Coupling II: applications to simulation

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Abstract

The constructive nature of probabilistic coupling ("to build Y using the randomness of X") makes it close in spirit to the task of constructing good stochastic simulations. Recently the link between coupling and simulation has been strengthened in striking ways, resulting in so-called "exact" or "perfect simulation". This talk will introduce these developments.

1 Introduction

Häggström (2002) includes discussion of some of these ideas at the level of a student monograph.

Other impending monographs which will discuss *CFTP*: Aldous and Fill (200x); Møller and Waagepetersen (2003); Roberts *et al.*

- http://www.warwick.ac.uk/statsdept/staff/WSK/talks/durham-lms2.pdf
  http://research.microsoft.com/~dbwilson/exact/

2 More on mixing

Recall (continuous-time!) random walk on n-dimensional Boolean hypercube.

Case $n = 1$: State space $\{0, 1\}$. Let X, Y start at 0, 1. How to couple them?

- We want $0 \to 1$ transitions at rate $1/\alpha$, and $1 \to 0$ transitions at rate $1/\alpha$.
- We want X and Y to meet (at time $T_{0,1}$ in notation of previous lecture).

> **Construction:** Supply (a) Poisson process (rate $1/\alpha$) of $0 \to 1$ transitions, (b) *ditto* of $1 \to 0$ transitions. Apply them where applicable to X, Y. **Clearly** X, Y have desired distributions.

Coupling happens at first instant of combined Poisson process, so $T_{0,1}$ is Exponential of mean $\alpha/2$.

Case $n > 1$: Couple each coordinate independently. Replace α by n to play fair. So (everything) coupled at time which is maximum of n independent Exponentials each of mean $n/2$.

> **Deduce** from coupling inequality that mixing occurs before time $n \log(n)/2$:-).

Questions:

Does it make sense to return first coupled value? (Yes here, no in "nearly every" other case.)

> But suppose I run algorithm from $[-T, 0]$ for increasing T, instead of from $[0, T]$ for increasing T? (This *will* work always: CFTP.)

Olle Häggström points out, mixing actually occurs around time $n \log(n)/4$. Can you find a coupling argument to show this?

3 *CFTP*

Theorem 1 *If coalescence is almost sure then* CFTP *delivers a sample from the equilibrium distribution of the Markov chain X corresponding to the random input-output maps $F_{(-u,v]}$.*

Proof: For each $[-n, \infty)$ use input-output maps $F_{(-n,t]}$

$$X_t^{-n} \quad = \quad F_{(-n,t]}(0) \qquad \text{for } -n \leq t\,.$$

Assume finite coalescence time $-T$ for F. Then (3 lines!)

$$\begin{aligned} X_0^{-n} &= X_0^{-T} \qquad \text{whenever } -n \leq -T\,; \\ \mathcal{L}(X_0^{-n}) &= \mathcal{L}(X_n^0) \end{aligned}$$

If X converges to an equilibrium π then

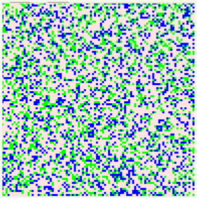
$$\operatorname{dist}_{tv}(\mathcal{L}(X_0^{-T}), \pi) = \lim_n \operatorname{dist}_{tv}(\mathcal{L}(X_0^{-n}), \pi) = \lim_n \operatorname{dist}_{tv}(\mathcal{L}(X_n^0), \pi) = 0$$

(dist_{tv} is total variation) hence giving the required result. □

3.1 Classic *CFTP*

The original Propp and Wilson (1996) idea showed how to make exact draws from the critical Ising model. A rather simpler application uses the heat-bath bath sampler to make exact draws from the sub-critical Ising model.

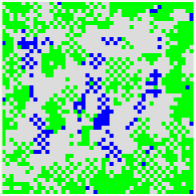
Classic *CFTP* for the Ising model (simple, sub-critical case). Heat-bath dynamics run from past; compare results from maximal and minimal starting conditions.



Green denotes both spin up; *blue* denotes both spin down; *misty-rose* denotes disagreement between maximal and minimal.

Approaches based on Swendsen-Wang ideas work for critical case. Huber (2003).
Under conditioning by noisy data, the difficulties caused by phase-transition phenomena disappear.

Classic *CFTP* for the Ising model conditioned by noisy data. Without influence from data ("external magnetic field") this Ising model would be supercritical.



