A Comparison of Markov-Functional and Market Models: The One-Dimensional Case

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Abstract

The LIBOR Markov-functional model is an efficient arbitrage-free pricing model suitable for callable interest rate derivatives. We demonstrate that the one-dimensional LIBOR Markov-functional model and the separable one-factor LIBOR market model are very similar. Consequently, the intuition behind the familiar SDE formulation of the LIBOR market model may be applied to the LIBOR Markov-functional model.

The application of a drift approximation to a separable one-factor LIBOR market model results in an approximating model driven by a one-dimensional Markov process, permitting efficient implementation. For a given parameterisation of the driving process, we find the distributional structure of this model and the corresponding Markov-functional model are numerically virtually indistinguishable for short maturity tenor structures over a wide variety of market conditions, and both are very similar to the market model. A theoretical uniqueness result shows that any accurate approximation to a separable market model that reduces to a function of the driving process is effectively an approximation to the analogous Markov-functional model. Therefore, our conclusions are not restricted to our particular choice of driving process. Minor differences are observed for longer maturities, nevertheless the models remain qualitatively similar. These differences do not have a large impact on Bermudan swaption prices.

Under stress-testing, the LIBOR Markov-functional and separable LIBOR market models continue to exhibit similar behaviour and Bermudan prices under these models remain comparable. However, the drift approximation model now appears to admit arbitrage that is practically significant. In this situation, we argue the Markov-functional model is a more appropriate choice for pricing.

1 Introduction

The problem of pricing callable exotic interest rate derivatives, such as the Bermudan swaption, is one of the most important problems in option pricing theory, being of great practical importance in the marketplace. The LIBOR market model of interest rate dynamics, developed by Brace, Gątarek & Musiela [1997], Milterson, Sandmann & Sondermann [1997], and to a lesser extent the corresponding swap-based market model developed by Jamshidian [1997], have now become some of the most popular models for pricing such derivatives. They are generally considered to have more desirable theoretical calibration properties than short-rate models such as the Vasicek-Hull-White model (Hull & White [1990]). In particular, they offer a much more flexible parameterisation that allows experienced users to calibrate the model to a larger set of calibrating instruments whilst ensuring that the model exhibits realistic dynamics. However, the high dimensionality of the full market model specification means that in practice it is usually desirable to approximate the model or the exercise boundary in some way to obtain an efficient pricing algorithm.

One popular technique for obtaining an approximation to the market model is to estimate the drift of the market model over large time steps. For example, Pelsser, Pietersz & van Regenmortel [2004] describe accurate approximations for the drift of a LIBOR market model based on a Brownian bridge (see Section 2). Also see Hunter, Jäckel & Joshi [2001] and Kurbanmaradov, Sabelfield & Shoenmakers [2002]. The application of such a drift approximation leads to gains in efficiency if we assume the instantaneous volatility structure of the market model is of a *separable* form, since this allows the market model to be approximated by a model driven by a low-dimensional Markov process under the terminal measure (following Pelsser et al. [2004]). For one of the first references on separability see Carverhill [1994]. For a one-factor LIBOR market model we say the model is *separable* if the instantaneous volatility function of each LIBOR is proportional to a common instantaneous volatility function σ_t. It is straightforward to show that under such a model the drift-approximated forward LIBORs may be written as a function of a one-dimensional Markov process of the form

$$x_t := \int_0^t \sigma_s \, dW_s \,. \tag{1}$$

We will henceforth refer to this approximate pricing model as the *drift approximation model.* We shall see that the concept of separability introduced in the construction of this efficient model provides the link between market models and Markov-functional models.

