Riemannian centres of mass

and a new look at the classical central limit theorem

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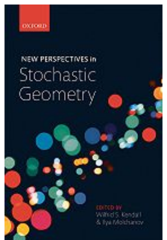
PLAN OF TALK

Riemannian barycentres

Riemannian barycentres have a history as old as that of measure-theoretic probability itself!

- Early years (Cartan 1929; Fréchet 1948);
- Careful description of properties in case of Riemannian manifolds (Karcher 1977);
- Appearing in applied statistics (Ziezold 1977, 1989, 1990, 1994, in the context of quasi-metric spaces).

Partly surveyed in WSK and Le (2010) (in W.S. Kendall and I.S. Molchanov, eds., *New Perspectives in Stochastic Geometry*, OUP, 2010, 608pp).
Follow-up: WSK and Le (2011) aimed to clarify the central limit theory for empirical barycentres.



Definitions

The definition of a Riemannian barycentre follows directly from mean-square-loss minimization definition.

- Conventional discrete sample mean / manifold-valued

$$\overline{y} = \operatorname*{argmin}_x \sum_y |x-y|^2\,; \quad \mathcal{E}_e[Y] = \operatorname*{argmin}_x \sum_y \operatorname{dist}(x,y)^2\,;$$

- Conventional expectation / manifold-valued

$$E[Y] = \operatorname*{argmin}_x \mathbb{E}\left[|x-Y|^2\right]\,; \quad \mathcal{E}[Y] = \operatorname*{argmin}_x \mathbb{E}\left[\operatorname{dist}(x,Y)^2\right]\,;$$

- It is convenient to have notation for the “energy” which is being minimized here, for example

$$\phi(x) \;=\; \frac{1}{2}\mathbb{E}\left[\operatorname{dist}(x,Y)^2\right]\,.$$

Uniqueness theorem(I)

A basic question:

- When are barycentres unique?
- It suffices to think about spheres; more specifically, hemispheres.

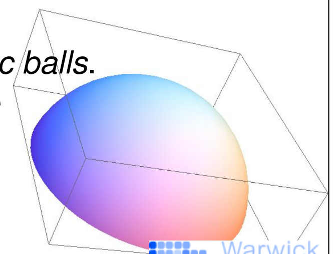
Uniqueness theorem(II)

Theorem

(WSK 1991) *Suppose that $S^{n-1}_{+,h}$ is a small hemisphere (intersection of sphere with half-space $x_1 \geq h$). Pick $\widetilde{h} \in (0, h)$. The following is convex if $2\nu\widetilde{h}^2(h^2 - \widetilde{h}^2) \geq 1$, and strictly convex outside the diagonal if strict inequality holds:*

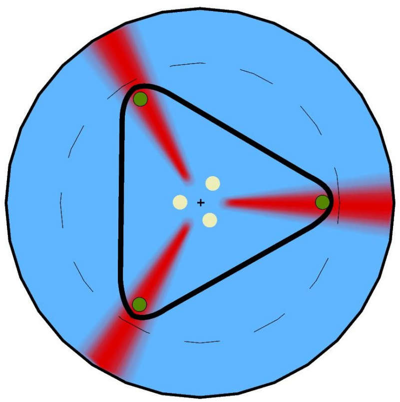
$$\Phi_{\nu,\widetilde{h}}(x, y) \quad = \quad \left(\frac{|x-y|}{2(x_1 y_1 - \widetilde{h}^2)}\right)^{\nu+1}$$

A direct modification holds for *regular geodesic balls*. In case of unit curvature upper bound, replace $p = \frac{1}{2}|x-y|^2$ by $1 - \cos \operatorname{dist}(x, y)$ and $q = x_1 y_1 - \widetilde{h}^2$ by $\cos \operatorname{dist}(x, \mathbf{o}) \cos(dist(y, \mathbf{o})) - \widetilde{h}^2$, for **o** the "north pole".



Failure of uniqueness

Uniqueness can fail in interesting ways if curvature varies.



(Negative curvature is blue, positive curvature is red.)

Laws of large numbers (I)

- Consider empirical barycentres of independent samples $X_1, \ldots, X_n$ from metric space $\mathcal{X}$.
- Ziezold (1977): strong law of large numbers for i.i.d. samples from separable finite quasi-metric spaces $\mathcal{X}$:

$$\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \arg\min_x \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \operatorname{dist}(X_i, x)^2} \subseteq \arg\min_x \mathbb{E}\left[\frac{1}{2} \operatorname{dist}(X_1, x)^2\right].$$

- Significant links with set-valued expectations in stochastic geometry (Stoyan and Molchanov 1997).
- Consistency under local compactness (Bhattacharya and Patrangenaru 2003).

