Probability coupled with Geometry

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Introduction (I)

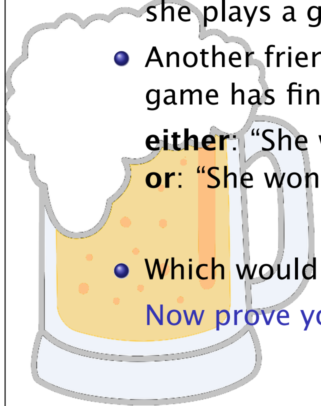
Wikipedia (2013) on (probabilistic) coupling:
"A proof technique that allows one to compare two unrelated variables by 'forcing' them to be related in some way".

The notion of probabilistic coupling dates back to work of Wolfgang Döblin, carried out before the Second World War, right at the beginning of measure-theoretic probability.

Thematic problem: Construct random processes to start differently but end identically.

Other flavours of coupling problems:
representation, approximation, monotonicity, connection, simulation, . . .

Introduction (II)



A story to illustrate application of probabilistic coupling:

- You and a friend walk into a bar.
- You stay in the snug. She goes to the main bar, where she plays a game: toss n coins, win if r heads or more.
- Another friend comes through from main bar, after the game has finished, and says something,

  either: "She won the first toss";
  or: "She won at least one toss".

  (Why don't people speak more clearly?!)
- Which would be better news for your friend?
  Now prove your answer without calculation!

Doeblin's coupling for Markov chains

(run independently till first meeting)

Doeblin (1938)'s construction:

- if X is a finite-state-space Markov chain for which the convergence theorem applies (irreducible, aperiodic);
- construct two copies X, $\widetilde{X}$ which move independently till they couple, and then move synchronously.

If T is the coupling time, and $\widetilde{X}$ starts in equilibrium, then the coupling inequality uses $\mathbb{P}[T > t]$ to estimate total variation distance of $\mathcal{L}(X_t)$ from equilibrium.

Clearly this is inefficient.

We need to do much better if coupling is to be widely useful in theory and applications.

Total variation distance

Total variation distance measures the extent to which probability measures agree:

Definition

The *total variation distance* between two probability measures P and Q is given by

$$\mathrm{dist}_{\mathrm{TV}}(P,Q) \quad = \quad \max\{P[A]-Q[A] \;:\; \text{measurable } A\}\,.$$

- Definition is symmetric!
  (clue: use complements $P[A] = 1 - P[A^c]$);
- If P, Q are discrete then

$$\mathrm{dist}_{\mathrm{TV}}(P,Q) \quad = \quad \frac{1}{2}\sum_x |P[\{x\}]-Q[\{x\}]|\,;$$

- If P, Q have densities f, g then

$$\mathrm{dist}_{\mathrm{TV}}(P,Q) \quad = \quad \int_{f>g}(f(x)-g(x))P[\,\mathrm{d}x]\,.$$

Two basic coupling questions

We aim to answer:

- When should we expect there to be a practical coupling which couples as fast as possible?
- When can there exist highly *unsuccessful* couplings? ("Shy couplings", staying more than $\varepsilon > 0$ apart from each other for all time.)

In both cases the answer involves significant geometry.

Very often, probability questions mesh with geometry in non-trivial and beautiful ways!

Coupling: maximal or immersion?

Coupling is used for many purposes, but the objective which organizes much of the theory is,

Whether a coupling is successful?
and how quickly might success occur?

Successful coupling:

$$\mathbb{P}\left[X_t = \widetilde{X}_t \quad \text{for all large enough } t\right] \quad = \quad 1\,.$$

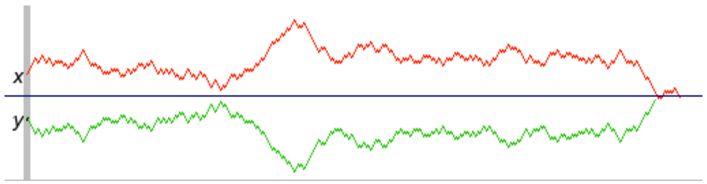
Choices for possible constructions:

- maximal coupling (*eg:* Pitman 1976): potential theory, achieves maximal rate: $\mathbb{P}[T \geq t] = \mathrm{dist}_{\mathrm{TV}}(\pi, \mathcal{L}(X_t))$;
- or immersion coupling (prescriptions of marginals respect a specific filtration): often easier to work with;
- other possibilities . . . .

