$$

Now L increases only when D vanishes: since $\mathbb{P}\left[\, D(t) = 0\, \right] = 0$ for all positive t we see that $\mathrm{d}_I L$ has support confined to a random set of Lebesgue measure zero. Consider then the drifts of both sides of Eq. (11):

$$\mathrm{d}L \quad = \quad 2Y\,\texttt{drift}\ \mathrm{d}V + (\,\mathrm{d}_I M)^2 - \mathrm{sgn}(D)\,\texttt{drift}\ \mathrm{d}_I D\,.$$

The constraint on the support of $\mathrm{d}_I L$ means that $\mathbb{I}_{[Y=0]}\,\mathrm{d}L = \mathbb{I}_{[D=0]}\,\mathrm{d}L = \mathrm{d}L$; since the zero-set of D is Lebesgue-null and its drift is absolutely continuous we can deduce that $\mathbb{I}_{[Y=0]}\,\texttt{drift}\ \mathrm{d}_I D = \mathbb{I}_{[D=0]} G\,\mathrm{d}t = 0$; and finally we may apply the martingale representation theorem for Brownian filtrations to show that $(\,\mathrm{d}_I M)^2 = |F|^2\,\mathrm{d}t$ is also absolutely continuous and therefore $\mathbb{I}_{[Y=0]}(\,\mathrm{d}_I M)^2 = \mathbb{I}_{[D=0]}|F|^2\,\mathrm{d}t = 0$. Hence

$$\mathrm{d}L \quad = \quad 0$$

as required. $\square$

Reverting to our previous notation, it follows that the changing of sign of $\tilde{D}$ makes it impossible to define $\tilde{Z}^{\pm}$ as semimartingales. However Eq. (10) can be viewed as a formula connecting the coefficients of a quadratic to the sum and product of its roots $\tilde{Z}^+$ and $\tilde{Z}^-$. We therefore consider instead the processes α and β, where

$$\begin{aligned} \alpha \quad &= \quad \tilde{Z}^+ + \tilde{Z}^- \quad = \quad -B_1 \\ \beta \quad &= \quad (\tilde{Z}^+ - \tilde{Z}^-)^2 \quad = \quad D \quad = \quad (B_1)^2 - 4B_2\,. \end{aligned}$$

These are well-defined even at collision times and so can be used to give a representation of $\tilde{Z}^+$ and $\tilde{Z}^-$ in terms of semimartingales, except for an inevitable ambiguity whenever $\tilde{Z}^+$ and $\tilde{Z}^-$ coincide. Arbitrary choices of branches-of-root must be made in order to resolve this ambiguity. One simple way runs as follows: complex conjugate parts of the root-processes arise when (B_1, B_2) makes excursions into the region

$\{x^2 < 4y\}$; for each such excursion we require the corresponding portion of $\tilde{Z}^+$ to lie on the $+i$ half of the complex plane with probability $\frac{1}{2}$ independently of the past, otherwise on the $-i$ half, and set $\tilde{Z}^-$ to be the complex conjugate of $\tilde{Z}^+$. This particular resolution has the advantage of symmetry: however it can lead to a requirement for "uncountable amounts of randomness", if for example we want to construct co-adapted solutions Z of $Z^2 + B = 0$ for all possible initial values $Z(0)$. An alternative approach involves connecting the larger real root-process fragments with the excursions into the $+i$ half of the complex plane. This method breaks symmetry but does not require extraneous randomness. Whichever way is chosen, the ambiguity is resolved. (But note there is no pressing reason to favour one resolution before the other.)

We now have a complete expression for the root-processes of $\tilde{Q}(z)$, given by the original $\tilde{Z}^+$ and $\tilde{Z}^-$, together with their re-parameterization at the collision times and an essentially *ad hoc* decision as to how to continue the root-processes through collisions.

This gives a picture of how the root-processes interact. The two roots must either be both real or both complex. If they are both real then they will continue to be real until they first collide: leaving a collision point they may either become complex and move together as complex conjugates, or continue on as real roots. The root-processes can be thought of as making paired excursions into the complex plane from the real line, or (in more picturesque terms) as performing a "dance of the roots" changing to and from the real line onto the full complex plane. The process β tracks the excursions onto the complex plane: when it is negative, the roots must lie in the complex plane.

3.3 Explosions and the complex projective plane

Using the previous examples as a guide, we can now give a full treatment of the root-processes of a polynomial process. A root-process of the polynomial process defined by Eq. (8) must be a stochastic process Z that depends continuously upon the coefficients $X_1, \ldots, X_{n+1}$ and such that $P(Z) \equiv 0$. We have seen, however, that in general *ad hoc* rules may have to be imposed in order to define such a process for all time. Furthermore, for this sort of definition to make sense, the root-process has to take values in an algebraically closed field (that is to say, $\mathbb{C}$ rather than $\mathbb{R}$). Finally, since the polynomial changes with time, we must also deal with the times at which the leading coefficient vanishes. At such times, one root of the polynomial can be thought of as having become infinite: to make this rigorous, we introduce the notion of homogeneous polynomials on projective space.

We let $\mathbb{C}P^1$ denote complex projective space and write $[x, y]$ for the equivalence class of $(x, y) \in \mathbb{C}^2$ (see for example [26]). For any polynomial $p(z) = a_1 z^n + \cdots + a_{n+1}$, there is an associated homogeneous polynomial in two variables, which we denote by $\tilde{p}(z, w)$, and which is defined by

$$\tilde{p}(z, w) \quad = \quad a_1 z^n + a_2 z^{n-1} w + \cdots + a_n z w^{n-1} + a_{n+1} w^n \, .$$

Thus a zero $\alpha \in \mathbb{C}$ of $p(z)$ corresponds to a zero $(\alpha, 1) \in \mathbb{C}^2$ of $\tilde{p}(z, w)$, and so a zero of $\tilde{p}(z, w)$ on $\mathbb{C}P^1$ away from the point $[1, 0]$ corresponds to a zero of $p(z)$ on $\mathbb{C}$; a zero of $\tilde{p}(z, w)$ at $[1, 0]$ corresponds in intuitive terms to a zero of $p(z)$ at the point $\alpha = \infty$.

We now define a root-process more formally, so as to take account of the above.

Definition 3.10 *A* root-process *of a polynomial process $P(z) = \{P(z, t) : t \geq 0\}$ is a stochastic process Z such that*

(i) $Z(t) \in \mathbb{C}P^1$ *for all* $t \geq 0$.

(ii) $Z(t)$ *is continuous in* t*, for all* $t \geq 0$.

(iii) $\tilde{P}(Z(t), t) \equiv 0$ *for all* $t \geq 0$*, where* $\tilde{P}(Z(t), t)$ *is the homogeneous polynomial (defined on* $\mathbb{C}P^1$*) associated with* $P(z, t)$ *for each* t *and* $Z(t)$ *is here understood as taking values in the complex projective plane.*

Proposition 3.11 *Let* $P(z) = \{P(z, t) : t \geq 0\}$ *be a polynomial process such that the polynomial* $P(z, 0)$ *has an isolated root* z_0*. Then* $P(z)$ *has a root-process* Z *begun at* z_0*, which is well defined at least until the stopping time* τ*, where*

$$\tau \quad = \quad \inf\left\{ t \ : \ P'(Z(t), t) = \left|\frac{\partial P(z)}{\partial z}\right|_{z=Z(t)} = 0 \right\} . \tag{12}$$

Moreover, Z *is a semimartingale on the interval* $[0, \tau)$.