Laws of large numbers (II)

Two equivalent variant Lindeberg conditions for finite-energy independent random variables $X_1, X_2, \ldots$ from $\mathcal{X}$. Let $\phi_n(x)$ be the aggregate energy at $x \in \mathcal{X}$. The following are equivalent:

Local: As $n \to \infty$, for each $\varepsilon > 0$,

$$\frac{1}{\phi_n(x)} \sum_{m=1}^{n} \mathbb{E}\left[\frac{1}{2}\operatorname{dist}(X_m, x)^2 \;;\; \frac{1}{2}\operatorname{dist}(X_m, x)^2 > \varepsilon\phi_n(x)\right] \to 0$$

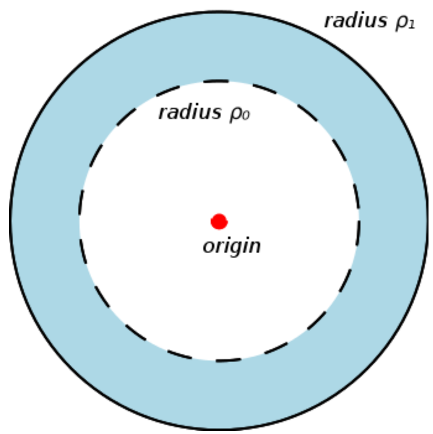
Semi-global: As $n \to \infty$, for each $\varepsilon > 0$,

$$\frac{1}{n\phi_n(x)} \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}\left[\frac{1}{2}\operatorname{dist}(X_i, X_j)^2 \;;\; \frac{1}{2}\operatorname{dist}(X_i, X_j)^2 > \varepsilon\phi_n(x)\right] \to 0\,.$$

Second form is *semi*-global, as $\phi_n(x)$ still depends on $x \in \mathcal{X}$.

Laws of large numbers

visualizing the result



We want the mean aggregate energies to be uniformly strictly higher in rings away from the origin.
Then we can prove a weak law of large numbers for empirical barycentres.

Laws of large numbers (III)

- Let $\mathcal{X}$ be a locally compact separable metric space. Consider independent non-identically distributed X_1, X_2, $\ldots$ from $\mathcal{X}$ with $\mathbb{E}\left[\operatorname{dist}(X, \mathbf{o})^2\right] < \infty$.
- Suppose for some $\rho_1 > 0$, for every positive $\rho_0 \leq \rho_1$, we have $(1+\kappa)\phi_n(\mathbf{o}) < \inf\{\phi_n(y) : \rho_0 \leq \operatorname{dist}(y, \mathbf{o}) \leq \rho_1\}$.
- Suppose a local variant Lindeberg condition holds.
- (Theorem) For any sequence of local minimizers[1] of empirical energy lying within ball$(\mathbf{o}, \rho_1)$, there is a subsequence which converges to $\mathbf{o}$ in probability[2].

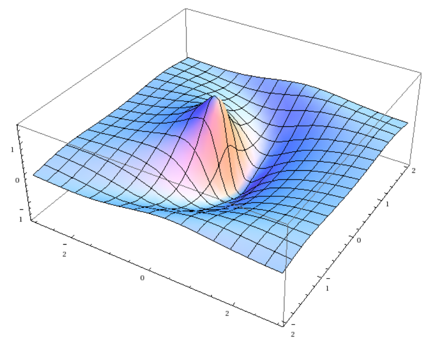
[1] Minimizers of empirical energy restricted to ball$(\mathbf{o}, \rho_1)$

[2] So local minimizers also local minima of unrestricted empirical energy.

Optimizing the empirical energy

Best way to locate point where empirical energy is at minimum:

- Seek out where gradient vanishes;
- Do this by pretending the energy surface is quadratic;



- Newton's method for cumulative empirical energy $\frac{1}{2}\sum_{i=1}^{n}\operatorname{dist}(x, X_i)^2$ (cf: Groisser 2004).

Central Limit Theorem (I)

We are now in a position to consider a CLT for non-identically distributed random variables on a Riemannian manifold $\mathbb{M}$. Consider independent $X_1, X_2, \ldots$ all with barycentre at $\mathbf{o}$.

- Gradient $\operatorname{grad}_x \frac{1}{2}\operatorname{dist}(x,y)^2$:
  $\operatorname{dist}(x,y)\operatorname{grad}_x \operatorname{dist}(x,y) = -\operatorname{Exp}_x \chi_x^{(y)}$,
  where $\chi_x^{(y)}$ is vector pointing from x to y.
- Hessian:
  define $\operatorname{Hess}_x \frac{1}{2}\operatorname{dist}(x,y)^2$ using covariant derivatives.
- Euclidean counterparts:
  $\operatorname{grad}_x \frac{1}{2}\operatorname{dist}(x,y)^2 \approx x - y$,
  $\operatorname{Hess}_x \frac{1}{2}\operatorname{dist}(x,y)^2 \approx$ identity matrix.

