Interfaces between Probability and Geometry

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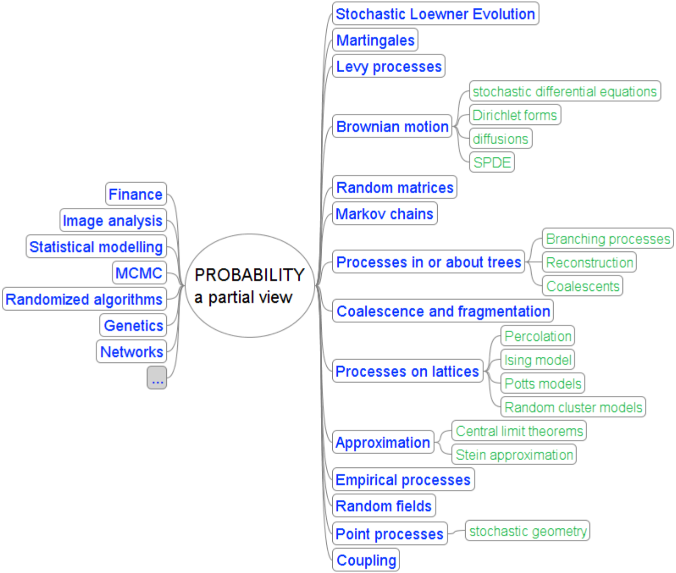
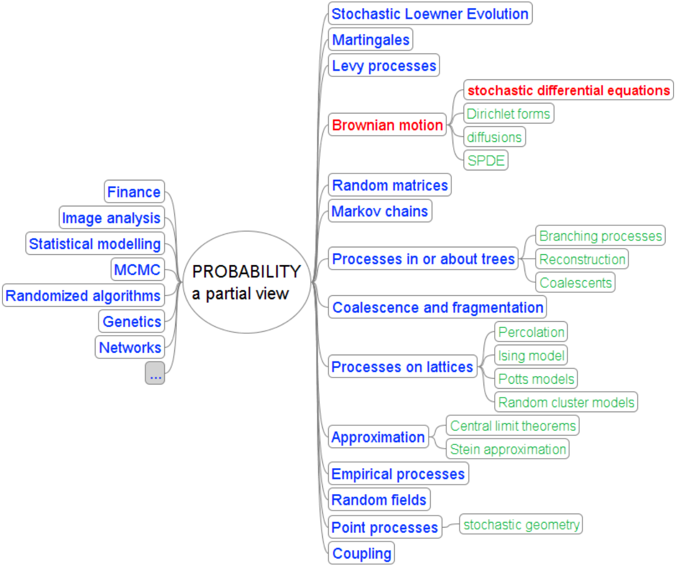
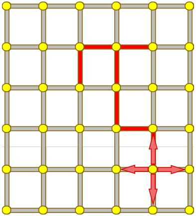
Introduction



Brownian motion



Brownian motion (I)

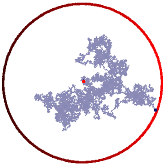


"Infinitesimal random walk" relates to numerical analysis: Courant, Friedrichs, and Lewy 1928.

$$f(x) = \frac{1}{4}\left(f(x+n) + f(x+e) + f(x+s) + f(x+w)\right)$$

converging in the limit to

$$\Delta f = 0$$
$$f(x) = \mathbb{E}\left[f(B_T) | B_0 = x\right]$$



(martingales, stochastic calculus, . . .)

Brownian motion (II)

From random walk to Brownian motion and beyond:

- extends idea of Central Limit Theorem;
- stochastic modelling;
- complex analysis *via* Lévy's Theorem;
- stochastic Itô calculus (finance!).

Explicit representation $f(x) = \mathbb{E}\left[f(B_T)|B_0 = x\right]$ useful:

- in case of irregular boundaries;
- when generalizing to diffusions;
- as part of general approach to random processes on fractals.

Significant generalizations:

- from stochastic differential equations (SDE) to stochastic partial differential equations (SPDE);
- measure-valued diffusions (let BM reproduce and die);
- nonlinear elliptic variational problems (need to generalize martingales, expectations).

Brownian motion (III)

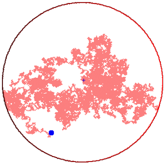
Neumann heat kernel

Reflecting Brownian motion:

- prevent all moves outside of the domain D;
- related to heat flow in insulated domain;
- probability (heat) kernel $p_t^D(x, y)$ is probability density of B_t at Y if B is begun at x.

QUESTION: does $p_t^D(x, y)$ depend monotonely on D?