Proof: For some $X_1, \ldots, X_{n+1} \in \mathcal{S}$, the polynomial process $\tilde{P}$ defined on $\mathbb{C}P^1$ can be written as a function of the coefficients as in Remark 3.3,

$$\tilde{P}(z) \quad = \quad F(X_1, \ldots, X_{n+1}, z) \qquad \text{for all } z \in \mathbb{C}P^1 .$$

Since z_0 is an isolated zero, $\partial F(z_0)/\partial z \neq 0$, and we may apply the implicit function theorem [31] to F at the point $(X_1(0), \ldots, X_{n+1}(0), z_0)$. This gives the existence of open sets around $(X_1(0), \ldots, X_{n+1}(0))$ and z_0, which we call A and B respectively, for which there exists a unique $\mathcal{C}^2$-function $g : A \to B$ with the property that $F(x, g(x)) = 0$ for all $x \in A$. Since F is smooth, the function g will also be smooth. It follows from the continuity of $X_1, \ldots, X_{n+1}$ that there will be an interval of time $[0, \delta)$ for which $(X_1(t), \ldots, X_{n+1}(t))$ stays in the open set A. Thus for $t \in [0, t + \delta)$, we define the root-process Z by

$$Z_t \quad \equiv \quad g(M_t) .$$

Applying Itô's formula, we see that this defines Z as a semimartingale on $[0, \delta)$. This argument can now be repeated to give another interval of time for which Z is well defined. The uniqueness of the function g ensures that Z is well defined on the overlap of these two intervals. Following this construction, we can define Z until it hits the locus

$$\left\{ z : P'(z) = \frac{\partial F}{\partial z} = 0 \right\} .$$

We note also that τ is indeed a stopping time, since it is simply the hitting time of a closed set by the semimartingale (Z, P).

□

Continuation of Z beyond time τ requires choices of branch-of-root, as noted in the simple case of the Brownian quadratic above. The resolution is relatively simple in the case of the Brownian polynomials treated here: for general polynomial-valued semimartingales a careful treatment would need stochastic Galois theory as indicated in the concluding Section 6 below.

Remark 3.12 From Proposition 3.11, it follows that we can now calculate the stochastic differential of a root-process Z up to the first time τ such that the derivative of the polynomial evaluated at the root-process $P'(Z(\tau))$ is zero. Since we can in general factorise a polynomial $f(z) = a_1 z^n + \cdots + a_n z + a_{n+1}$ by any root α so that $f(z) = (z - \alpha)g(z)$, where $g(z)$ is a polynomial of degree $n - 1$, we can write $f'(z) = (z - \alpha)g'(z) + g(z)$ and so we see that the derivative of the polynomial can only be zero at a root α in the case where the root is repeated, although we must bear in mind the possibility that the root becomes a repeated root at infinity. Thus, rather than considering whether the derivative of the polynomial becomes zero, we may consider the equivalent but more intuitive idea of whether two root-processes collide. We also note that if a root-process is initially either real or complex then it will stay respectively real or complex until the first time τ of collision with some another root-process. In the case of a complex root-process Z of a polynomial with real coefficients, there will exist a complex-conjugate root-process W which is defined such that $W(t) = \overline{Z}(t)$ at all times $t \leq \tau$.

Remark 3.13 Hammersley [9] raises the concern that when dealing with the complex roots of random polynomials there is no ordering of the roots which can be chosen to be continuous with respect to the geometry of the complex plane. Of course when dealing with real roots of random polynomials this concern does not arise, since the usual ordering on the real line can be used so long as the roots are initially distinct. For Brownian polynomials we must consider complex roots and so this concern must be addressed. However the ordering need only be determined at time 0; this ordering is then propagated by evolution of the root-processes subject to our choice of how to continue root-processes at collision times. We assume here that (in the case of roots being initially distinct) some arbitrary initial ordering of the root-processes is chosen, such as the lexicographic ordering using real and imaginary coordinates. To include the case of multiple roots one needs to work with processes defined on the quotient space

$$\frac{\mathbb{C} \times \ldots \times \mathbb{C}}{\text{permutations of roots}}$$

of unordered sets of roots (with multiplicity), although for simplicity we do not take this route here.

We conclude this subsection by proving a result which describes when root-processes can explode and which gives a coordinate system to describe the polynomials on complex space, so that their stochastic differentials can be calculated using the results from Section 2.

Lemma 3.14 *Let $P(z)$ be the polynomial process*

$$P(z) \quad = \quad X_1 z^n + X_2 z^{n-1} + \cdots + X_n z + X_{n+1} \,,$$

where the coefficients $X_1, \ldots, X_{n+1} \in \mathcal{S}$ are locally bounded semimartingales. Then some root-process Z hits $[1, 0] \in \mathbb{C}P^1$ if and only if X_1 hits 0.

Proof: To show that the roots of the process will not explode if $X_1 \neq 0$ we use the fact that the martingale coefficients X_r are locally bounded. We can use Lemma 3.6(i) and (ii) to bound the behaviour of the roots in terms of the coefficients. Indeed, at time t all the roots of the polynomial $P(z)$ will be contained in the circle

$$\left\{ z : |z| < 1 + \max_{r=2,\ldots,n+1} \left| \frac{X_r}{X_1} \right| \right\},$$

and so the root-processes are (locally) bounded if X_1 is bounded away from 0. Conversely, from Lemma 3.6 (iii), as the leading coefficient becomes small in relation to the other coefficients so one root must become arbitrarily large. □

Two corollaries follow immediately.

Corollary 3.15 *If the coefficients of a monic polynomial-valued semimartingale do not explode almost surely, then the root-processes will almost surely not explode.*

Corollary 3.16 *If a root-process of a polynomial-valued semimartingale explodes then, provided that it is the only root-process to explode at that time, it can be described using the coordinate system for* $\mathbb{C}P^1$ *defined by* $\phi_1([\alpha, 1]) = \alpha$ *and* $\phi_2([1, \beta]) = \beta$.

Using this, we can identify points of $\mathbb{C}P^1$ away from $[1, 0]$ with points in $\mathbb{C}$. Thus although the root-processes are defined as processes in complex projective space nevertheless away from explosion times they can be regarded as complex-valued stochastic processes living in $\mathbb{C}$. For simplicity, we will generally treat them as such, and only explicitly mention projective space when it becomes necessary to deal with explosions of the processes.

4 Complex Brownian polynomials

Complex Brownian polynomials are easier to study than real Brownian polynomials. This is analogous to the situation for random polynomials with normally distributed coefficients, and is a consequence of the discontinuity introduced by restriction of the coefficients to the real axis (see [9] for a discussion). In Proposition 4.1 below, the stochastic differentials of the root-processes of a complex polynomial process are calculated over time intervals for which Proposition 3.11 guarantees that they are well defined. We then show that for certain processes, in particular for complex Brownian polynomials, the description thus obtained will almost surely be valid for all time.

Proposition 4.1 *Let* $P(z)$ *be a (non-constant) complex polynomial process, defined for all* $t \geq 0$ *by*

$$P(z) \quad = \quad M_1 z^n + M_2 z^{n-1} + \cdots + M_n z + M_{n+1} \,,$$

where $M_1, \ldots, M_{n+1}$ *are complex local semimartingales. Also, let* Z *be a root-process of* P. *Then* Z *is a complex semimartingale over stopping-time intervals in which it neither explodes nor collides with another root-process, and over such stopping-*

time intervals its stochastic differential is given by:

$$
\begin{aligned}
dZ \quad = \quad & -\frac{1}{P'(Z)}\sum_{r=1}^{n+1} Z^{n+1-r}\,\mathrm{d}_I M_r + \\
& +\frac{1}{P'(Z)^2}\sum_{r=1}^{n+1}\sum_{s=1}^{n+1}(n+1-r)Z^{n-r}Z^{n+1-s}\,\mathrm{d}\langle M_r,\overline{M}_s\rangle \\
& -\frac{P''(Z)}{2P'(Z)^3}\sum_{r=1}^{n+1}\sum_{s=1}^{n+1} Z^{n+1-r}Z^{n+1-s}\,\mathrm{d}\langle M_r,\overline{M}_s\rangle\,.
\end{aligned}
$$