The LIBOR Markov-functional model (Hunt, Kennedy & Pelsser [2000]) is an alternative pricing model for callable interest rate derivatives. It can fit Black's formula for caplets (or indeed any arbitrage-free European caplet formula) in a similar fashion to the LIBOR market model but it has the advantage that derivative prices can be calculated just as efficiently as under a Gaussian short-rate model such as the Vasicek-Hull-White model (Hull & White [1990]). This is an important consideration for practitioners. Efficient implementation is possible because under a Markov-functional model all discount bond prices are at any time a function of some low-dimensional Markov process, hence it is only necessary to keep track of this driving process in the pricing algorithm. Note that in contrast with the LIBOR market model, one cannot easily write a simple SDE for the behaviour of the relevant LIBORs under the Markov-functional model and this relatively non-standard model formulation makes its properties somewhat less transparent.

In this article, our aim is to understand the relationship between Markov-functional models with a one-dimensional driving process and one-factor separable LIBOR market models, and thereby shed light on the implicit behaviour of rates under the Markov-functional model. All our models will be calibrated to Black's formula for the relevant caplet prices. In the following we shall not enter into any debate on the appropriateness of the assumption of separability or the use of a single factor model in any particular pricing problem, since our focus here is to understand the relationship between the three models. The reader may find these relatively contentious issues discussed elsewhere.[1]

Although we focus on a comparison of one-dimensional models in this article, it is also interesting to look at what happens in n dimensions. Note that a standard k-factor LIBOR market model has dimension given by n, the number of LIBORs (as it is Markovian only in n dimensions), whereas the dimension of a Markov-functional model is simply the dimension of the driving process x. If a k-factor market model is separable, then we know there exists a k-dimensional Markovian model that approximates the behaviour of the market model, such as the drift approximation model (Pelsser et al. [2004]). It is natural to ask how a k-dimensional Markov-functional model compares with such an approximation.

In the one-dimensional case, the construction of a Markov-functional model is effectively unique (in so far as it is uniquely specified on a grid, for a given driving process, assuming Black caplet prices; see discussion towards the end of Section 3). This suggests a comparison with a one-factor separable LIBOR market model because this model may be approximated by a one-dimensional Markovian model with a driving process x_t of the form (1). Choosing x_t to be the same driver as for the LIBOR Markov-functional model, the uniqueness result indicates that as long as the approximating model does not admit noticeable arbitrage the two models must be effectively the same. Thus the Markov-functional model will also be a very close match to the corresponding LIBOR market model. In this case, even though the separable one-factor LIBOR market model is theoretically Markovian only in n dimensions, its behaviour will resemble that of a one-dimensional model.

We expect that a similar relationship may be found between Markov-functional models and LIBOR market models in two or three dimensions, although here there will not be such a straightforward link between the models. This is because the specification of a Markov-functional model in higher dimensions is much more flexible (it typically requires the additional specification of a "pre-model" that describes the correlation structure of the model; see Bennett [2005]).

In the following, we perform a comparison of one-factor versions of the separable LIBOR market model and both the associated one-dimensional drift approximation model and the corresponding one-dimensional Markov-functional model. For both the drift approximation and Markov-functional models it is possible to study the numerical distributional structure explicitly as all the relevant LIBORs at each exercise date are a known function of the driving process at that date. Such a distributional study is more informative than a simple comparison of exotic derivative prices, as the latter are effectively only summary statistics (albeit important ones in practice).

Under normal market conditions the distributions of LIBORs under the separable LIBOR market model and the associated drift approximation model appear extremely close to those under the analogous Markov-functional model with the same driving process. For short maturities the three models are virtually indistinguishable numerically. We find that at each exercise date the log of each LIBOR is essentially linear in the log of the terminal LIBOR in all cases (similar conclusions are drawn when investigating the analogous link between the one-factor swap Markov-functional model and the corresponding one-factor separable swap market model). One would therefore expect exotic derivative prices calculated using these LIBOR models to be very similar.

Practitioners will find within Section 4 a discussion of the appropriate choice of model in the context of Bermudan swaption pricing. As a general rule, the computation of Bermudan swaption prices under the LIBOR Markov-functional model is found to be approximately 100 times faster than under the corresponding LIBOR market model.[2] Pelsser & Pietersz [2005] obtain similar relative execution times; they state that performing the same price/delta/vega analysis on a Bermudan swaption takes 92 seconds using a swap market model and only 3 seconds for a swap Markov-functional model.