- yes at large times;
- not necessarily at very short times;
- yes for convex *well-separated* domains (WSK 1989);
- **no** for general convex domains (Bass and Burdzy 1993).



("fly in convex room" example)

Brownian motion (IV)

Stochastic Loewner Evolution (SLE)

Consider $\mathbb{H}$ the upper half-plane, and the differential equation

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}$$

defined for Brownian motion B begun at 0, with $g_0(z) = z \in \overline{\mathbb{H}}$.

This fails to be defined after time $T_z = \sup\{t : |g_s(z) - \kappa B_s| > 0 \text{ for } s \leq t\}$.

Then $K_t = \{z \in \overline{\mathbb{H}} : T_z \leq t\}$ is (chordal) Stochastic Loewner Evolution SLE_κ, closely linked to many stochastic boundaries, the study of which won a Fields medal for Wendelin Werner in 2006.

Brownian motion: preparations and directions

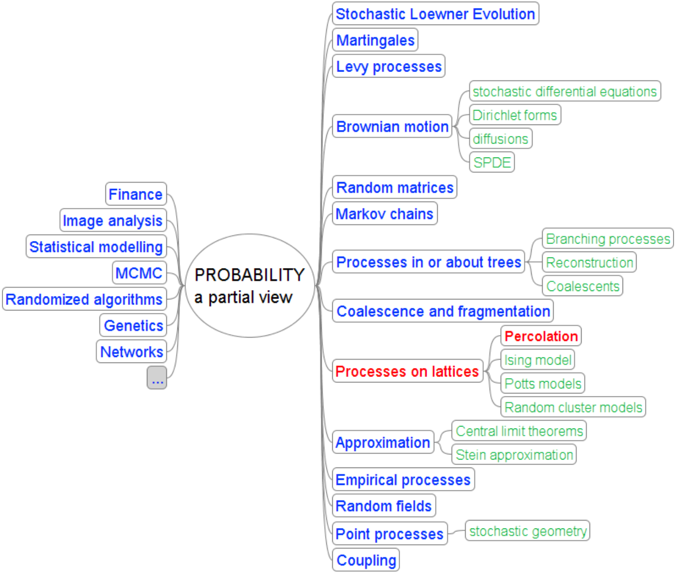
Preparations:

- measure theory, measure-theoretic probability, analysis;
- Øksendal (2003) (book on SDE);
- Revuz and Yor (1999) (book on martingales and Brownian motion).

(Random) directions:

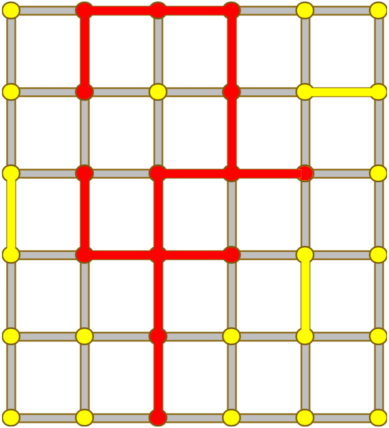
- What might be the statistical consequences of domain non-monotonicity of the Neumann heat kernel?
- How might the theory of SLE be used in applied probability to model interfaces?

Percolation



Percolation (I)

Consider bond percolation: pipe segments are independently open with probability p, closed with probability $1 - p$. As p increases, when does an infinite open cluster first appear?



History: Broadbent and Hammersley (1957)

Clearly a critical p_H should exist.

Theorem (Kesten, 1980): $p_H = 1/2$.

Percolation (II)

There are many other kinds of percolation

- site percolation;
- triangular lattice site percolation
  (Cardy's formula in critical case, SLE_6);
- Poisson-Voronoi site percolation
  (Bollobas and Riordan 2006, $p_H = 1/2$);
- Percolation on Cayley graphs of groups ...

Percolation (III)

Percolation and non-amenability

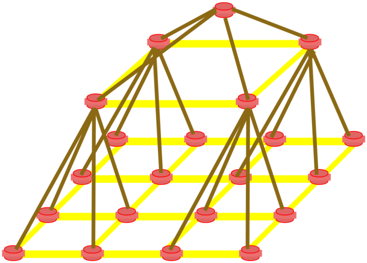
Grimmett and Newman (1990) study bond-percolation in $T \times \mathbb{Z}^d$ where T is a regular tree. (Hard to visualize!) Tree-bonds have different probabilities from space-bonds.

They show there are at least two phase transitions: 0-to-many-infinite-clusters, then many-to-1-infinite-cluster.