Proof: Proposition 3.11 ensures that a root-process Z exists and is well defined over stopping-time intervals as specified. We calculate $\mathrm{d}_I Z$ on such stopping time intervals by considering $P(z)$ to be a function of $M_1,\ldots,M_{n+1}$ and $z\in\mathbb{C}$ as in Remark 3.3. Let $F:\mathbb{C}^{n+1}\times\mathbb{C}\to\mathbb{C}$ be defined by $F(a_1,\ldots,a_{n+1},z)=a_1z^n+\cdots a_nz+a_{n+1}$, for $a_1,\ldots,a_{n+1},z\in\mathbb{C}$. Writing $\partial F/\partial z=F_z$, $\partial F/\partial a_r=F_r$ and using Itô's formula from Proposition 2.11 (and the fact that F is holomorphic), we find that

$$
\begin{aligned}
\mathrm{d}_I P(Z) \quad = \quad & \mathrm{d}_I F(M_1,\ldots,M_{n+1},z) \\
= \quad & \sum_{r=1}^{n+1} F_r(Z)\,\mathrm{d}_I M_r + F_z(Z)\,\mathrm{d}_I Z + \frac{1}{2}\sum_{r=1}^{n+1}\sum_{s=1}^{n+1} F_{rs}(Z)\,\mathrm{d}\langle M_r,\overline{M}_s\rangle \\
& +\frac{1}{2}\sum_{r=1}^{n+1}\left[F_{rz}(Z)\,\mathrm{d}\langle M_r,\overline{Z}\rangle + F_{zr}(Z)\,\mathrm{d}\langle Z,\overline{M}_r\rangle\right] \\
& +\frac{1}{2}F_{zz}(Z)\,\mathrm{d}\langle Z,\overline{Z}\rangle\,.
\end{aligned}
$$

Now calculate the partial derivatives of F and note that $\mathrm{d}\langle M_r,\overline{Z}\rangle = \mathrm{d}\langle Z,\overline{M}_r\rangle$ so as to obtain the equation

$$
\begin{aligned}
\mathrm{d}_I P(Z) \quad = \quad & \sum_{r=1}^{n+1} Z^{n+1-r}\,\mathrm{d}_I M_r \\
& +P'(Z)\,\mathrm{d}_I Z + \sum_{r=1}^{n+1}(n+1-r)Z^{n-r}\,\mathrm{d}\langle Z,\overline{M}_r\rangle + \frac{1}{2}P''(Z)\,\mathrm{d}\langle Z,\overline{Z}\rangle\,.
\end{aligned}
$$

We now recall that the process Z is defined so that $P(Z)\equiv 0$, and so the equation above allows us to find an expression for $\mathrm{d}_I Z$. The bracket processes $\mathrm{d}\langle Z,\overline{M}_r\rangle$ and $\mathrm{d}\langle Z,\overline{Z}\rangle$ can further be expressed in terms of $\mathrm{d}\langle M_r,\overline{M}_s\rangle$ by multiplying by $\mathrm{d}_I M_s$ and $\mathrm{d}_I Z$ respectively. Thus,

$$
\begin{aligned}
\mathrm{d}\langle Z,\overline{M}_s\rangle \quad &= \quad -\frac{1}{P'(Z)}\sum_{r=1}^{n+1} Z^{n+1-r}\,\mathrm{d}\langle M_r,\overline{M}_s\rangle \\
\mathrm{d}\langle Z,\overline{Z}\rangle \quad &= \quad \frac{1}{P'(Z)^2}\sum_{r=1}^{n+1}\sum_{s=1}^{n+1} Z^{n+1-r}Z^{n+1-s}\,\mathrm{d}\langle M_r,\overline{M}_s\rangle\,.
\end{aligned}
$$

Putting these expressions together completes the proof of the proposition. □

In the case of conformal martingale coefficients, the formula for the stochastic differentials is easily obtained from Proposition 4.1; we consider here the case where the coefficients are complex Brownian motions, to arrive at the first of our main theorems: we restate this as it was given in the introduction:

Theorem 1.1 restated: *Let $P(z)$ be the (non-constant) complex Brownian polynomial defined by*

$$P(z) \quad = \quad B_1 z^n + B_2 z^{n-1} \cdots + B_n z + B_{n+1} \,,$$

where the coefficients B_i are independent complex Brownian motions. The generic case (in the sense of Lebesgue measure induced by the coefficients) is for roots to be distinct; moreover the region of multiple roots is polar with respect to the complex Brownian polynomial. If $P(z)$ has distinct roots at time zero and if Z is a root-process then Z is a conformal martingale with stochastic differential defined almost surely for all positive time by

$$\mathrm{d}_I Z \quad = \quad -\sum_{r=1}^{n+1} \frac{Z^{n+1-r}}{P'(Z)} \,\mathrm{d}_I B_r \,.$$

Proof: The stochastic differential $\mathrm{d}_I Z$ can be calculated directly from Proposition 4.1, until the first time such that $P'(Z) = 0$ or a root-process explodes. From Corollary 3.15, we know that root-processes almost surely only explode if the leading coefficient is zero: since the leading coefficient is a complex Brownian motion, this will almost surely never happen. Thus to prove the theorem it is enough to show that $P'(Z)$ is almost surely never zero.

It follows from this result that $\mathrm{d}_I \langle Z, \overline{B}_r \rangle = 0$ for all r and therefore that Z is a conformal martingale, using the equivalent formulation following Definition 2.8. Moreover we can similarly deduce that

$$P'(Z) \quad = \quad nB_1 Z^{n-1} + (n-1)B_2 Z^{n-2} \cdots + 2B^{n-1} Z + B_n$$

is a conformal martingale. It must be non-constant, as we have excluded the case of constant P. It follows from Proposition 2.10 that $P'(Z)$ almost surely never hits zero if it does not start from zero, and leaves zero immediately never to return if it does start from zero. (A similar argument shows that Z itself can never explode to ∞.) Therefore the stochastic differential equation for Z is defined for all time, save possibly time 0 if $P'(Z)$ is zero at time 0. However this would imply a repeated root, which we have excluded from consideration.

The genericity of the set of repeated roots follows by noting if the coefficient Brownian motions $B_1, B_2, \ldots, B_n, B_{n+1}$ are started with rotationally symmetric Gaussian distributions of common variance 1, and means $1, 0, \ldots, 0, -1$ respectively, then $P(z)$ at some future time $t > 0$ will be statistically identical to what it would be at time $t+1$ if the coefficient Brownian motions were started as before but with variance 0. In the second case the roots start initially as roots of unity, and so begin distinct, and so remain distinct for all time. The Lebesgue measure genericity follows by considering that in the first case the initial coefficient values have a joint density everywhere positive. □

Remark 4.2 If we replace the complex Brownian coefficients $B_1, \ldots, B_{n+1}$ by conformal martingales $M_1, \ldots, M_{n+1}$ such that $\langle M_r, \overline{M}_s \rangle = 0$ for all r, s then $P'(Z)$

remains a conformal martingale, by Corollary 2.12. Thus, the theorem holds in this case also, so long as the set of repeated roots is still polar in the new set-up. (This includes in particular the monic case, in which B_1 is replaced by the constant value 1.)

Finally in this section we consider what can be said if $P(z)$ *does* have repeated roots at time zero. Since $B_1, Z_1, \ldots, Z_n$ is simply a reparametrization of $B_1, \ldots, B_{n+1}$, it is a diffusion and so our question is essentially to identify the entrance boundary behaviour of a Markov chain. As a consequence of the zero-one law for Euclidean Brownian motion, given the values of the roots and of B_1 there is just one way for the diffusion to behave: there is just one entrance boundary point for each B_1 and set of initial values of roots.

First note that, given repeated roots $Z_1 = Z_2 = \ldots = Z_k$ at time 0 (say), there is no way to select one of the paths $Z_1|_{[0,t)}, \ldots, Z_k|_{[0,t)}$ without using information from the future at time 0! This follows by a standard measure-theoretic argument. For to be able to do this we would have to be able to separate the *germs* at 0 of these paths (equivalence classes of paths agreeing from time 0 up to path-dependent times). Now the Markov property together with the positive density of Gaussian distributions make it clear that no events in the σ-algebra generated by $\{B_t(s), \ldots, B_{n+1}(s) : s > t\}$ can separate paths $Z_1|_{[0,t/2)}$, $Z_2|_{[0,t/2)}$, and therefore such events cannot separate the corresponding germs. But the property "cannot separate germs at 0 of Z_1, Z_2" is preserved by intersection, taking complements, and countably infinite unions. Therefore it follows that germs cannot be separated by events in the σ-algebra generated by $\{B_t(s), \ldots, B_{n+1}(s) : s > 0\}$; and therefore to distinguish Z_1 from Z_2 we have to have information about Z_1, Z_2 strictly belonging to times in the future of 0.

