

A Perfect Queue

Swiss Statistics Seminar

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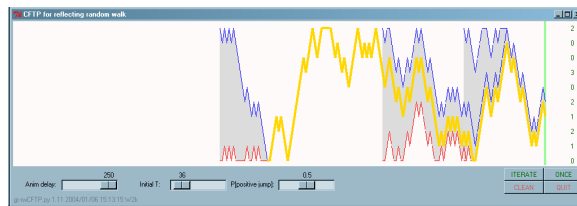


- Introduction
- CFTP and queues of finite capacity
- Dominated CFTP
- M/G/c Queues
- Conclusion
- References



Perfect simulation: the basic idea.

Propp and Wilson (1996) Coupling from the Past (CFTP).
 Very basic case: reflected simple random walk on state-space $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.



Ingredients:

- Past → Present, not Present → Future;
- Monotonicity;
- Capture all starts using upper and lower processes;
- Re-use randomness.



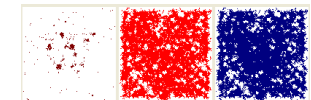
Examples of perfect simulation

(“Backwards from the past” constructions to analyse very general queues were introduced by [Lynes 1962](#).)

1. Dead Leaves example:



2. Simple image analysis:



3. Use of small sets ([Murdoch and Green 1998](#)):




WSK (2015): “Introduction to CFTP using R”.



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CFTP for “bounded” queues

1. Classic *CFTP* requires “bounded” state-space.
2. So focus initially on **queues with finite capacity** (“turn customers away when waiting room is full”). Seek to find exact draws from equilibrium.
3. Murdoch and Takahara (2006) study *CFTP* for: $M/M/1/c$, $M/M/c/c$, $GI/M/c/c$, $GI/G/c/c$. Here c means, if arrivals find c people already in the system then they go away never to return.
4. Integrated Masters Dissertation, Liu (2015): $M/D/1/c$ based on residual total workload.
Problem: Need to alter upper process very carefully to allow for failure of monotonicity!




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The trouble with $M/D/1/c$

Monotonicity can fail!

- Suppose Q_1 is at capacity, with current customer almost having completed service;
- and suppose Q_2 has one spare place, so Q_2 is “smaller” than Q_1 .
- Someone arrives! There is room in Q_2 but not in Q_1 .
- So Q_2 can jump above Q_1 . This can break monotonicity.
- **Fix:**
use $M/D/1/c + 1$ for upper process,
use $M/D/1/c - 1$ for lower process.

Failure of boundedness would be much trickier.




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Classic CFTP and unbounded state-spaces

What to do if the state-space is “unbounded”? Example: the Lindley recursion for $GI/G/1$ provides a representation that nearly delivers *CFTP*, but unboundedness means we can’t determine when to stop (see Foss and Tweedie 1998).


- “Unbounded” state-spaces can sometimes be small! (Key: is the chain **uniformly ergodic**? If so then classic *CFTP* is possible *in principle*; Foss and Tweedie 1998. Practical example from point process theory: Häggström, van Lieshout, and Møller 1999)
- One could **truncate and hope**. (Somewhat misses point of perfect simulation.)
- Murdoch (2000): **induce uniform ergodicity** by occasional proposals using independence sampler.
- **Dominated CFTP** (*domCFTP*): next section.



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Basic idea for dominated *CFTP*

- Classic *CFTP* fails when we can’t construct both upper and lower processes.
- So try to replace these by random processes in statistical stationarity (“dominating processes”).
- **If** we can guarantee a coupling which maintains “sandwiching” for the resulting envelope processes, and if they coalesce to run as the target process, **then** we can rescue the *CFTP* algorithm.
- We’ll need to be able to do the following:
 - draw from joint stationary distribution of the dominating processes;
 - simulate them **backwards** in time;
 - use their paths to simulate envelope processes **forwards** in time to coalesce as target processes;
 - check for coalescence, and ensure this happens on a reasonable time-scale.



