Brownian confidence bands

(Statistical MTV at Warwick)

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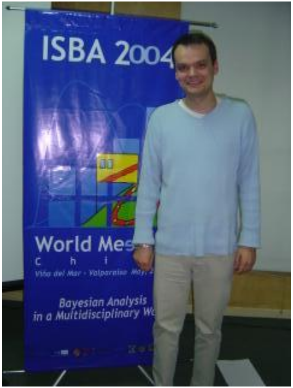
3 November 2005

Introduction . . .

This is joint work with Christian Robert and Jean-Michel Marin: informal convergence diagnostics for Monte Carlo leading to an apparently novel application of Brownian local time and thus to an unresolved question.



Monte Carlo

IID sampling from π gives an estimate and CLT:

$$\widehat{\mathfrak{I}}_n = \frac{1}{n}\sum_{i=1}^{n} h(X_i) \quad \longrightarrow \quad \mathfrak{I} = \int h(x)\pi(\mathrm{d}\,x)$$

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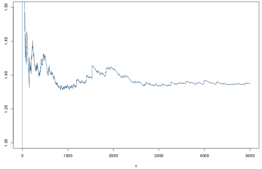

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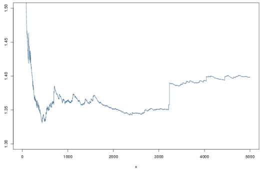

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$$\min_{u,v} \int_0^1 (u(t) + v(t))\,\mathrm{d}\,t \quad \text{subject to}$$
$$\mathbb{P}\left[-v(t) \leq W(t) \leq u(t) \text{ for all } t \in (0,1)\right] = 1 - \alpha\,,$$
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(using symmetry to produce one-sided version)

$$\min_{u \geq 0} \mathbb{P}\left[W(t) \geq u(t) \text{ for some } t \in (0,1)\right]$$

$$\text{subject to } \int_0^1 u(t)\,\mathrm{d}\,t = \kappa\,.$$

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$$\min_{u \geq 0} \left(\mathbb{E}\left[L_u(1)\right] = \frac{1}{2} \int_0^1 \exp\left(-\frac{u(s)^2}{2s} \frac{\mathrm{d}\, s}{\sqrt{2\pi s}} \right) \right)$$

$$\text{subject to } \int_0^1 u(t)\mathrm{d}t = \kappa\,.$$

The “Kobayashi Maru” strategy (II)

There is an “explicit” solution u^* involving the ProductLog or Lambert W function ($W(z)$ solves $We^W = z$ *maximally*),

$$u^*(s) \quad = \quad \begin{cases} \sqrt{sW(\kappa^2 s^2)} & \text{if } \kappa s \leq 1/\sqrt{e}\,, \\ 0 & \text{otherwise,} \end{cases}$$

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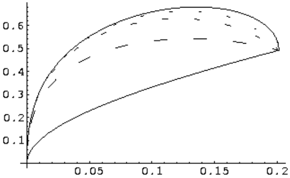

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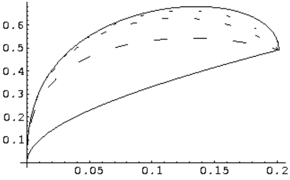

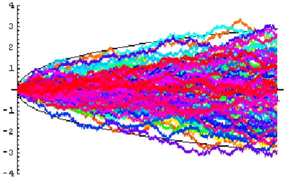

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WHY and HOW does the local time problem
agree so well with
the original exceedance-probability version?

References

Kendall, W. S., J. Marin, and C. P. Robert (2004, November).
Brownian confidence bands on Monte Carlo output, .
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