The issue here is exactly the issue that arises if one tries to identify a specific representative Θ of the class of Θ processes occurring modulo 2π in the polar representation (R, Θ) of two-dimensional Brownian motion.

We now discuss the question of how the repeated roots behave for small time. For the sake of simplicity we treat only the monic case

$$P(z) \quad = \quad z^n + B_2 z^{n-1} + \ldots + B_n z + B_{n-1}$$

with all of $B_2(0) = \ldots = B_{n+1} = 0$: other cases can be treated in similar ways. Individual roots cannot be identified until positive time has passed: therefore we obtain information about the evolution of the roots for small time by considering simple stochastic polynomials whose sets of roots approximate rescalings of the set of roots of $P(z)$ for small time.

Begin by scaling the roots using the Bessel process S, where $S^2 = |B_2|^2 + \ldots + |B_{n+1}|^2$. Setting $w = z/S^{1/n}$, we obtain the following equation for scaled roots z:

$$w^n + S^{(n-1)/n}\left(\frac{B_2}{S}\right) w^{n-1} + \ldots + S^{1/n}\left(\frac{B_n}{S}\right) w + \left(\frac{B_{n+1}}{S}\right) \quad = \quad 0\,. \quad (13)$$

The joint distribution of

$$\frac{B_2}{S}, \ldots, \frac{B_{n+1}}{S}$$

does not depend on either time t or the Bessel process S: moreover $S(t) \to 0$ as $t \to 0$. Therefore we can appeal to the continuous dependence of roots on coefficients (Lemma 3.6(i)) and deduce that for small t the roots of Eq. (13) are approximated by those of

$$w^n + \left(\frac{B_{n+1}}{S}\right) \quad = \quad 0\,.$$

So the collection of root processes $(Z_1, Z_2, \ldots, Z_n)$ of the original polynomial process $P(z)$ can be approximated for small time t by

$$\left\{\frac{Z_1}{S^{1/n}}, \ldots, \frac{Z_n}{S^{1/n}}\right\} \quad \to \quad \left\{\omega \cdot \left(\frac{B_{n+1}}{S}\right)^{1/n} \;:\; \omega^n + 1 = 0\right\}. \tag{14}$$

However this is not a complete account: as time t tends to zero the ratio $|B_{n+1}|/S$ of stochastic processes can visit very close to zero; while the approximation Eq. (14) translates into an asymptotic for random processes, it does so at the price of not excluding repeat visits to the locus where all roots vanish. (The problem is related to criticality of the 2-dimensional Bessel process: Bessel processes of higher dimension have lower functions such as $\sqrt{t^{1+\epsilon}}$ for all positive ϵ, whereas 2-dimensional Bessel processes can visit much closer to zero for small times: see the treatment in Itô and McKean [13, §4.12], also [23, 30].)

We can of course exclude the question of repeat visits by noting polarity as mentioned in Theorem 1.1: however a slightly more detailed picture can be obtained by using a different scaling. Replace S by R where

$$R^2 \quad = \quad |B_n|^{2n/(n-1)} + |B_{n+1}|^2 .$$

Scaling as before, but now with $u = z/R^{1/n}$, we obtain

$$\begin{aligned} u^n + R^{(n-1)/n}\left(\frac{B_2}{R}\right)u^{n-1} \quad + \quad \ldots + R^{2/n}\left(\frac{B_{n-1}}{R}\right)u^2 & \\ + \quad R^{1/n}\left(\frac{B_n}{R}\right)u + \left(\frac{B_{n+1}}{R}\right) \quad &= \quad 0\,. \end{aligned} \tag{15}$$

Since we have

$$\left|\frac{B_n}{R^{(n-1)/n}}\right|^{2n/(n-1)} + \left|\frac{B_{n+1}}{R}\right|^2 \quad = \quad 1 \tag{16}$$

we are assured that for Eq. (15) one or both of the last two coefficients will always be non-trivial, even though the work reported in [13, §4.12] shows that for arbitrarily small times the last coefficient may be very small. Moreover the same work shows us we can neglect all other terms: using upper and lower functions,

$$\begin{aligned} \limsup_{t\to 0} \frac{B_j(t)}{\sqrt{t^{1-\epsilon}}} \quad &= \quad 0 \\ \limsup_{t\to 0} \frac{\sqrt{t^{1+\epsilon}}}{|B_n(t)|^2 + |B_{n+1}|^2} \quad &= \quad 0 \end{aligned}$$

while we can use the Hölder inequality to show

$$\begin{aligned} R^2 \quad = \quad |B_n|^{2n/(n-1)} + |B_{n+1}|^2 \quad &\geq \quad |B_n|^{2n/(n-1)} + |B_{n+1}|^{2n/(n-1)} \\ &\geq \quad \left(\frac{|B_n(t)|^2 + |B_{n+1}|^2}{2^{1/(n-1)}}\right)^{n/(n-1)} . \end{aligned}$$

It follows that for sufficiently small ϵ all but the leading and the last two terms of Eq. (15) will tend to zero considered as stochastic processes. Thus we can improve on the asymptotic of Eq. (14) and assert that as time tends to zero so the set of scaled root processes

$$\left\{ \frac{Z_1}{R^{1/n}}, \ldots, \frac{Z_n}{R^{1/n}} \right\}$$

is uniformly approximated in time by the set of roots of

$$u^n + R^{1/n} \left(\frac{B_n}{R} \right) u + \left(\frac{B_{n+1}}{R} \right) \quad = \quad 0 \,. \tag{17}$$

A little care is still needed with the interpretation of this last asymptotic. Roots of Eq. (17) can coincide pairwise only when it can be solved simultaneously with the equation for roots of the derivative polynomial

$$n u^{n-1} + R^{1/n} \left(\frac{B_n}{R} \right) \quad = \quad 0 \,. \tag{18}$$

Such solutions can happen only when the ratio B_n^{n-1}/B_{n-1}^n visits the point $(n-1)^{n-1}/(-n)^n$; the ratio is a conformal martingale so almost surely for all positive time there will be no such solution. However *close* approaches can be made, and so the approximation does not identify the *individual* scaled roots process-by-process with corresponding approximating roots of Eq. (17). This is inevitable: consideration of the original polynomial $P(z)$ and the locus of repeated roots (which is the closure of a locus of complex codimension 1 corresponding to double roots) shows that pairs of roots of $P(z)$ itself will for small time approach each other in the manner of a BES(2) process approaching zero. BES(k) rescalings (for $k \geq 3$) cannot hope to disentangle such close approximations. Note however that roots of Eq. (17) would coincide in triples only if it could be solved simultaneously with not only Eq. (18) but also

$$n(n-1) u^{n-1} \quad = \quad 0 \,. \tag{19}$$

Since $R^{1/n} B_n / R$, B_{n+1}/R cannot both be small, by Eq. (16), this cannot happen even approximately.

Notice further that exact solutions of Eq. (17), even when available, will not be illuminating: case $n = 3$ corresponds to the notoriously un-illuminating solution of the general cubic (under the standard transformation removing the second term).

5 Real Brownian polynomials

We turn now to consider polynomial processes based on real Brownian coefficients.