Central Limit Theorem (II) – geometric technicalities

- Taylor expansion of gradient of cumulative empirical energy $\frac{1}{2}\sum_{i=1}^n \operatorname{dist}(x, X_i)^2$:

$$\Pi_{x,\mathbf{o}} \sum_{i=1}^n Y_i(x) = \sum_{i=1}^n Y_i(\mathbf{o}) - \sum_{i=1}^n H_i(\mathbf{o})\gamma_x'(0) + \Delta_n(x)\gamma_x'(0)\,.$$

- Here $\Pi_{x,\mathbf{o}}$ is parallel transport, Y_i is the individual gradient field, H_i is the individual Hessian field, γ_x is minimal geodesic from $\mathbf{o}$ to x, and Δ_n is a correction term.
- If x is empirical barycentre then

$$0 \approx \sum_{i=1}^n Y_i(\mathbf{o}) - \sum_{i=1}^n H_i(\mathbf{o})\gamma_x'(0)$$

Central Limit Theorem (III)

Hence the following vector points towards the empirical barycentre:

$$\left(\frac{1}{\sqrt{2\phi_n(\mathbf{o})}}\sum_{i=1}^n H_i(\mathbf{o})\right)^{-1} \cdot \frac{1}{\sqrt{2\phi_n(\mathbf{o})}}\sum_{i=1}^n Y_i(\mathbf{o})\,.$$

Requirements:

- Cut-locus of $\mathbf{o}$ must be avoided;
- Control of $\Delta_n(x)$ (control local geometry *via* local mean variation of Hessian);
- $\phi_n(\mathbf{o})$ grows at least linearly, and bounds the sum of mean squared Frobenius norms of $H_1, \ldots H_n$, so that inverse matrix multiplier converges to constant in probability;
- Limit inverse matrix multiplier has an inverse;
- There is a vector-valued CLT for the normalized sum of gradients.

Classical innovations(I)

- We need a classical CLT for a sequence of independent non-identically distributed vector-valued random variables.
- A Lindeberg condition is needed. However the classical, coordinate-wise, Lindeberg condition is unnatural!
- We prefer a variant Lindeberg condition on the lengths of the random vectors: as $n \to \infty$ for each $\varepsilon > 0$,

$$\frac{1}{\phi_n}\sum_{m=1}^n \mathbb{E}\left[\frac{1}{2}\|Y_m\|^2\,;\,\frac{1}{2}\|Y_m\|^2 > \varepsilon\phi_n\right] \to 0$$

  where $\phi_n = \sum_{i=1}^n \mathbb{E}\left[\frac{1}{2}\|Y_m\|^2\right]$.
- But this is not sufficient to ensure convergence to a normal distribution.
  (Simple counterexample: consider sequences of bivariate normal distributions!)

A visual clue

Two problems which amount to the same thing:

- Measuring the distance between two probability distributions;
- Moving sand in the most efficient way possible

$$W_1(\mu,\nu) = \inf\{\mathbb{E}[\|U-V\|] : \mathcal{L}(U)=\mu, \mathcal{L}(V)=\nu\}$$
$$\widetilde{W}_1(\mu,\nu) = \inf\{\mathbb{E}[1\wedge\|U-V\|] : \mathcal{L}(U)=\mu, \mathcal{L}(V)=\nu\}$$

Classical innovations(II)

- Counterexample begs the question: perhaps we can show the variant Lindeberg condition forces normalized sum to become more normal even if it doesn't actually converge?
- "Central approximation" rather than "Central limit".
- Key idea: Use "truncated Wasserstein distance" (or "Dudley distance"):

$$\widetilde{W}_1(\mu,\nu) = \inf\{\mathbb{E}[1\wedge\|U-V\|] : \mathcal{L}(U)=\mu, \mathcal{L}(V)=\nu\}$$

- Metrizes weak convergence (Müller 1997; Villani 2003).
- Using Kantorovich-Rubinstein duality (Villani 2003, Remark 7.5(i))

$$\widetilde{W}_1(\mu,\nu) = \sup\{\int f\,\mathrm{d}(\mu-\nu) : f \text{ Lip}(1) \text{ for distance } 1\wedge\|x-y\|\},$$

  we can generalize the classical characteristic functions proof of the Lindeberg-Feller CLT.
- We can even prove the Feller converse!

Classical innovations(III)

We therefore have a clear picture of requirements for barycentre CLT:

$$\left(\frac{1}{\sqrt{2\phi_n(\mathbf{o})}}\sum_{i=1}^{n} H_i(\mathbf{o})\right)^{-1}\cdot\frac{1}{\sqrt{2\phi_n(\mathbf{o})}}\sum_{i=1}^{n} Y_i(\mathbf{o}).$$

converges to normality in sense of truncated Wasserstein distance when:

- Cut-locus of **o** is avoided;
- Control of $\Delta_n(x)$ *via* local mean variation of Hessian;
- Linearly growing $\phi_n(\mathbf{o})$ bounds the sum of mean square Frobenius norms of $H_1, \ldots H_n$, so that inverse matrix multiplier converges to constant in probability;
- Limit inverse matrix multiplier has an inverse;
- The variant Lindeberg condition holds.

Conclusion

- Clarification of requirements for barycentre CLT;
- A surprising new development for the classical Feller-Lindeberg theorem;
- My student Ashish Kumar has checked the results carry through for triangular arrays;
- Question: Desirable to find a simple example to show arbitrarily slow convergence for Lindeberg CLT;
- Question: Infinite dimensions?
- Question: Metric spaces instead of manifolds? (cf: hurricanes!)
- Question: Stable laws?

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