Green denotes both spin up; *blue* denotes both spin down; *grey* denotes disagreement between maximal and minimal. In the summary image, *orange* denotes coalesced spin up, image spin down while *pink* denotes coalesced spin down, image spin up.

3.2 Falling leaves

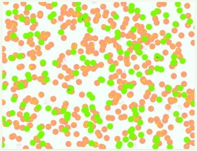
Kendall and Thönnes (1999) describe a visual and geometric application of *CFTP* in mathematical geology: this particular example being well-known to workers in the field previous to the introduction of *CFTP* itself.

Occlusion *CFTP* for the falling leaves of Fontainbleau.
(Why "occlusion"? the *CFTP* builds up the result directly.)

3.3 Dominated *CFTP* for area-interaction point processes

Dominated CFTP replaces the deterministic maximum by a known random process run backwards in time, providing starts for upper- and lower-envelope processes guaranteed to sandwich a valid simulation. It works, for example, on both attractive and repulsive area-interaction point processes (Kendall 1998; Kendall and Møller 2000).

Application of Dominated *CFTP* for attractive area-interaction point process with geometric marking using Poisson processes in disks (Kendall 1997).

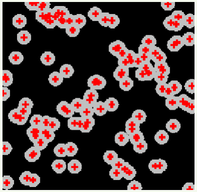


See also Huber (1999)'s notion of a "swap move". If birth proposal is blocked by just one point, then replace old point by new in a *swap*, with swap probability p_{swap} which we are free to choose. Hence "bounding chain", "sure/not sure" dichotomy.

3.4 Fast attractive area-interaction *CFTP*

Häggström, van Lieshout, and Møller (1999) describe fast *CFTP* for attractive area-interaction point processes using special features.

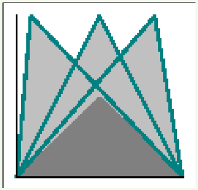
Gibbs' sampler *CFTP* for the attractive area-interaction point process as a marginal of a two-type soft-core repulsion point process.



3.5 Small-set *CFTP*

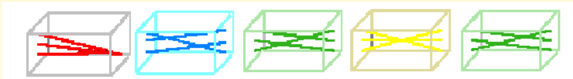
Green and Murdoch (1999) showed how to use small sets to carry out *CFTP* when the state-space is continuous with no helpful ordering. Their prescription includes the use of a partition by several small sets, to speed up coalescence.

Small set *CFTP* in nearly the simplest possible case: a triangular kernel over $[0, 1]$.



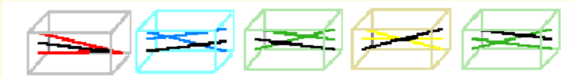
4 Fill's method (FMMR)

The alternative to *CFTP* is *Fill's algorithm* (Fill 1998; Thönnes 1999), at first sight quite different, based on the notion of a *strong uniform time* T. Fill et al. (2000) establish a profound link. We explain using "blocks" as input-output maps for a chain.



First recall that *CFTP* can be viewed in a curiously redundant fashion as follows:

- **Draw from equilibrium** $X(-T)$ and run forwards;
- continue to increase T until $X(0)$ is coalesced;
- return $X(0)$.



Key observation: By construction, $X(-T)$ is independent of $X(0)$ and T so . . .

- Condition on a convenient $X(0)$;
- Run X backwards to a fixed time $-T$;
- Draw blocks conditioned on the X transitions;
- **If** coalescence **then** **`return`** $X(-T)$ **else** **`repeat`**.

> "It's a kind of magic . . ."
> Queen

Is there a *dominated* version of Fill's method?

5 Price of perfection

Coupling of couplings:...

$$|p_t(x_1,y) - p_t(x_2,y)| \quad \leq$$
$$\leq \quad |\mathbb{P}[X_1(t)=y|X_1(0)=x_1] - \mathbb{P}[X_2(t)=y|X_2(0)=x_2]| \leq$$
$$|\mathbb{P}[X_1(t)=y|\tau>t, X_1(0)=x_1] - \mathbb{P}[X_2(t)=y|\tau>t, X_2(0)=x_2]|$$
$$\times\, \mathbb{P}[\tau>t|X(0)=(x_1,x_2)]$$