Proposition 5.1 *Let $X_1, \ldots, X_{n+1}$ be real semimartingales, and let $P(z)$ be the non-constant polynomial process defined by $P(z) = X_1 z^n + X_2 z^{n-1} + \cdots + X_n z + X_{n+1}$ for all $z \in \mathbb{C}$. Then a root-process Z satisfies the following stochastic differential* *equation over stopping-time intervals in which $P'(Z)$ is non-vanishing:*

$$\begin{aligned}
\mathrm{d}_I Z &= -\sum_{r=1}^{n+1} \frac{Z^{n+1-r}}{P'(Z)} \mathrm{d}_I X_r \\
&\quad + \frac{1}{P'(Z)^2} \sum_{r=1}^{n+1} \sum_{s=1}^{n+1} \Big((n+1-r) Z^{n-r} Z^{n+1-s} \\
&\qquad\qquad - \frac{Z^{n+1-r} Z^{n+1-s} P''(Z)}{2P'(Z)} \Big) \mathrm{d}\langle X_r, X_s \rangle \,.
\end{aligned}$$

Proof: This proposition is proved using the same method as Proposition 4.1. The difference here is that the function F here is not holomorphic and so the version of Itô's formula given in Proposition 2.13 is used. □

The aim of this section is to extend the description given of the root-processes of a real Brownian polynomial by Proposition 5.1 so that it is valid for all time. However, the method used for the complex case cannot now be used: we have already seen in Examples 3.5 and 3.8 that both collisions and explosions have a positive probability of occurring. In this section we build towards a proof of our second main theorem, which we restate here:

Theorem 1.3 restated: *Let $P(z)$ be the (non-constant) real Brownian polynomial defined by*

$$P(z) \quad = \quad B_1 z^n + B_2 z^{n-1} + \cdots + B_n z + B_{n+1} \,.$$

where the coefficients B_i are independent real Brownian motions. Root-processes Z of $P(z)$ cannot be constructed so as to possess the semimartingale property for all time: however a root-process is a semimartingale over time intervals in which it neither collides with other root-processes of $P(z)$ nor explodes, and then its stochastic differential satisfies

$$\mathrm{d}_I Z \;=\; -\sum_{r=1}^{n+1} \frac{Z^{n+1-r}}{P'(Z)} \mathrm{d}_I B_r + \sum_{r=1}^{n+1} \frac{Z^{2(n+1-r)}}{P'(Z)^2} \left(\frac{n+1-r}{Z} - \frac{P''(Z)}{2P'(Z)} \right) \mathrm{d}t \,. \tag{20}$$

(Note: in the above Z may be complex, over periods in which it is tracking one half of a conjugate pair of roots.) Collisions almost surely only occur between pairs of real root processes or between complex conjugate root-processes. If (Z_1, Z_2) is a pair of root-processes defined up to and colliding at time τ, then their definition can be extended continuously across the collision time τ (up to necessary ambiguities relating to branch-of-root) by re-expressing the roots in the coordinate system given by elementary symmetric functions:

$$\alpha = Z_1 + Z_2 \quad \textit{and} \quad \gamma = Z_1 Z_2 \,.$$

These processes possess the semimartingale property over stochastic intervals containing only collisions between Z_1 and Z_2. At explosion times we can change coordinates *using $W = 1/Z$ and so re-express the root-process Z in terms of the corresponding root-process $W = 1/Z$ of the Brownian polynomial*

$$P^*(w) \quad = \quad B_1 + B_2 w + \cdots + B_n w^{n-1} + B_{n+1} w^n \,,$$

which will almost surely not explode at such times.

As in the proof of Theorem 1.1, the main difficulty is in dealing with the behaviour at collision and explosion times. Away from collision points, the stochastic differential (20) is easily calculated using Proposition 5.1. The first stage of the proof is to show that only those types of collisions mentioned in Theorem 1.3 can occur.

Notation

For all nonnegative integers n, let $\mathbb{R}(z,n)$ denote the space of all polynomials of degree n in the complex variable z and with coefficients in $\mathbb{R}$.

$$\mathbb{R}(z,n) \quad \equiv \quad \left\{F(z) = a_1 z^n + a_2 z^{n-1} + \cdots + a_n z + a_{n+1} : a_1, \ldots, a_{n+1} \in \mathbb{R}\right\} \,.$$

We may define a smooth structure on $\mathbb{R}(z,n)$ via the coordinate system Φ which is given by

$$\begin{aligned}
&\Phi : \mathbb{R}(z,n) \to \mathbb{R}^{n+1} \,, \\
&\Phi(F(z)) = (a_1, \ldots, a_{n+1}) \,, \\
&\quad \text{for } F(z) = a_1 z^n + a_2 z^{n-1} + \cdots + a_n z + a_{n+1} \,.
\end{aligned}$$

Thus, Φ gives the space $\mathbb{R}(z,n)$ the structure of $\mathbb{R}^{n+1}$. In particular we note that the space $\mathbb{R}(z,n)$ is $n+1$-dimensional. We define below the subsets of $\mathbb{R}(z,n)$ which P must hit when its roots collide: we call these collision spaces.

Definition 5.2 **(i)** *We set L to be the space of all polynomials in $\mathbb{R}(z,n)$ with repeated roots:*

$$L = \{f(z) \in \mathbb{R}(z,n) : f(\omega) = f'(\omega) = 0, \textit{ for some } \omega \in \mathbb{C}\} \;;$$

(ii) *We define L_1 to be the set of all polynomials in $\mathbb{R}(z,n)$ with a repeated real root:*

$$\begin{aligned}
L_1 \quad = \quad &\big\{f(z) \in \mathbb{R}(z,n) : f(z) = (z-x)^2 g(z) \\
&\quad \textit{for some } x \in \mathbb{R}, g(z) \in \mathcal{R}(z, n-2)\big\} \,;
\end{aligned}$$

(iii) *We define L_2 to be the set of all polynomials in $\mathbb{R}(z,n)$ with two pairs of repeated real roots:*

$$L_2 = \left\{f(z) \in \mathbb{R}(z,n) : f(z) = (z-x)^2 (z-y)^2 g(z)\right\}$$

where $x, y \in \mathbb{R}$ and $g(z) \in \mathcal{R}(z, n-4)$;

(iv) *We define L_3 to be the set of all polynomials in $\mathbb{R}(z,n)$ with a repeated root of order 3:*

$$L_3 = \left\{f(z) \in \mathbb{R}(z,n) : f(z) = (z-x)^3 g(z)\right\}$$

where $x \in \mathbb{R}$ or $\mathbb{C}$ and $g(z) \in \mathcal{R}(z, n-3)$;

(v) *We define L_4 to be the set of all polynomials with a repeated complex root. In factorising these polynomials, we note that if a polynomial has one repeated complex root, then the conjugate of that root will also be a repeated root of the polynomial. Thus:*

$$L_4 = \left\{ f(z) \in \mathbb{R}(z,n) : f(z) = (z-\omega)^2 (z-\overline{\omega})^2 g(z) \right\}$$

where $\omega \in \mathbb{C}$ and $g(z) \in \mathcal{R}(z, n-4)$.

Lemma 5.3 *The Hausdorff dimensions of the collision spaces are bounded as follows:*

$$\begin{aligned} \dim_H(L_1) &\leq n\,, & \dim_H(L_2) &\leq n-1\,, \\ \dim_H(L_3) &\leq n-1\,, & \dim_H(L_4) &\leq n-1\,. \end{aligned}$$

Proof: We have observed that the dimension of the space $\mathbb{R}(z,n)$ is $n+1$. Using this and the fact that smooth functions (or more generally, locally Lipschitz functions) cannot increase Hausdorff dimension [7, Corollary 2.4, p.30], the lemma follows immediately from considering the maps defined below, which map smoothly onto the collision spaces.

$$\begin{aligned}
\Psi_1 &: \mathbb{R} \times \mathcal{R}(z,n-2) \to L_1\,, & \Psi_1(x, g(z)) &= (z-x)^2 g(z) \\
\Psi_2 &: \mathbb{R}^2 \times \mathcal{R}(z,n-4) \to L_2\,, & \Psi_2(x, y, g(z)) &= (z-x)^2 (z-y)^2 g(z) \\
\Psi_3 &: \mathbb{R}^3 \times \mathcal{R}(z,n-3) \to L_3\,, & \Psi_3(x, g(z)) &= (z-x)^3 g(z) \\
\Psi_4 &: \mathbb{R}^2 \times \mathcal{R}(z,n-4) \to L_4\,, & \Psi_4(x, y, g(z)) &= (z-x-iy)^2 (z-x+iy)^2 g(z)
\end{aligned}$$

□

If we apply the coordinate system Φ to the polynomial process $P(z)$ at each time t then we obtain an n-dimensional real stochastic process $\Phi(P)$: this is clearly the Brownian motion associated with P. We now consider whether this Brownian motion will hit the subsets of $\mathbb{R}^n$ which correspond (via Φ) to the collision spaces. By using the observed Hausdorff dimension of the various sets involved, we show that the only way in which the roots of $P(z)$ can collide at a particular time is if $P(z)$ has a single pair of real roots at that time. This can only occur *via* the collision of two real roots, or *via* the degeneration to reality of a complex conjugate pair of roots.

