Shy and unusual Brownian couplings

28th EMS, Piraeus

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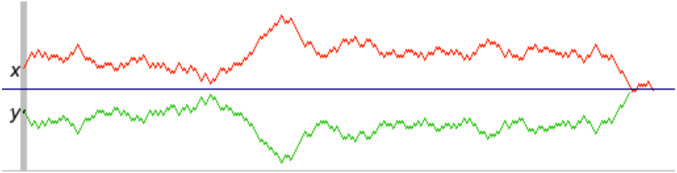
Wikipedia: Coupling (Probability)

Look up "coupling" on Wikipedia. After dis-ambiguation:

Wikipedia (2010) describes the *thematic* case of synchronous and reflection couplings for random walk.

Reflection Coupling:

Make one process meet other by doing mirror-opposite!



Lindvall (1982) "On coupling of Brownian motions."

Coupling isn't always about making processes meet, but the history of the subject (and much theory and application) centres around this case.

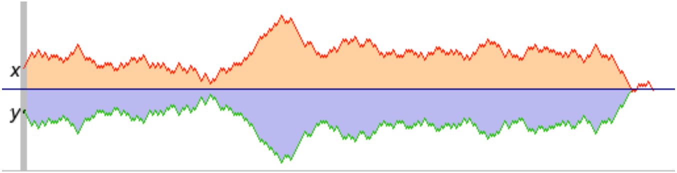
Summary of this talk:

Some possible topics to add to the Wikipedia entry(!)

Two General Coupling Questions

- How *much* can one couple[1]?

  (Path functionals as well as particles?)



- When can one *avoid* coupling?

  (Thematic: can one couple[1] reflected BM in compact domains so as to stay substantially far apart?)

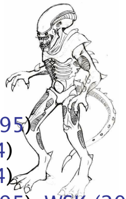
▸ SHY ANIMATION (I)

Begin by brief review of progress on first question: this concerns *unusual* couplings.

[1] co-adaptively – a technical (though important) point.

Coupling Path functionals

Survey of some known results:



Path and functionals	Couplings	
Brownian motion[2] B	refl	Lindvall (1982)
$B, \int B \, \mathrm{d}\, t$	refl + sync	(Ben Arous et al. 1995)
$B, \int B \, \mathrm{d}\, t, \int\int B \, \mathrm{d}\, s \, \mathrm{d}\, t$	refl + sync	WSK and Price (2004)
$B, \int B \, \mathrm{d}\, t, \ldots, \int \ldots \int B \, \mathrm{d}\, s \ldots \mathrm{d}\, t$	Morse-Thue	WSK and Price (2004)
$BM(\mathbb{R}^2)$, stochastic area[3]	refl + sync	(Ben Arous et al. 1995), WSK (2007)
$BM(\mathbb{R}^n)$, $\binom{n}{2}$ stochastic areas	refl + rotate	WSK (2007)

Coupling single stochastic area: ▸ HEISENBERG ANIMATION

- WSK (2009b) results extend to bounds on speed of coupling for (multiple) stochastic areas.
- Couple all invariant diffusions on nilpotent Lie groups?

  all hypoelliptic diffusions?

[2] One-dimensional Brownian motion

[3] Stochastic area: $\int B_i \, \mathrm{d}\, B_j - B_j \, \mathrm{d}\, B_i$

Shy-ness (I)

Now for the second question.

Shy-ness clearly relates to convexity . . .

- Evidently shy coupling *can* occur in an annulus.

  ▸ SHY ANIMATION (II)

- However it is reasonable to suppose that domain-convexity precludes shy coupling.

Convex C^2 planar domain, regularity [4]	(Benjamini et al. 2007)
Convex planar domain	WSK (2009a)
Convex domain in $\mathbb{R}^n$, regularity[3]	WSK (2009a)

- WSK (2009a) method of proof: potential theory; view coupling as a degenerate problem in stochastic control; find an appropriate function which is a supermartingale under all couplings.

  Can one say anything more about shy-ness?

[4] supporting lines touch boundary only at isolated points

Shy-ness (II)

Can one say anything more? YES!

- Bramson suggested (2008, personal communication): no shy-ness in any planar simply-connected domain!
- Bramson, Burdzy, and WSK (2010) prove this, so long as domain is Lipschitz and satisfies "uniform exterior sphere condition".
  (*Nearly* required for strong reflecting BM: (Saisho 1987).)
- Special case of stronger result: no shy-ness in Cat(0) (regular) domains!