Birth-death chain

Nonlinear birth and death processes: I

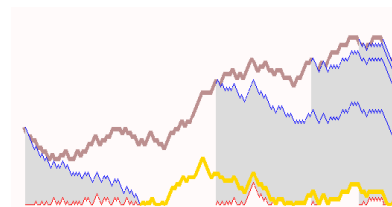
Adapted from WSK (1997).

- X is nonlinear immigration-death process:
 - $X \rightarrow X - 1$ at rate μX ;
 - $X \rightarrow X + 1 \dots \alpha_X$ where $\alpha_X \leq \alpha_\infty < \infty$.
 No maximum (not uniformly ergodic), so no classic CFTP!
- Bound by linear immigration-death process Y :
 - $Y \rightarrow Y - 1$ at rate μY ;
 - $Y \rightarrow Y + 1 \dots \alpha_\infty$.
- Produce X from Y by censoring births and deaths:
 - if $Y \rightarrow Y - 1$ then $X \rightarrow X - 1$ with c.prob. X/Y ;
 - if $Y \rightarrow Y + 1 \dots X \rightarrow X + 1 \dots \alpha_X / \alpha_\infty$.

The domCFTP construction

Nonlinear birth and death processes: II

- Given trajectory of Y , build trajectories of X starting at every $0 \leq X_0 \leq Y_0$ and then staying below Y .
- Because Y is reversible, with known equilibrium (detailed balance!) we can simulate Y backwards, then run forwards with sandwiched X realizations.
- Identify “golden thread” of X started arbitrarily far back in time. (Animation describes analogous random walk.)



Dominated CFTP: a recipe

Basic ingredients:

- dominating process:
 - draw from equilibrium;
 - simulate backwards in time.
- sandwiching:
 - Lower₁ ≲ Lower₂ ≲ ... ≲ Targets ≲ ... ≲ Upper₂ ≲ Upper₁
- coalescence:
 - eventually a Lower and an Upper process must coalesce to produce a Target process.
- computability:
 - is the price tag of Perfection too high?

Generalities

- Corcoran and Tweedie (2001) produce a general recipe for domCFTP for Harris chains. Roughly speaking:
 - Identify small set to which dominating process returns;
 - Check for regeneration at each visit;
 - Check domination maintained if no regeneration.
- WSK (2004) used this idea to show domCFTP is possible in principle for all geometrically ergodic chains:
 - Use Foster-Lyapunov criterion to build set-valued dominating process;
 - Model using a discrete-time D/M/1 queue!
 - Ensure ergodicity by sub-sampling;
 - Exploit regeneration (uses small-set theory).
- Connor and WSK (2007b, 2007a) extends this to some non-geometrically ergodic positive-recurrent chains.

Super-stable M/G/c and domCFTP (I)

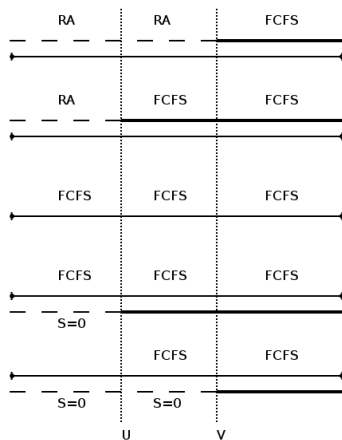
Sigman (2011) pioneered *domCFTP* for multi-server queues.
Key step: find amenable dominating process.

- Restrict to **super-stable** case (queue remains stable if we remove all but one server).
- Workload of $M/G/1$ queue as stable dominating process.
- Same $M/G/1$ workload if PS not FCFS.
- But $M/G/1[PS]$ is **dynamically reversible** (so we can reverse time in equilibrium).
- Recover $M/G/1[FCFS]$ from workload $M/G/1[PS]$.
- $M/G/c$: FCFS workload smaller than $M/G/1[FCFS]$ (FCFS is optimally efficient: Wolff 1977, 1987).
- In particular, domination is **sample-wise** if each service time is assigned **at initiation of service**.
- Coalescence forced when $M/G/1[PS]$ empties. (Finite mean if finite second moment of service time.)