Proposition 5.4 *Given two root-processes Z_1 and Z_2 of the Brownian polynomial process P at a specific positive time t, almost surely the next collision will not be between Z_1 and Z_2 unless either of the cases below hold*

(i) *Both Z_1 and Z_2 are real;*

(ii) *The roots Z_1 and Z_2 are a pair of complex conjugate roots.*

Any collision involving three or more root-processes will almost surely never happen; collisions and explosions will almost surely never occur at the same time.

Proof: To prove the proposition, we consider the Brownian motion

$$B = (B_1, \ldots, B_{n+1}) = \Phi(P)$$

associated with P. At the time of root collision this must lie in one of the above spaces. If three (or more) root-processes collide, then P must hit the collision space L_3. If two non-conjugate complex root-processes collide, then at the time of collision then P must hit the space L_4. If a real root-process collides with a complex root-process, then it must also collide with the conjugate of the complex root-process, and so again P must lie in the space L_3. Thus to show that only the types of collision (i) or (ii) above can occur (where P lies in L_2) it is sufficient to show that P will almost surely never hit spaces L_3 or L_4.

However, if P hits either L_3 or L_4 then the Brownian motion B must hit the spaces $\Phi(L_3)$ or $\Phi(L_4)$ respectively. Since the coordinate system Φ is smooth, we know from Lemma 5.3 that the Hausdorff dimension of the spaces that B must hit have the following bounds:

$$\begin{aligned} \dim_H \Phi(L_3) &\leq \dim_H \Phi(\Psi_3(\mathbb{R}^3 \times \mathcal{R}(z, n-3)) \\ \dim_H \Phi(L_4) &\leq \dim_H \Phi(\Psi_4(\mathbb{R}^2 \times \mathcal{R}(z, n-4)) . \end{aligned}$$

Each of these dimensions is less than or equal to $n-1$. This means that such a collision can occur only if the $n+1$-dimensional Brownian motion B hits a space of codimension ≤ 2 and it is well known that this will almost surely never happen (for a proof see for example [14, Theorem 4, p. 245]). Two root-processes clearly cannot explode at the same time, since for this to happen, the planar Brownian motion (B_1, B_0) would have to hit the origin. Similarly, if two root-processes collide at the same time as an explosion then the Brownian motion would have to hit space L at a time when the leading coefficient was zero; this again would mean that B would have to hit a space of dimension less than or equal to $n-1$, and so almost surely this would never happen. □

Proposition 5.5 *Let $Z_1, \ldots, Z_n$ be distinct root-processes of the Brownian polynomial $P(z)$ and suppose that Z_1 and Z_2 are adapted to the filtration and collide at positive time τ. Then Z_1 and Z_2 cannot be semimartingales in any stopping-time interval containing τ. However they can be extended continuously over the collision point (up to the necessary ambiguity), in the coordinate system given by*

$$\alpha = Z_1 + Z_2 \quad \textit{and} \quad D = \alpha^2 - 4Z_1Z_2 .$$

Proof: For some neighbourhood of time around τ, Proposition 5.4 ensures that Z_1 and Z_2 will be the only two root-processes to collide and neither of them can explode. Thus in a deleted neighbourhood of τ we can write P in the form:

$$P(z) \quad = \quad (z - Z_1)(z - Z_2)S(z)$$

where $S(z)$ is a polynomial process of degree n whose root-processes almost surely neither collide nor explode. Indeed, we can write

$$S(z) \quad = B_1 \quad \prod_{i=3}^{n} (z - Z_i) .$$

So, using this representation of P and the method of Example 3.8 we re-parameterize the colliding roots in the following way:

$$\begin{aligned}
\alpha &= Z_1 + Z_2 = -\sum_{r=2}^{n} Z_r - B_2/B_1 \\
\gamma &= Z_1 Z_2 = \frac{(-1)^n B_n/B_1}{\prod_{r=3}^{n} Z_r} \\
D &= \alpha^2 - 4\gamma\,.
\end{aligned}$$

It is clear that α is well defined for all time; γ is well defined unless one of the roots (say Z_3) is zero at a collision point. This means that the last coefficient B_n must be zero. Examining the dimension of the space that the associated Brownian motion would have to hit, we see that this will almost surely never happen.

Notice that D is always real in the vicinity of a collision between Z_1 and Z_2, since near there they are always either both real or else complex conjugates of each other.

We now check that the conditions of Lemma 3.9 apply for $Y = |Z_1 - Z_2|$ and D. The filtration conditions apply immediately, and clearly $Y^2 = |D|$. Moreover D is locally a smooth function of an $n+1$-dimensional Brownian motion, and therefore $\mathbb{P}\left[\, D(t) = 0 \,\right] = 0$ for all positive t (since the relevant locus has Lebesgue measure zero).

So the proof is finished if we can show that $L = |D| - \int \operatorname{sgn}(D)\, \mathrm{d}_I D$ cannot be constant in an interval in which D changes sign. But we can write

$$\mathrm{d}_I Z_k \quad = \quad -\frac{1}{B_1} \sum_{j=1}^{n+1} \frac{Z_k^{n+1-j}}{\prod_{j \neq k}(Z_k - Z_j)}\, \mathrm{d}_I B_j + \mathtt{drift}\ \mathrm{d}_I Z_k$$

(using $P'(Z_k) = \prod_{j \neq k}(Z_k - Z_j)$ and so

$$\begin{aligned}
(\,\mathrm{d}_I D)^2 &= 4(Z_1 - Z_2)^2 (\,\mathrm{d}_I Z_1 - \,\mathrm{d}_I Z_2)^2 \\
&= \frac{4(Z_1 - Z_2)^2}{B_1^2} \sum_{j=1}^{n+1} \left(\frac{Z_1^{n+1-j}}{\prod_{j \neq 1}(Z_1 - Z_j)} - \frac{Z_2^{n+1-j}}{\prod_{j \neq 2}(Z_2 - Z_j)} \right)^2 \mathrm{d}t \\
&= \frac{4}{B_1^2} \sum_{j=1}^{n+1} \left(\frac{Z_1^{n+1-j}}{\prod_{j \neq 1,2}(Z_1 - Z_j)} - \frac{Z_2^{n+1-j}}{\prod_{j \neq 1,2}(Z_2 - Z_j)} \right)^2 \mathrm{d}t\,.
\end{aligned}$$

Since near τ we know B_1 doesn't hit zero and Z_1, Z_2 collide with no other root processes, it follows that $(\,\mathrm{d}_I D)^2/\,\mathrm{d}t > 0$ near τ, and is a locally continuous function of $B_1, Z_1, \ldots, Z_n$. This allows us to localize and argue as immediately before Lemma 3.9, that L cannot be constant in an interval in which D changes sign.

Thus $|Z_1 - Z_2|$ cannot be a semimartingale in stopping-time intervals containing τ. Since $\alpha = Z_1 + Z_2$ *is* a semimartingale in such stopping-time intervals, if sufficiently small to exclude other collisions, it follows that neither Z_1 nor Z_2 can be semimartingales, as required. □

Proof: (of Theorem 1.3)
The proof of Theorem 1.3 follows from Lemma 5.3, Proposition 5.4 and Proposition 5.5, noting that the root-processes of the polynomial process $P^*(w)$ will almost

surely not explode at the same time as those of $P(z)$, since the planar Brownian motion (B_1, B_{n+1}) almost surely doesn't hit the origin.