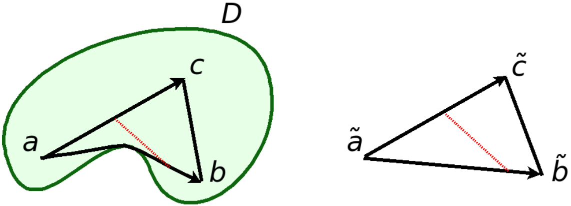
▸ SHY ANIMATION (III)

Bramson, Burdzy, and WSK (2010):
Shy Couplings, CAT(0) Spaces, and the Lion and Man.

Cat(0) spaces

Consider a connected open subset D of Euclidean space.

- Furnish it with the intrinsic metric; the distance between two points is the least length of a connecting path lying completely in D.
- We say D is a Cat(0) domain if intrinsic geodesic triangles are skinnier than comparable Euclidean triangles.



- Health warning: Itô analysis of intrinsic distance produces singularities in drift away from zero!

The Lion and Man

- Rado (in 1930's) proposed the Lion and Man problem: Lion X chases Man Y around disk. Both move at unit speed, are arbitrarily agile, and tireless. Can the Lion catch the Man?
- Obviously yes; X to centre of disk, Y moves as far away as possible and keeps running, X can capture Y by moving on circle of half radius.
- Never trust an argument containing the word "obviously". Besicovitch showed that if Y moves slightly away from boundary then Y can avoid X for ever (pretty argument centering around standard criterion for convergence / divergence of $\sum n^{-\alpha}$).
- The Lion gets arbitrarily close, but never actually catches up with Man. But what has this to do with shy coupling?

Shy-ness (II) ideas of proof

Idea of proof is simple, but careful new Cat(0) geometry arguments are required to establish appropriate regularity.

- Use Cat(0) version of celebrated Lion-and-Man problem;
- Derive vector-field $\chi(X, Y)$ from pursuit strategy;
- Impose large multiple of χ on SDE for coupled reflecting BMs (WSK 2009a):

$$\begin{aligned} \mathrm{d}\,X &= \mathrm{d}\,B && + n\chi(X,Y)\,\mathrm{d}\,t && -\nu_X\,\mathrm{d}\,L^X\,, \\ \mathrm{d}\,Y &= (\mathbb{J}^\top\,\mathrm{d}\,B + \mathbb{K}^\top\,\mathrm{d}\,A) && + n\mathbb{J}^\top\chi(X,Y)\,\mathrm{d}\,t && -\nu_Y\,\mathrm{d}\,L^Y\,; \end{aligned}$$

- Weak convergence, time-change ⇒ deterministic Lion-and-Man ⇒ X gets close to Y for large n;
- Use Cameron-Martin-Girsanov theorem to translate vector-field into change-of-measure;
- Deduce positive chance for X, Y to break shy-ness *however coupled*.

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Further Questions

These results suggest some significant foundational questions for Coupling (Probability)[5].

- How much further could one go in "unusual" coupling – coupling path functionals of Brownian motion?
- Can one go beyond Cat(0) for failure of shy coupling? (We think the answer is Yes . . .)
- Can one say anything about domains in which shy coupling occurs?
- Bold conjecture:
  impossible to be shy in simply-connected domains?
- Can one develop a theory for *non-co-adapted* shy coupling?

THE END

[5] Co-adaptive coupling! (unless otherwise stated . . .)

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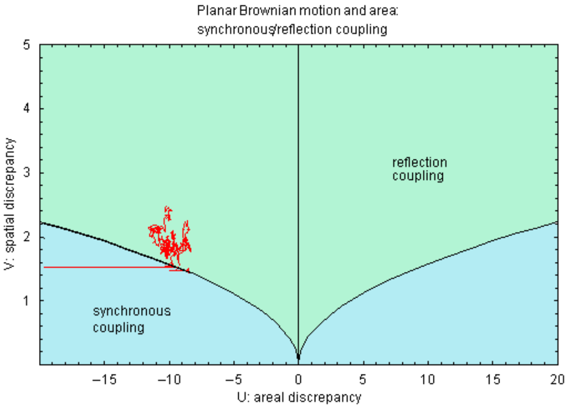
Coupling BM and stochastic area



WSK (2009b) distributional asymptotics for coupling time: $(U_0^2/V_0^2)\times$ reciprocal of Gamma; uses Lamperti (1972).

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