Coupling, local times, immersions

Developments in Coupling: University of York

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Introduction: the thematic problem of coupling

Wikipedia (2010) on (probabilistic) coupling:
"A proof technique that allows one to compare two unrelated variables by 'forcing' them to be related in some way".

Thematic problem:
Construct processes to start differently but end identically.
Other flavours of coupling problems:
representation, approximation, monotonicity, connection . . .

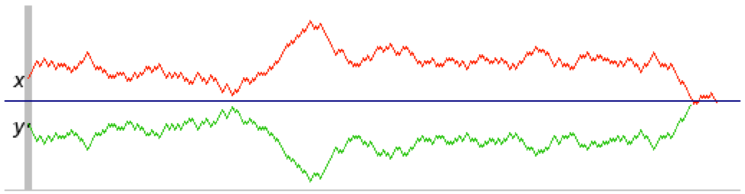
- Maximal coupling (Griffeath 1975; Pitman 1976; Goldstein 1978)

  Contrast: *Shift-coupling (Thorisson 1994)*
- Co-adapted or *immersed* coupling

  Contrast: *Shy coupling (Benjamini, Burdzy, and Chen 2007, et seq.)*
- Co-immersed coupling (Émery 2005)

Brownian Reflection Coupling

Notions of synchronous and reflection couplings for random walk / Brownian motion.



Lindvall (1982) preprint
"On coupling of Brownian motions".

Brownian motion is a very special case: an immersed coupling which is maximal.

Contrast Ornstein-Uhlenbeck process
(Connor 2007, PhD. Thesis).

Examples: specific examples

Restrict attention to diffusions.

Elliptic case
- Brownian motion
- Diffusions
- Riemannian Brownian motion

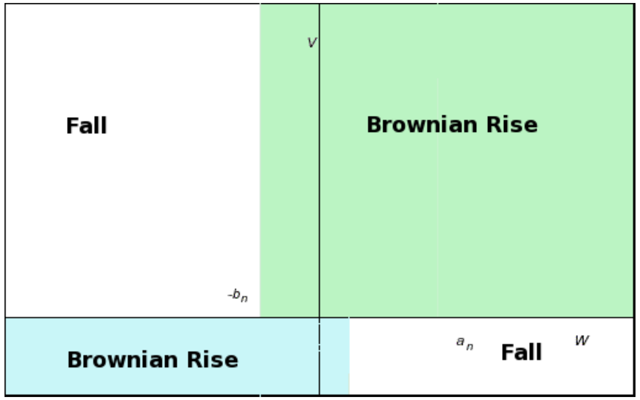
  "Efficient coupling" (Burdzy and WSK 2000)
  Work on "maximal Markovian couplings" by Kuwada, Sturm, *et al.*

Hypo-elliptic case
- BM + time integral(s)
- BM^2 + Lévy stochastic area
- $\text{BM}^n + \binom{n}{2}$ Lévy stochastic areas

and beyond ... ?

Simplest hypoelliptic example:

Brownian motion and time integral



Horizontal axis: $W = B - A$;
Vertical axis: $V = \int W \, \mathrm{d}\, t$.

Why couple Local Time?

This talk is about:
coupling real Brownian motion and local time at zero.

Motivating reasons:

1. Develop further intuition about coupling functionals of Brownian motion;
2. The example is amenable to calculations, and yields exact answers, so may be a useful model for other situations;
3. The example highlights the difference between immersed and co-immersed coupling;
4. There is a significant application to the theory of filtrations.

Local Time: Representation

- Recall Tanaka formula:
  if X is standard Brownian motion then

$$\mathrm{d}\,|X| \quad = \quad \operatorname{sgn}(X)\,\mathrm{d}\,X + dL^{(0)}\,.$$

- Consequence:

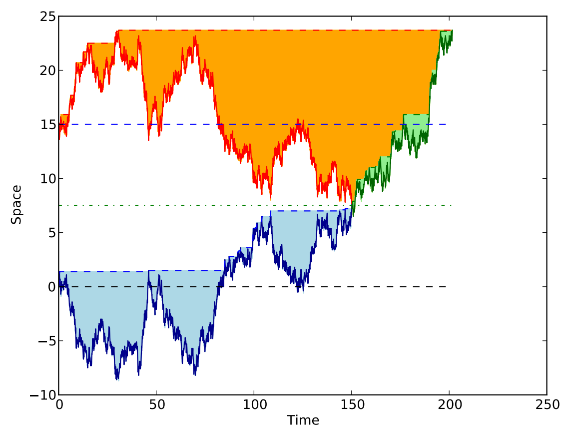
$$\mathcal{L}\left(L^{(0)} - |X|, L^{(0)}\right) \quad = \quad \mathcal{L}\,(B, S)$$

  where B is standard Brownian motion and $S_t = \sup\{B_s : s \leq t\}$.
- Take $S = L^{(0)}$ and $B = -\int \operatorname{sgn}(X)\,\mathrm{d}\,X$.
- Strictly speaking, the Brownian motions and local time must begin at 0. But we can fix this up.
- Think of this as providing a new coordinate chart for the coupling problem. It suffices to couple real Brownian motion and its supremum process.