Stable M/G/c and domCFTP (I)

Connor and Kendall (2015): dominate with $M/G/c[RA]$.
RA = "random assignment", so c copies of $M/G/1$, independent fractional arrival processes.

- Evidently stable if and only if $M/G/c$ is stable!
- **Domination!** (extend Asmussen 2003, Chapter XII)



Super-stable M/G/c and domCFTP (II)

Natural objections to what has been achieved so far

The Sigman (2011) result is pioneering, but only partial.

1. Coalescence is ensured by running backwards in time till dynamically-reversed $M/G/1[PS]$ empties. But this will be inefficient if the target $M/G/c$ workload is such that $M/G/1$ is nearly unstable.
2. Worse, the interesting case for $M/G/c$ is exactly when the $M/G/1$ is **not** stable. (When else would you pay for more than one server?!)
3. Sigman (2012) describes an importance-sampling approach when the corresponding $M/G/1$ is not stable. Unfortunately mean termination time is infinite.

Stable M/G/c and domCFTP (I)

Algorithm 1 (coalescence when $M/G/c[RA]$ empties):
NB: Processes run backwards are crowned with a tilde.

1. Consider $(M/G/1[PS])^c$ process \tilde{Y} , run backwards in statistical equilibrium. Draw from $\tilde{Y}(0)$.
2. Simulate c components of reversed-time $(\tilde{Y}(\tilde{t}) : \tilde{t} \geq 0)$ over $[0, \tilde{\tau}]$, where $\tilde{\tau}$ is smallest reversed time such that all components are empty at $\tilde{\tau}$.
3. Use $(\tilde{Y}(\tilde{t}) : \tilde{t} \in [0, \tilde{\tau}])$ to construct (dynamic) time reversal, and so build $(Y(t) : \tau \leq t \leq 0)$, an $M/G/c[RA]$ process (set $\tau = -\tilde{\tau}$).
4. Use Y to evolve X , an $M/G/c[FCFS]$ process, over $[\tau, 0] = [-\tilde{\tau}, 0]$, started in the empty state.

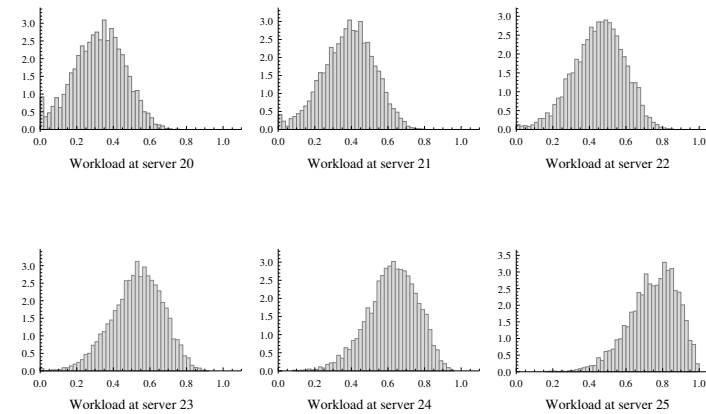
Now return $X(0)$.