Lindvall's reflection coupling for Brownian motion

(do the opposite till first meeting)

Lindvall (1982): How to couple two Brownian motions?



Generate first Brownian motion X beginning at x;
Locate y, initial point for second Brownian motion;
Construct line of reflection, hence reflection map H;
Generate second Brownian motion Y using reflection of first;
Couple in higher dimensions by reflecting in hyperplane . . . ;
This coupling is an immersion coupling (no "cheating") and maximal.

Coupling Brownian motion on manifolds

be geometric about doing the opposite till first meeting

When the diffusion is Brownian motion on a Riemannian manifold, the "mirror-map" H can be chosen accordingly.

Reflect the Brownian noise using

(a) stochastic development,
(b) a geodesic connecting the two diffusions.

Coupling can then be analyzed in terms of geometry, relating success of coupling to curvature bounds.

Also applicable if there is *drift* (smoothly directed bias).

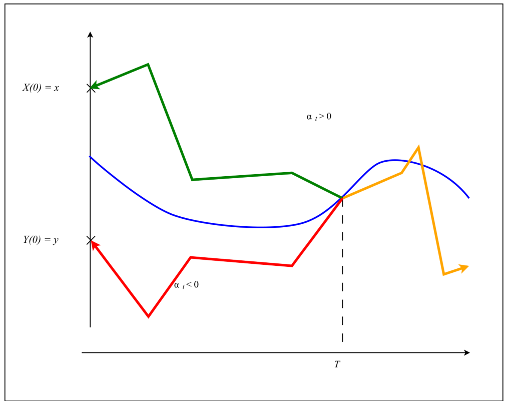
Can this ever be a maximal immersion coupling?

Basic tools of the trade

- Brownian motion on Riemannian manifold: limit ($\Delta t \to 0$) of random walk, uniform jumps $\sqrt{\Delta t}$ at $k\Delta t$;
- *Stochastic calculus* represents limit as solving stochastic differential system driven by (flat) Brownian motion.
- *Drift* can be induced by biasing the choice of random direction in a way that depends on current location.
- Corresponds to perturbing stochastic differential system by a (typically but not necessarily smooth) vectorfield.
- For small $t > 0$, if BM+drift moves from x at time 0 to y at time t then its path approximates a geodesic.
- $\{$ Riemannian Brownian motions with smooth drift $\} \equiv$ $\{$ Euclidean elliptic diffusions, smooth coefficients $\}$.
- Related to potential theory of $\frac{1}{2}\sum_{i,j} a_{ij}\partial_i\partial_j + \sum_k b_k\partial_k$ for smooth b, smooth positive-definite a.

Pitman's construction of maximal coupling (I)

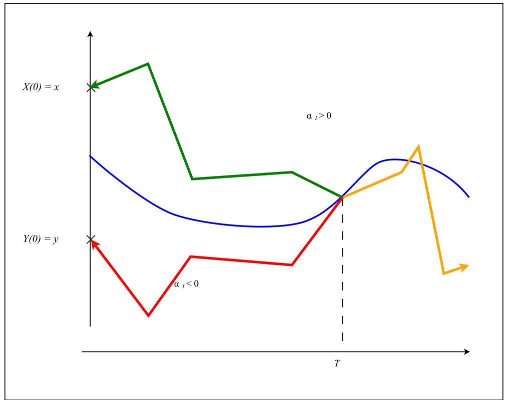
Construct maximally coupled diffusions X, Y begun at x, y.



Interface $\alpha = 0$ where $\alpha(z) = p_t(x,z) - p_t(y,z)$.

Pitman's construction of maximal coupling (II)

Riemannian Brownian motion with smooth drift.