As in the quadratic case, the ambiguity of branch-of-root can be resolved by choosing the branch at random with probability $\frac{1}{2}$ each way, or else for example by connecting the larger of the pair of colliding roots in real mode with the $+i$-excursion in conjugate mode. What is the difference between resolving this ambiguity for the case of real Brownian coefficients as opposed to the case of complex Brownian coefficients with the polynomial started with repeated roots? In the real case the possibility of collisions means that the root-processes are not uniquely defined, and therefore a particular root-process is singled out using information pertaining to positive times (*eg* whether the root is the larger of two colliding real roots) which has to be used progressively as time progresses and as collisions force the use of extra information. In the complex case a root-process is uniquely defined as soon as we have specified its trajectory up to some small positive time, but we still need information delivered up to some positive time in order to identify the root. □

Remark 5.6 Topological considerations imply that there *will* be collision times as well as explosion times. Intuitively, this is clear: if not then the polynomial process would always have the same number of real and complex roots. More formally we see this by again considering the sets of $\mathbb{R}(z, n)$ that will by hit by the associated Brownian motion: the Brownian motion will almost surely enter the open cone of $\mathbb{R}(z, n)$ in which the polynomial has exactly r pairs of conjugate roots (for $0 \leq 2r \leq n$) in finite time.

Finally, as in Section 4, one may consider the entrance boundary for repeated roots. Exactly the same kind of analysis holds: in summary, in the typical monic case

$$x^n + B_1 x^{n-1} + \ldots + B_{n+1}$$

we use the scaling Bessel process R, where

$$R^2 \quad = \quad |B_{n-1}|^{2n/(n-2)} + |B_n|^{2n/(n-1)} + |B_{n+1}|^2 ,$$

and relate the scaled root processes $Z_i / R^{1/n}$ to the roots of

$$v^n + \left(\frac{B_{n-1}}{R^{(n-2)/n}}\right) v^2 + \left(\frac{B_n}{R^{(n-1)/n}}\right) v + \left(\frac{B_{n+1}}{R}\right) .$$

Now the non-criticality of the BES(3) process $\sqrt{B_{n-1}^2 + B_n^2 + B_{n+1}^2}$ allows us as before to use lower function analysis to show negligibility of the remaining terms.

6 Further remarks and generalizations

6.1 Remark on the calculations

Initially, the root-differentials of the monic Brownian quadratic and cubic were calculated using computer algebra: the implementation *Itovsn3* of stochastic calculus in the computer algebra package *REDUCE* was used [15]. These calculations suggested a general form for the stochastic differential in the case of a monic Brownian polynomial $P(z)$ of degree n, which was then derived using the algebraic method sketched below.

The Brownian coefficients can be expressed as homogeneous polynomials of the root-processes. Thus, assuming the existence of n root-processes, and using Itô's formula, a system of $\frac{1}{2}n(n+1)$ equations can be found in the $\frac{1}{2}n(n+1)$ infinitesimal quadratic covariations of the system. By solving this system the infinitesimal quadratic covariations of the roots are obtained. The drifts can be calculated in a similar way. As would be expected, the condition for there to be a consistent solution to the system is that root-processes do not collide.

These calculations give an alternative expression of the stochastic differentials in terms of all n root-processes. If $Z_1, \ldots, Z_n$ are the n distinct root-processes of $P(z)$, then

$$
\begin{aligned}
\mathrm{d}_I Z_j \quad = \quad & -\sum_{r=1}^{n} \frac{Z_j^{n+1-r}}{B_1 \prod_{s \neq j}(Z_j - Z_s)} \mathrm{d}_I B_r \\
& + \frac{1}{B_1} \frac{Z_r^n \, \mathrm{d}\langle B_1, \overline{B_1} \rangle}{B_1 \prod_{s \neq j}(Z_j - Z_s)} + \sum_{r=1,\, r \neq j}^{n} \frac{\mathrm{d}\langle Z_r, \overline{Z_j} \rangle}{(Z_j - Z_r)} \,.
\end{aligned}
$$

This formula is of course equivalent to the expression (2) found in Theorem 1.3: this may be proved directly using the fact that for any polynomial $p(z)$, with roots x_r, $p'(x_r) = \prod_{j \neq r}(x_r - x_j)$.

The above form Eq. (21) of writing the stochastic differentials suggests connections with work on the eigenvalues of matrix-valued diffusions. The latter arise in connection with problems from particle physics, where the eigenvalues represent interacting particles [3, 5, 28]. Eigenvalues of matrix diffusions are also studied in order to find skew product representations of Brownian motion on certain manifolds of matrices [25, 24]. Typically, the eigenvalues λ_i, $i = 1, \ldots, n$ satisfy a stochastic differential equation such as that given in [3]

$$
\mathrm{d}_I \lambda_r \quad = \quad \frac{1}{\sqrt{n}} \mathrm{d}_I B_r - \frac{1}{2} \lambda_r \, \mathrm{d}t + \frac{1}{2n} \sum_{s \neq r} \frac{1}{\lambda_r - \lambda_s} \, \mathrm{d}t \,,
$$

where the B_r are independent Brownian motions. However in the cases studied in the papers already mentioned the eigenvalues never collide.

6.2 An algebraic context for root-processes

The above discussion suffices for our present purposes, but is somewhat tied to the particular case of Brownian polynomials. There is a more general, algebraic, context for general root-processes of polynomial-valued semimartingales which one of us plans to investigate further [19]. We sketch out the arguments here since they shed further light on the essential ambiguity of choice of branch-of-root when constructing root-processes.

Issues concerning roots of polynomials were of course the driving motivation behind the formulation of Galois theory, which is all to do with extending algebraic fields by adding in the roots of polynomials with coefficients belonging to the original field. How can Galois theory be used to deal with roots of semimartingale polynomials?

The first step is to make sense of algebraic fields made up of modest generalizations of semimartingales (since we need to include inverses, we cannot hope to build non-trivial instances of fields using only genuine semimartingales). We can do this using a "field of fractions" construction: given an algebra of semimartingales, we can make sense of X/Y for X and Y in the algebra exactly when the algebra is an *integral domain*: which is to say, when the relationship

$$XY \quad \equiv \quad 0$$

forces the deduction that one of X, Y must vanish identically.

This certainly forces restrictions on the algebra of semimartingales to be considered. For example, it cannot contain both of B and $|B|$ for B a real Brownian motion, since then we would have

$$(B + |B|) \times (B - |B|) \quad = \quad B^2 - |B|^2 \quad \equiv \quad 0\,.$$

However it can be shown [19] that $\mathcal{P} = \mathcal{P}_n$ is an integral domain, where $\mathcal{P}_n$ is the smallest class of semimartingales containing $B_0(t) = t$, B_1, ..., B_n (for B_1, ..., B_n independent real Brownian motions) which is closed under scalar multiplication, addition, multiplication, and stochastic integration (thus if X, $X \in \mathcal{P}_n$ then $\int X \, \mathrm{d}_I Y \in \mathcal{P}_n$). The harder, non-commutative, analogue of this is now well-known to workers in quantum probability theory [1, 20]. There are at least two simple proofs for the commutative case of $\mathcal{P}$: either a potential-theoretic approach based on Itô calculus applied to

$$\frac{1}{(T-t)^{\beta/2}} \exp\left(-\frac{X(t)^\alpha}{2(T-t)}\right)$$

for $0 < \beta << \alpha << 1$ to show that

$$\mathbb{P}\left[\, X(T) = 0 \,\right] = 0 \quad \text{for } T > 0$$

and $X \not\equiv 0$, or an approach based on uniqueness of stochastic integration and representation by finite chaos expansions, showing that

$$\mathbb{P}\left[\, X(s) = 0 \text{ for } T < s < T + \epsilon \,\right] > 0$$

forces $X \equiv 0$. Both approaches use induction on the algebraic complexity of $X \in \mathcal{P}$, but differ how they extend beyond the case of $\mathcal{P}$ and in the strength of the property used to deduce that $\mathcal{P}$ is an integral domain.