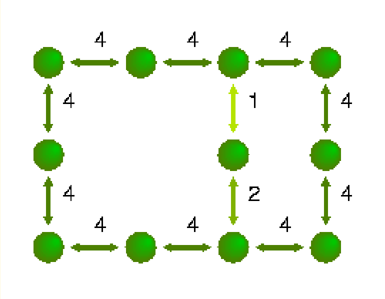
Suppose $|p_t(x_1,y) - p_t(x_2,y)| \approx c\exp(-\mu_2 t)$
while $\mathbb{P}[\tau>t|X(0)=(x_1,x_2)] \approx c\exp(-\mu t)$
Let X^* be a coupled copy of X but begun at (x_2,x_1):

$$|\,\mathbb{P}[X_1(t)=y|\tau>t, X_1(0)=x_1] - \mathbb{P}[X_2(t)=y|\tau>t, X_2(0)=x_2]\,|$$
$$= \quad |\,\mathbb{P}[X_1(t)=y|\tau>t, X(0)=(x_1,x_2)] -$$
$$\mathbb{P}[X_1^*(t)=y|\tau>t, X^*(0)=(x_2,x_1)]\,|$$

So let σ be time when X, X^* couple:

$$\leq \mathbb{P}[\sigma>t|\tau>t, X(0)=(x_1,x_2)] \quad (\approx c\exp(-\mu' t))$$

Thus $\mu_2 \geq \mu' + \mu$.



Burdzy and Kendall (2000), Mountford and Cranston (2000), Burdzy and Chen (2002), also Kumar and Ramesh (2001).

5.1 Domination and small sets

Corcoran and Tweedie (2001) describe how to mix dominated *CFTP* and small set *CFTP*. The upper envelope process must be formulated carefully ....
The idea is close to Foster-Liapunov techniques for assessing geometric ergodicity *etc* for Markov chains. Foster-Liapunov uses a Liapunov function V to deliver a controlled supermartingale off a small set:

$$\mathbb{E}\left[V(X_{n+1})|X_n\right] \quad \leq \quad \lambda V(X_n) + \beta\, \mathbb{I}\left[X_n \in C\right] .$$

Temptation: define dominating process using V. There is an interesting link – Rosenthal (2002) draws it even closer – but:

Existence of Liapunov function
doesn't ensure dominated *CFTP*

There are perverse examples satisfying the supermartingale inequality, but failing the stochastic dominance required of $V(X)$ by dominated *CFTP* ... :-(.[1]

[1] *Later:* I have discovered how to fix this using sub-sampling.

6 Combinations and variations

(Other things I'd have liked to talk about . . .)

6.1 Layered Multishift *CFTP*

(Wilson (2000b) and further work by Corcoran and Schneider (2002)) Issue: how to draw simultaneously from Uniform$(x, x+1)$ for all $x \in \mathbb{R}$, and to try to couple the draws? Answer: draw a uniformly random *unit span integer lattice*, ... Now think about more general distributions!

6.2 Read-once randomness

Wilson (2000a) shows how to avoid a conventional requirement of *CFTP*, to re-use randomness used in each cycle.

6.3 Perfect simulation for Peirls' contours *etc*

(Ferrari et al. 2002). The Ising model can be reformulated in an important way by looking only at the *contours* (lines separating ± 1 values). In fact these form a "non-interacting hard-core gas", permitting (theoretically) Ferrari et al. (2002) to apply their variant of perfect simulation (*Backwards-Forwards Algorithm*).

6.4 Randomness Recycler

Fill and Huber (2000) introduce a quite different form of perfect simulation! Overleaf is how they apply their *Randomness Recycler* algorithm to the problem of drawing a random independent subset X of a graph G, weighted exponentially by number of points in X.

Start: $V = \emptyset$, $x \equiv 0$. End: $V = G$, x indicates X membership.

```
while V ≠ G:
    v ← choice(G − V)    # Choose v from G \ V
    V.add(v)
    if uniform(0, 1) ≤ 1/(1 + alpha):
        x[v] ← 0    # Skip v with prob 1/(1 + α)
    else:
        x[v] ← 1    # or tentatively include it ...
        nbd ← [ ]    # ... iterate thro' neighbours
        for w ∈ neighbourhood(v):    # Valid?
            nbd.append(w)
            if x[w] = 1:    # If not valid ...
                x[w] ← 0    # ... remove all
                x[v] ← 0    # "contaminated" vertices
                V ← V − [v] − nbd
                break    # and move on
```

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