Varadhan asymptotics: interface $\alpha_0 = 0$ is "perpendicular bisector" of geodesic $x \to y$;
Markovian maximal coupling: same for $\alpha_t = 0$, all t;
Reflection coupling forces involutive isometry.

Geometry, Killing vectorfields, and a characterization

- Use work of Myers and Steenrod (1939):
- Markovian maximal couplings between all point pairs; manifold $\mathbb{M}$ must be maximally symmetric;
- Interfaces disconnect $\mathbb{M}$, so $\mathbb{M}$ is simply connected;
- Drift vectorfields must be related by isometries, hence (using Jacobi vectorfield theory) either Killing or scaling.

Theorem (Markovian maximal couplings force rigidity)

Given an elliptic diffusion, smooth coefficients, on $\mathbb{M}$, dim≥ 2, supports Markovian maximal couplings from all starts.
Then intrinsic diffusion geometry of $\mathbb{M}$ is (a) hypersphere, or (b) Euclidean space, or (c) hyperbolic space.
Cases (a), (c): drift vectorfield is Killing (twist from rotations).
Case (b): sum of Killing vectorfield and scaling vectorfield.

Shy couplings

When are there highly *unsuccessful* couplings? (“shy couplings”, staying more than $\varepsilon > 0$ apart for all time)?

- Restrict attention to reflecting Brownian motions in a smooth domain, and immersion (Markovian) couplings.
- It is easy to construct a shy coupling in an annulus, so consider simply connected domains. ▸ ANNULUS
- Even the simplest case (domain is a disk) is tricky, because of the

  perverse coupling: given $B = (B_1, B_2)$, let coupling be $\widetilde{B} = (B_2, B_1)$.
- Then $|B - \widetilde{B}|$ has positive drift until one of B, $\widetilde{B}$ hits a boundary. ▸ REFLECTION ▸ PERVERSE

  More general domains seem out of reach! ▸ VIDEOGAME

The Lion and Man

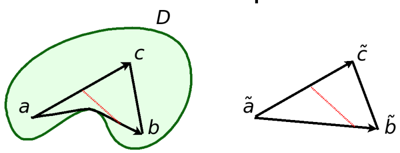
A problem in recreational mathematics:

- Richard Rado (1925) proposed the Lion and Man problem: Lion X chases Man Y around disk. Both move at unit speed, are arbitrarily agile, and tireless. Can the Lion catch the Man?
- Obviously yes; X to centre of disk, Y moves as far away as possible and keeps running, X can capture Y by moving on circle of half radius.
- Never trust an argument containing the word “obviously”. Besicovitch showed that if Y moves slightly away from boundary then Y can avoid X for ever (pretty argument revolving around standard criterion for convergence / divergence of $\sum n^{-\alpha}$).
- The Lion gets arbitrarily close, but never actually catches up with Man. What has this to do with shy coupling?

From disks to CAT(0) spaces

CAT(0) is an integrated form of a curvature constraint.
Consider a connected (open) subset D of Euclidean space.

- Furnish with the intrinsic metric; distance between two points is least length of connecting path lying completely in D.
- Say D is a CAT(0) domain if intrinsic geodesic triangles skinnier than comparable Euclidean triangles.



- GOOD: Lion-Man pursuit works for CAT(0) domains (Alexander, Bishop, and Ghrist 2006; Alexander, Bishop, and Ghrist 2010).
- BAD: stochastic calculus generally doesn’t work well with intrinsic geometry.
- GOOD: but the drift argument above does work well!