For Cayley graphs, existence of two phase transitions is related to whether the underlying group is *non-amenable* (cannot possess an invariant mean).

Percolation (IV)

For particular tree-like graphs arising in quad-tree-based image analysis, WSK and Wilson (2003) show two phase transitions using methods related to *hyperbolicity*.



Percolation: preparations and directions

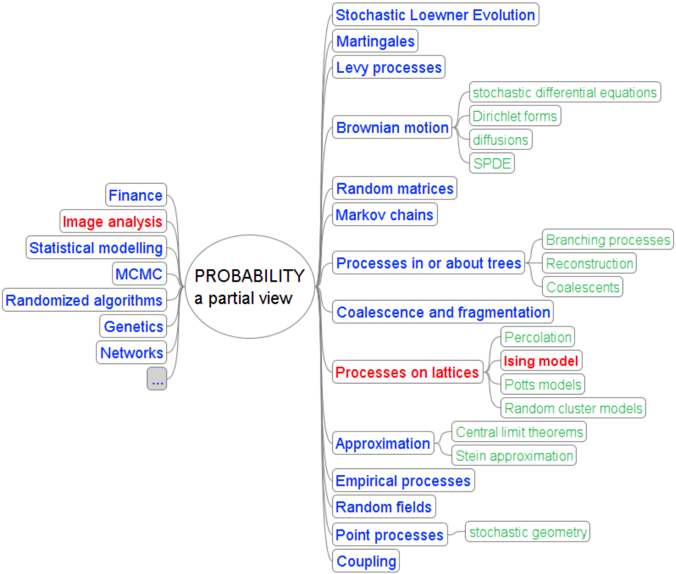
Preparations:

- first-year undergraduate probability(!) and plenty of mathematical analysis;
- Grimmett (1999) (book on percolation, mostly Euclidean);
- Benjamini and Schramm (1996) “Percolation beyond $\mathbb{Z}^d$, many questions and a few answers”.

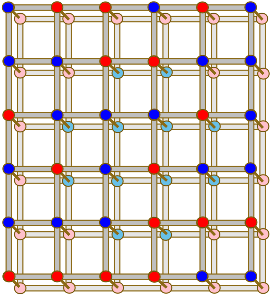
(Random) directions:

- Investigate whether the Bollobas and Riordan (2006) result has consequences for spatial epidemics.
- Can uniqueness-of-infinite-cluster transition be related to notions of metric-space hyperbolicity for graphs?

The Ising model



The Ising model (I)



- Each node is spin-up ($+$, $\sigma = +1$) or spin-down ($-$, $\sigma = -1$);
- Probability of configuration proportional to:

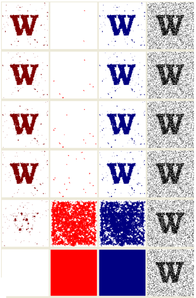
$$\exp\left(-\frac{1}{kT}\left(-J\sum_{ij}\sigma_i\sigma_j\right)\right) = \exp\left(\beta\sum_{ij}\sigma_i\sigma_j\right) .$$

- Phase transition for infinite lattice: computable in dimension 2;
- Local specification *versus* model on infinite grid;
- Image analysis: superimpose a second fixed/conditioned lattice, linked by different β.

The Ising model (II)

Image analysis with Ising models:

- "heat-bath" algorithm (MCMC);
- Coupling from the Past (CFTP);
- Good image reconstruction:
  low "temperature" plus
  strong influence of image;
- Recent theoretical evidence suggests fast convergence (of modified algorithm);
- More sophisticated approaches: use "quad-trees" (superimposed coarser grids). Issue: identify phase transitions;
- Use the Fortuin-Kastelyn inequalities to estimate using percolation!



Ising model: preparations and directions

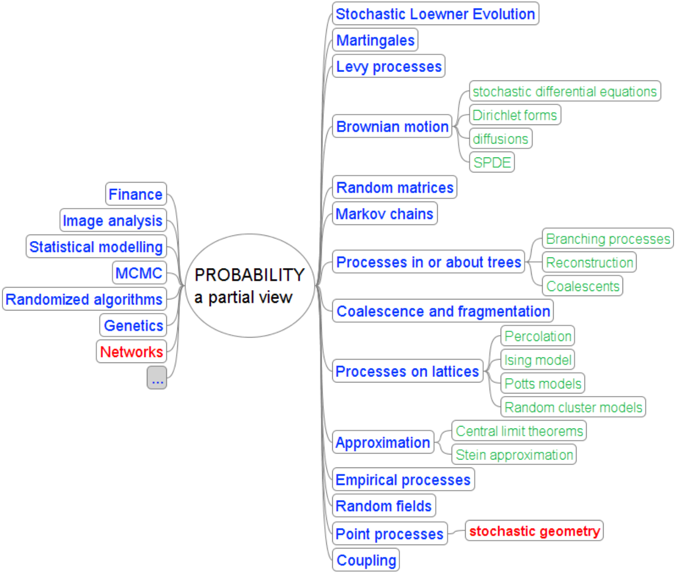
Preparations:

- once again, first-year undergraduate probability(!) and plenty of mathematical analysis;
- Kindermann and Snell (1980) (excellent introduction, freely available on web!);

(Random) directions:

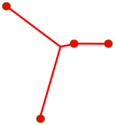
- The quad-tree work is now focussed on getting an understanding of when interfaces propagate.
- SLE is conjectured to be related to the behaviour of the Ising model at criticality.

Networks and stochastic geometry



Networks and stochastic geometry (I)

- What is the shortest network to connect up the red sites using the grid?
- The answer is the rectilinear Steiner Tree.
- In general, computation of this is an NP-complete algorithm!
- The Euclidean variant is also NP-complete of course.
- Steiner trees are good for minimal length connection, but bad for efficient connections. How much more expensive to do better?

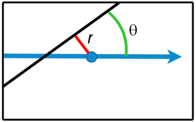


Networks and stochastic geometry (II)

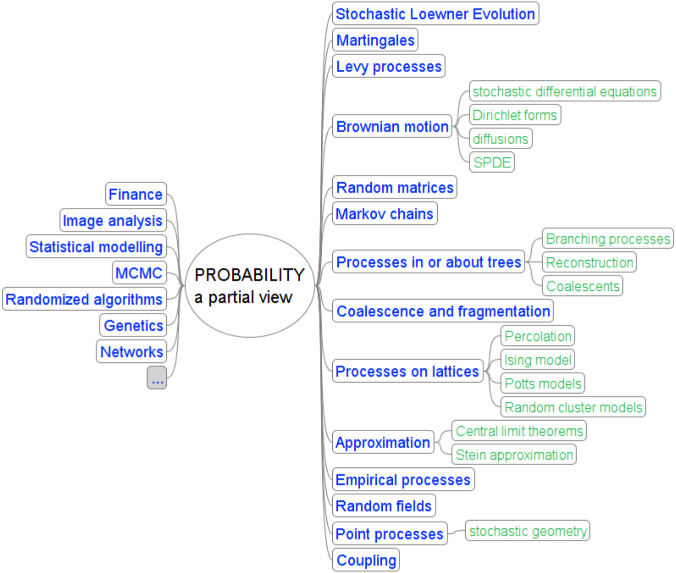
Aldous and WSK 2008: take the Euclidean Steiner tree for n locations in a rectangle $[0, \sqrt{n}]^2$;

- total Steiner tree network length is of order n;
- we can reduce average connection length to within about $O(\log n)$ of best possible, as follows:
  - add a very small number of extra random lines to generate efficient long-range connections;
  - add a small amount of extra connectivity to ensure one can move efficiently onto the lines using the Steiner tree;
- total extra network length is just $o(n)$!

Random line: choose uniform random orientation θ, choose uniform random (signed) distance r from reference point.



Conclusion


PROBABILITY
a partial view
Finance
Image analysis
Statistical modelling
MCMC
Randomized algorithms
Genetics
Networks
...
Stochastic Loewner Evolution
Martingales
Levy processes
Brownian motion
stochastic differential equations
Dirichlet forms
diffusions
SPDE
Random matrices
Markov chains
Processes in or about trees
Branching processes
Reconstruction
Coalescents
Coalescence and fragmentation
Processes on lattices
Percolation
Ising model
Potts models
Random cluster models
Approximation
Central limit theorems
Stein approximation
Empirical processes
Random fields
Point processes
stochastic geometry
Coupling

Conclusion (continued)

There is plenty to do in probability theory, whether you want:

- deep and difficult theories with powerful links to other areas of mathematics;
- stochastic techniques and concepts which underly the finance industry;
- or pretty and challenging applied problems with surprising results.

Bibliography

This is a rich hypertext bibliography. Journals are linked to their homepages, and stable URL links (as provided for example by JSTOR or Project Euclid) have been added where known. Access to such URLs is not universal: in case of difficulty you should check whether you are registered (directly or indirectly) with the relevant provider. In the case of preprints, icons , , , linking to homepage locations are inserted where available: note that these are less stable than journal links!.

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