For short maturities the prices of standard annual Bermudan swaptions under the separable LIBOR market model (computed using the least squares method of Longstaff & Schwartz [2001]), the drift approximation model and the corresponding Markov-functional model are virtually identical over all scenarios.

For longer maturities and high volatilities we begin to notice slight numerical differences between these models, although qualitatively the models still exhibit similar characteristics and prices of Bermudan swaptions under these models remain comparable (in fact it is arguable that these differences would be acceptable to practitioners). It is under these conditions that the high dimensionality of the LIBOR market model becomes apparent - the scatter plot of a given LIBOR versus the terminal LIBOR (at a given exercise date) tends to exhibit significant dispersion and is therefore no longer well represented by a single functional form. The drift approximation model now admits arbitrage that is numerically significant and is therefore inappropriate for derivative pricing under these extreme scenarios. Indeed, it is under these unusual conditions that any low dimensional approximation to the market model (or the exercise boundary within a market model) is likely to fail. The choice of an exact model such as the Markov-functional model, which admits an efficient arbitrage-free numerical implementation without the need for approximation, would seem preferable to the use of an approximation whose limitations may be unknown.

2 The market model and separability

We begin by describing the standard construction of the market model and the drift approximation model under which a separable LIBOR market model is approximated by a model driven by a low-dimensional Markov process.

Notation and definitions

In this section our notation for the LIBOR market model is introduced. Let D_{tT} denote the time-t value of a zero-coupon discount bond with maturity T. Let $T_1 < T_2 < \ldots < T_{n+1}$ be a sequence of dates and for $i = 1, \ldots, n$ define the corresponding forward LIBORs

$$L_t^i := \frac{D_{tT_i} - D_{tT_{i+1}}}{\alpha_i D_{tT_{i+1}}}\,, \tag{2}$$

where the α_i are the accrual factors.

We develop all models in this paper under the terminal measure $\mathbb{F}$, which is the equivalent martingale measure associated with the numéraire $D_{\cdot T_{n+1}}$.

For later reference it is convenient to define the numéraire-rebased discount bonds

$$\hat{D}_{tT} := \frac{D_{tT}}{D_{tT_{n+1}}}. \tag{3}$$

Note immediately from (2) and (3) that

$$\hat{D}_{tT_i} = (1 + \alpha_i L_t^i)\hat{D}_{tT_{i+1}}. \tag{4}$$

The LIBOR market model

Under the one-factor LIBOR market model (Brace et al. [1997], Milterson et al. [1997]), each of the forward LIBORs L^i solve an SDE of the form

$$dL_t^i = \mu_t^i L_t^i \, dt + \sigma_t^i L_t^i \, dW_t, \tag{5}$$

for some instantaneous volatility functions σ_t^i, where W is a standard Brownian motion.[3]

If the model is to be arbitrage-free under $\mathbb{F}$, the drift term μ_t^i must have the following form, for $1 \leq i < n$,

$$\mu_t^i(L^{i+1}, \dots, L^n) = -\sum_{j=i+1}^{n} \frac{\alpha_j L_t^j}{1 + \alpha_j L_t^j} \sigma_t^j \sigma_t^i. \tag{6}$$

The drift μ^n of the terminal forward rate is zero since L^n is a martingale under $\mathbb{F}$.

For future reference, it is convenient to observe that the formal solution to (5) is given by

$$L_t^i = L_0^i \exp\left[\int_0^t \left(\mu_s^i - \frac{1}{2}\left(\sigma_s^i\right)^2\right) ds + \int_0^t \sigma_s^i \, dW_s\right]. \tag{7}$$

Drift approximations and separability

In this subsection we review the two essential steps introduced in Pelsser et al. [2004] that result in a tractable one-dimensional approximation to the true LIBOR market model (5).