Local Time: Reflected / Synchronized algorithm

A simple exercise in basic theory of Brownian motion!
We wish to couple (B, S) and $(\widetilde{B}, \widetilde{S})$. Suppose $B_0 > \widetilde{B}_0$.

- Reflection coupling till $B = \widetilde{B}$ at time T_1.
- Synchronous coupling till $B = \widetilde{B}$ hits the level $S_0 \wedge \widetilde{S}_0$ at time T_2 and then $S_{T_1} \wedge \widetilde{S}_0$ at T_3 (the coupling time).



Local Time: Rate of coupling (I)

We can compute the rate of coupling! Suppose that $B_0 > \widetilde{B}_0$.

- Coupling occurs at

$$T_3 = H^1(-\tfrac{1}{2}(B_0 - \widetilde{B}_0)) + H^1((S_0 \wedge \widetilde{S}_0) - \tfrac{1}{2}(B_0 - \widetilde{B}_0)) + H^1(S_{T_1} - (S_0 \wedge \widetilde{S}_0)),$$

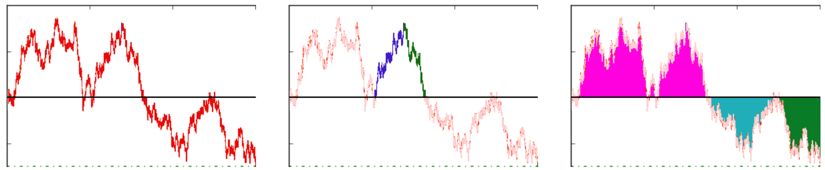
  where $H^1(a)$ has the law of hitting time of a for standard Brownian motion.
- The third hitting time depends on S_{T_1} hence in fact on the first hitting time.
- Compute $\mathbb{P}[T_3 < \text{Exponential}(\alpha)]$ using Markov and memory-less properties and excursion theory, to determine the moment generating function (MGF) of T_3.

Local Time: Rate of coupling (II)

Specifically,

- Suppose $S_0 = B_0 \geq \widetilde{S}_0$ as well as $B_0 > \widetilde{B}_0$.
- Set $b = \frac{1}{2}(B_0 - \widetilde{B}_0)$, and $\alpha^* = \sqrt{2\alpha}$.
- Compute using standard excursion-theoretic arguments as in the illustrative example supplied by Rogers and Williams (1987, Volume II, §56):

$$\mathbb{E}[\exp(-\alpha T_3)] = 1 - \sinh(\alpha^* b) \log\coth(\frac{1}{2}\alpha^* b).$$



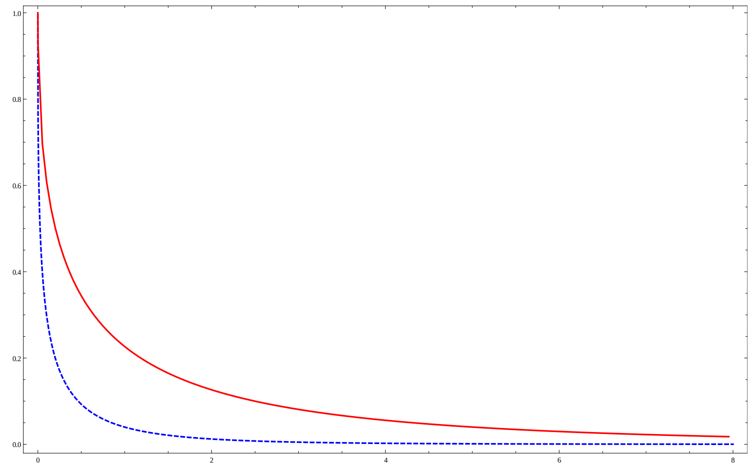
Height $n(x) = 1/(2|x|)$; BES^3 out; BES^3 back.

BES^3 hitting times and Poisson point processes ....

Local Time: Comparison with maximal coupling (I)

- The joint distribution of B_t and S_t can be computed explicitly using the reflection principle (Revuz and Yor 1991, Exercise (3.14) part 2).
- Hence we can compute the *meet* of the joint densities of (B_t, S_t) and $(\widetilde{B}_t, \widetilde{S}_t)$. (Sub-probability density which is minimum of two densities.)
- No coupling can happen faster than the total integral of the meet! (Aldous inequality.)
- Compute MGF of (one minus the total integral of) the meet numerically and compare to MGF of reflected / synchronized coupling time.