Shy-ness ideas of proof

- Bramson, Burdzy, and WSK (2013): vector-field $\chi(X, Y)$ from "greedy" pursuit strategy of Lion chasing Man;
- Impose large multiple of χ as drift for SDE for coupled reflecting BMs (WSK 2009):

$$\begin{aligned} \mathrm{d}\,X &= \mathrm{d}\,B && + n\chi(X,Y)\,\mathrm{d}\,t && -\nu_X\,\mathrm{d}\,L^X\,, \\ \mathrm{d}\,Y &= (\mathbb{J}^\top\,\mathrm{d}\,B + \mathbb{K}^\top\,\mathrm{d}\,A) && + n\mathbb{J}^\top\chi(X,Y)\,\mathrm{d}\,t && -\nu_Y\,\mathrm{d}\,L^Y\,; \end{aligned}$$

- Weak convergence, time-change ⇒ deterministic Lion-and-Man ⇒ X gets close to Y for large n;
- Recall basic tools: drift vector-field translates into change-of-measure;
- Deduce positive chance for X, Y to break shy-ness *however coupled.*

  Technical part:
  establish regularity of χ, make above quantitative.

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Results about shy-ness

Theorem

Let D be a bounded CAT(0) domain (with smooth boundary). Then there are no shy immersion couplings for reflected Brownian motion in D.

Remarkably, planar simply-connected domains are CAT(0)! Hence:

Corollary

Let D be a bounded planar simply-connected domain (with smooth boundary).
Then there are no shy immersion couplings for reflected Brownian motion in D.

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Conclusion

We have seen

- the search for maximal diffusion couplings which are also Markovian leads inexorably to considerations of Riemannian geometry, and thus to a clean and simple classification result: hyperspheres, Euclidean spaces, hyperbolic spaces and Killing and scaling vectorfields.
- seeking a clearer understanding of shy couplings leads to considerations of modern metric geometry, and thus (*via* the Lion and the Man) to CAT(0) spaces.
- probability questions often lead naturally to geometry: aim to let geometry flow from the probability question, rather than try to impose geometry.
- Even in these two themes, much scope for further work:
  - what about *almost* maximal couplings?
  - can we go beyond CAT(0)?

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References:

Alexander, S., R. L. Bishop, and R. Ghrist (2006).
Pursuit and evasion in non-convex domains of arbitrary dimension.
In *Proceedings of Robotics: Science and Systems*, Philadelphia, USA.

Alexander, S., R. L. Bishop, and R. Ghrist (2010).
Total curvature and simple pursuit on domains of curvature bounded above.
Geom. Dedicata 149(1), 275–290.

Banerjee, S. and W. S. Kendall (2014).
Rigidity for Markovian Maximal Couplings of Elliptic Diffusions.
arXiv 1412.2647, 42pp.

Warwick Statistics

Bramson, M., K. Burdzy, and WSK (2013, July).
Shy Couplings, CAT(0) Spaces, and the Lion and Man.
Annals of Probability 41(2), 744–784.

Doeblin, W. (1938).
Exposé de la Théorie des Chaînes simples constants de Markoff á un nombre fini d'États.
Revue Math. de l'Union Interbalkanique 2, 77–105.

Kuwada, K. (2009).
Characterization of maximal Markovian couplings for diffusion processes.
Electronic Journal of Probability 14, 633–662.

Lindvall, T. (1982).
On Coupling of Brownian Motions.
Technical report 1982:23, Department of Mathematics, Chalmers University of Technology and University of Göteborg.

Myers, S. B. and N. E. Steenrod (1939, April).
The Group of Isometries of a Riemannian Manifold.
Annals of Mathematics 40(2), 400–416.

Pitman, J. W. (1976).
On coupling of Markov chains.
Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 35(4), 315–322.

Saisho, Y. (1987).
Stochastic differential equations for multidimensional domain with reflecting boundary.
Probability Theory and Related Fields 74(3), 455–477.

Wikipedia (2013).
Coupling (probability) — Wikipedia, The Free Encyclopedia.

WSK (1986a).
Nonnegative Ricci curvature and the Brownian coupling property.
Stochastics and Stochastic Reports 19(1-2), 111–129.

WSK (1986b).
Stochastic differential geometry, a coupling property, and harmonic maps.
The Journal of the London Mathematical Society (Second Series) 33, 554–566.

WSK (2009, September).
Brownian couplings, convexity, and shy-ness.
Electronic Communications in Probability 14(Paper 7), 66–80.