Stable M/G/c and domCFTP (II)

Algorithm 2 (use upper and lower processes):

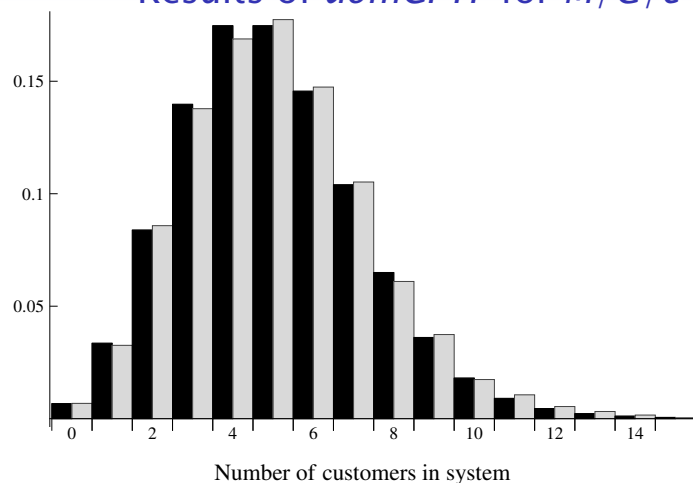
1. Consider a $(M/G/1[PS])^c$ process \tilde{Y} , run backwards in statistical equilibrium. Draw from $\tilde{Y}(0)$.
2. Fix suitable $\tilde{T} = -T$. Evolve queue for server j (independently of other servers) until first time $\tilde{\tau}_j \geq \tilde{T}$ that this server is empty, for $j = 1, \dots, c$.
3. Construct $M/G/1[FCFS]$ Y_j over corresponding reversed time interval $[-\tilde{\tau}_j, 0]$, for $j = 1, \dots, c$.
4. Produce lists \mathcal{L}_T^* , \mathcal{L}_T of service durations, arrival times.
5. Construct upper sandwiching process, $U_{[T,0]}$ over $[T, 0]$.
6. Construct lower sandwiching process, $L_{[T,0]}$ over $[T, 0]$.
7. Check for coalescence. Otherwise, extend.

Results of domCFTP for M/G/c (I)



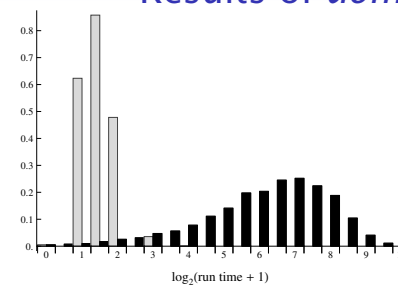
Equilibrium distribution of final 6 coordinates of Kiefer-Wolfowitz workload vector: arrival rate $\lambda = c = 25$, service durations $\text{Uniform}([0, 1])$. (5000 draws, Algorithm 2.)

Results of domCFTP for M/G/c (II)



Number of customers for $M/M/c$ queue in equilibrium when $\lambda = 10$, $\mu = 2$ and $c = 10$. Black bars summarize theoretical number of customers in system; light grey bars summarize 5000 draws, Algorithm 2. χ^2 -test: p -value 0.62.

Results of domCFTP for M/G/c (III)



Distribution of time taken to detect coalescence under Algorithms 1, 2 applied to $M/M/c$ queue, 5000 runs, $\lambda = 10$, $\mu = 2$, $c = 10$. Black bars show distribution of $\log_2(\tilde{\tau} + 1)$ for Algorithm 1 ($\tilde{\tau}$ is first time at which \tilde{Y} empties). Light grey bars show distribution of $\log_2(\tilde{T} + 1)$ for Algorithm 2, where \tilde{T} is smallest time needed to detect coalescence using binary back-off. (Algorithm 2 speed-up also suggested by calculations.)

Conclusion

It is highly feasible to produce *exact* simulations of stable *M/G/c* queues (if service distribution has finite second moment) using *domCFTP*.

Further possibilities:

- *M/G/c* behaviour is not encoded by first two moments of service distribution (Gupta et al. 2009).
- General input processes: Blanchet and Wallwater (2014), Blanchet et al. (2015).
- Omnithermal *domCFTP* for a range of values of *c*?
- Networks? Sigman (2013), Blanchet and Dong (2013).
- Compartment models? Connor and WSK are working on SIR epidemics.
- “Efficient simulation”?

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