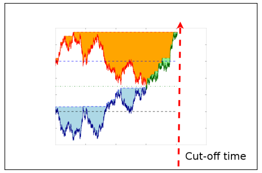
Local Time: Comparison with maximal coupling (II)

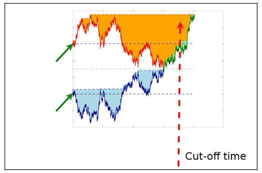


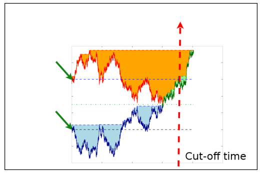
1. Solid red line: MGF of maximal coupling time;
2. Dotted blue line: MGF of reflected / synchronized coupling time.
3. Reflected / synchronized coupling is not maximal.

Local Time: Optimality of immersed coupling

- It seems “obvious” that the reflected / synchronized coupling is optimal amongst immersed couplings.
- Key idea: starting with synchronous coupling
  (a) risks effectively increasing $S_0 \wedge \widetilde{S}_0$,
  and (b) (more significantly) wastes time.







- Prove “bang-bang” approximation result to make this rigorous.
- So for this problem there is an optimal immersed coupling, which nevertheless is not maximal.

. . . but that is not all . . .

BKR diffusion

- The BKR diffusion (X, Y) (Benes, Karatzas, and Rishel 1991) arose in a study of a wide-sense control problem which has no strict-sense optimal control: X, Y satisfy

$$\operatorname{sgn}(X)\,\mathrm{d}X + \operatorname{sgn}(Y)\,\mathrm{d}Y \quad = \quad 0$$

  where $\mathrm{d}X$ and $\mathrm{d}Y$ must be Brownian differentials.
- Émery (2009) treated this as a case study, exemplifying general issues from filtration theory.
- The underlying filtration is in fact *Brownian*: Émery (2009) pointed out that an illuminating proof would follow if one could produce a successful *co-immersed* coupling of two distinct copies of (X, Y).
- It is straightforward to use the local time reflected / synchronized coupling to produce a immersed coupling for the BKR diffusion . . .

. . . but this is clearly not co-immersed.

BKR diffusion: the key question

Are there co-immersed couplings of Brownian motion and local time?

Sketch of a co-immersed coupling of Brownian motion and local time (I)

Without loss of generality, suppose $B_0 > \widetilde{B}_0$.
For simplicity take $S_0 = B_0 \geq \widetilde{S}_0$.

- Express earlier coupling strategy using Itô calculus:

$$H_t \quad = \quad \begin{cases} -1 & \text{for } t < T_1 = \inf\{s : B_s = \frac{1}{2}(B_0 + \widetilde{B}_0)\}\,, \\ +1 & \text{for } T_1 \leq t < T_3 = \inf\{s : B_s = S_{T_1}\}\,. \end{cases}$$

- The coupling $\mathrm{d}\widetilde{B} = H\,\mathrm{d}B$ is co-immersed for filtrations of B and $\widetilde{B}$, but not co-immersed for filtrations of X and $\widetilde{X}$ (all those ± 1 signs of excursions!).

  Indeed:

$$\mathrm{d}\widetilde{X} \quad = \quad \operatorname{sgn}(\widetilde{X}) H \operatorname{sgn}(X)\,\mathrm{d}X\,.$$

- Clue: delay choice of signs.

Sketch of a co-immersed coupling of Brownian motion and local time (II)

- Choose *deterministic* $\psi(t) > 0$ tending *fast* to zero.
- Introduce delays

$$\begin{aligned}\sigma_t &= \sup\{s < t : |X_s| > \psi_s\}, \\ \widetilde{\sigma}_t &= \sup\{s < t : |\widetilde{X}_s| > \psi_s\}.\end{aligned}$$

- Replace $\operatorname{sgn}(X_t)$ and $\operatorname{sgn}(\widetilde{X}_t)$ by $\operatorname{sgn}(X_{\sigma_t})$ and $\operatorname{sgn}(\widetilde{X}_{\widetilde{\sigma}_t})$.
- New "delayed" coupling is

$$d\widetilde{X} = \operatorname{sgn}(\widetilde{X}_{\widetilde{\sigma}}) H \operatorname{sgn}(X_\sigma)\, dX.$$

- Coefficient of dX adapted to both X and $\widetilde{X}$ filtrations.
- The coupling will fail at time T_3, but only by a small amount: keep re-running algorithm. If $\psi(t) \to 0$ quickly enough, then X will converge to $\widetilde{X}$ in finite time (compare coupling of Brownian time-integral!).

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Conclusion

1. Coupling of $(|X|, L^{(0)})$: a model example in which
   - immersed coupling is not maximal;
   - there is an optimal immersed coupling.
2. Application to BKR diffusion:
   - Distinction between immersed and co-immersed is important;
   - Coupling techniques are useful in modern filtration theory!

QUESTIONS?

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