We can thus define the field of "projective semimartingales" $\mathcal{R} = \mathcal{R}_n = \{X/Y : X, Y \in \mathcal{P}_n, Y \not\equiv 0\}$, including for example $1/B_1$.

Since $\mathcal{R}$ is a field, if $P(z)$ is a polynomial with coefficients in $\mathcal{R}$ (or, without loss of generality, in $\mathcal{P}$) then we can call on Galois theory [10] to deduce the existence of a field $\mathcal{R}(P)$ extending $\mathcal{R}$ so as to include the roots of $P(z)$. Moreover we can use this *stochastic Galois theory* to show that $\mathcal{R}(P)$ can be chosen to be as small as possible (the splitting field for $P(z)$ over $\mathcal{R}$) and that it then is unique up to algebraically isomorphism. We may realize such an extension in terms of root-processes ("branched semimartingales"), by using the methods of this paper to compute stochastic differentials of root-processes of minimal polynomials over stochastic intervals where they are isolated roots, and by making arbitrary (for example random choices of branch-of-root to resolve ambiguities where there are collisions of roots. Stochastic Galois theory imposes some constraints on these choices, since they should respect the symmetries of the Galois group of the field extension $[\mathcal{R}(P) : \mathcal{R}]$.

6.3 Generalizations

The above conclusions still hold when the generated Brownian motions are altered by added drifts. In particular Theorems 1.1 and 1.3 holds when the Brownian motions B_r are replaced by Ornstein-Uhlenbeck processes in stationary equilibrium. Now we can immediately read off the long-run behaviour of the resulting root-processes from known results concerning limiting behaviour of roots of random polynomials [2]. Just for example, in the case where random polynomials have complex normal coefficients [6], the roots concentrate on the unit circle as $n \to \infty$. We therefore can deduce that a specific root-process of a high-degree complex Ornstein-Uhlenbeck polynomial will spend most of its time near the unit circle. It would be interesting to explore the behaviour of the large deviations which must occur; [3, 28] are suggestive here.

We have already indicated above how this work might be generalized to root-processes for polynomial-valued semimartingales. The general perspective sketched out in Subsection 6.2 defines root-processes of general polynomial-valued semimartingales as "branched semimartingales", without recourse to complex projective spaces. The specific pairwise "dance of the roots" of Theorem 1.3 will not occur in general: consider the behaviour of the roots of the irreducible polynomial

$$P(z) \quad = \quad B^{p-1} + B^{p-2}z + \ldots + Bz^{p-2} + z^{p-1}$$

(for an independent real Brownian motion B and prime integer p: this is irreducible over the integral domain $\mathcal{P}(t, B)$ by the Eisenstein criterion [10, Theorem 3.J]). These form a system of $p-1$ complex roots which collide altogether whenever B hits zero. However a rather full picture may be constructed using Itô analysis as in Section 5, resolving the ambiguities arising from branch-of-root problems by careful development of the stochastic Galois theory sketched in Subsection 6.2.

Further work in this direction is planned, so as to exploit recent advances in computational algebra such as Gröbner basis algorithms [4], using the symbolic Itô calculus developed in [15, 16, 18]. (A modest and elementary example of Gröbner bases applied to stochastic calculus is to be found in [17].)

References

[1] S. Attal. Problèmes d'unicité dans les représentations d'opérateurs sur l'espace de Fock. In *Séminaire de Probabilités XXVI*, volume 1526, pages 619–632, 1992.

[2] A.T. Bharucha-Reid and M. Sambandham. *Random Polynomials*. Academic Press, New York, 1986.

[3] T. Chan. The Wigner Semi-Circle Law and eigenvalues of matrix-valued diffusions. *Probability Theory and Related Fields*, 93:249–272, 1992.

[4] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms*. Springer-Verlag, New York, second edition, 1997.

[5] F.J. Dyson. A Brownian-motion model for the eigenvalues of a random matrix. *J. Math. Phys.*, 3:1191–1198, 1962.

[6] A. Edelman and E. Kostlan. How many of the zeros of a random polynomial are real? *Bulletin of the American Mathematical Society*, 32:1–37, 1995.

[7] K. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*. John Wiley & Sons, Chichester, 1991.

[8] R.K. Getoor and M.J. Sharpe. Conformal Martingales. *Inventiones Mathematicae*, 16:271–308, 1972.

[9] J. Hammersley. The zeros of a random polynomial. In J. Neyman, editor, *Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability*, volume 2, pages 89–111. University of California Press, Berkeley, CA, 1956.

[10] I.N. Herstein. *Topics in Algebra*. Xerox College Publishing, Waltham, Massachusetts, 1964.

[11] N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes*. North-Holland / Kodansha, Amsterdam / Tokyo, first edition, 1981. (Second edition, 1989).

[12] K. Itô. Stochastic differentials. *Applied Mathematics and Optimization*, 1:374–381, 1975.

[13] K. Itô and H.P. McKean. *Diffusion Processes and Their Sample Paths*. Springer-Verlag, New York, second edition, 1974.

[14] J.-P. Kahane. *Some Random Series of Functions*. Cambridge University Press, second edition, 1985.

[15] W.S. Kendall. Symbolic Itô calculus: An overview. In N. Bouleau and D. Talay, editors, *Probabilités Numeriques*, pages 186–192, Rocquencourt, 1991. INRIA.

[16] W.S. Kendall. Doing stochastic calculus with *Mathematica*. In H. Varian, editor, *Economic and Financial Modeling with* Mathematica, pages 214–238. Springer-Verlag, New York, 1993.

[17] W.S. Kendall. A diffusion model for Bookstein triangle shape. *Advances in Applied Probability*, 30(2):317–334, 1998.

[18] W.S. Kendall. Symbolic Itô calculus: an ongoing story. Research Report 327, Department of Statistics, University of Warwick, 1998. To appear in *Statistics and Computing*.

[19] W.S. Kendall. Algebraic Reflections on Stochastic Calculus. In preparation, 1999.

[20] J.M. Lindsay. Independence for quantum stochastic integrators. In *Quantum probability & related topics*, pages 325–332, River Edge, NJ, 1991. World Scientific Publishing.

[21] T.J. Lyons. *Some Problems in Harmonic Analysis and Probabilistic Potential Theory*. D.Phil. thesis, University of Oxford, 1980.

[22] M. Marden. *The Geometry of the Zeros of a Polynomial in a Complex Variable*. American Mathematical Society, Providence, RI, second edition, 1960.

[23] M. Motoo. Proof of the law of iterated logarithm through diffusion equation. *Annals of the Institute of Statistical Mathematics*, 10:21–28, 1959.

[24] J.R. Norris, L.C.G. Rogers, and D. Williams. Brownian motion of ellipsoids. *Transactions of the American Mathematical Society*, 294:757–765, 1986.

[25] E.J. Pauwels and L.C.G. Rogers. Skew-product decompositions of Brownian motions. In R. Durrett and M.A. Pinsky, editors, *Geometry of Random Noise*, volume 73, pages 237–262. American Mathematical Society, Providence, RI, 1988.

[26] M. Reid. *Undergraduate Algebraic Geometry*, volume 12. London Mathematical Society, London, 1988.

[27] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*. Springer-Verlag, New York-Heidelberg-Berlin, 1991. (Third edition, 1999).

[28] L.C.G. Rogers and Z. Shi. Interacting Brownian particles and the Wigner law. *Probability Theory and Related Fields*, 95:555–570, 1993.

[29] L. Schwartz. *Semi-Martingales sur des Variétés, et Martingales Conformes sur des Variétés Analytiques Complexes*. Number 780 in Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1990.

[30] T. Shiga and S. Watanabe. Bessel diffusions as a one-parameter family of diffusion processes. *Zeitschrift fur Wahrscheinlichkeitstheorie*, 27:37–46, 1973.

[31] M. Spivak. *Calculus on Manifolds*. Addison-Wesley, 1965.

[32] M. Yor. Un exemple de processus qui n'est pas une semi-martingale. *Astérisque*, 52-53:219–221, 1978.

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