The first step requires some approximation for the drift of all unexpired forward rates at dates $T_1, \ldots, T_n$. There are a variety of increasingly sophisticated approximations available, such as predictor-corrector schemes (Hunter et al. [2001]) or Brownian-bridge approximations (Pelsser et al. [2004]).

The second step is the introduction of a restriction on the form of the instantaneous volatility functions, known as separability, which allows the drift-approximated forwards to be represented as functions of a low-dimensional Markov process.[4] This is the key to obtaining a model that approximates the behaviour of the original LIBOR market model and at the same time admits an efficient grid-based implementation.

For the first step, we use a method based on a Brownian-bridge (Pelsser et al. [2004]) to approximate the drift of each of the unexpired forward rates from time zero to a given time T_k ($k = 1, \ldots, n$).

As the drift of the nth forward rate is zero, $L^n_{T_k}$ is immediate given the value of $\int_0^{T_k} \sigma^n_s dW_s$ (see Equation (7)). Working back recursively from the nth forward rate down to the first, suppose that for a given $i < n$ we already have approximations for $L^{i+1}_{T_k}, \ldots, L^n_{T_k}$ and we wish to estimate $L^i_{T_k}$. Rewriting Equation (7) using (6),

$$L^i_{T_k} = L^i_0 \exp\left[H^i_{T_k} - \int_0^{T_k} \frac{1}{2} \left(\sigma^i_s\right)^2 ds + X^i_{T_k} \right], \tag{8}$$

where $X^i_{T_k}$ is given by

$$X^i_{T_k} := \int_0^{T_k} \sigma^i_s \, dW_s$$

and the corresponding integrated drift term is

$$H^i_{T_k} := -\int_0^{T_k} \sum_{j=i+1}^{n} \frac{\alpha_j L^j_s}{1+\alpha_j L^j_s} \sigma^j_s \sigma^i_s \, ds \,. \tag{9}$$

It is clear that $L^i_{T_k}$ may be estimated given the value of $X^i_{T_k}$ if we have an approximation for the integrated drift $H^i_{T_k}$.

If each $L^j_{T_k}$ $(j > i)$ has already been estimated, then the value of L^j_s for any $s \in (0, T_k)$ may be approximated by the mean at time s of the generalised geometric Brownian bridge that joins L^j_0 and $L^j_{T_k}$, given by

$$m^j_s = L^j_0 \left(\frac{L^j_{T_k}}{L^j_0} \right)^{(\xi^j_s / \xi^j_{T_k})} \exp\left\{ \left(\frac{\xi^j_s}{2\xi^j_{T_k}} \right) \left(\xi^j_{T_k} - \xi^j_s \right) \right\} \qquad 0 \le s \le T_k \,,$$

where

$$\xi^j_s := \int_0^s (\sigma^j_u)^2 \, du$$

(see Appendix A of Pelsser et al. [2004] for details).

The Brownian-bridge approximation for $H^i_{T_k}$ is computed by substituting m^j_s for all terms L^j_s appearing in the integrand of (9) and evaluating the integral numerically using a suitable quadrature method. We can now compute the values of all the unexpired LIBORs $L^i_{T_k}$ given the values of $X^i_{T_k}$.

The second step of this pricing approach is the essential ingredient required for efficient implementation. This is a condition on the specification of instantaneous volatilities, known as *separability*. Separability has appeared in the literature several times in the context of requiring certain processes to be Markovian, see for example Carverhill [1994] and references contained in Pelsser et al. [2004]. It is this condition that allows us to make the connection between market models and Markov-functional models.

Definition (Separability)**.** A collection of instantaneous volatility functions σ^i is *separable* if there exists an instantaneous volatility function σ such that

$$\sigma_t^i = \gamma^i \sigma_t$$

for some constants γ^i, for $0 \leq t \leq T_i,\ i = 1, \ldots, n$.[5]

If the volatility structure is separable then the stochastic integral $X_{T_k}^i$ appearing in Equation (8) may be written

$$X_{T_k}^i = \int_0^{T_k} \sigma_s^i dW_s = \int_0^{T_k} (\gamma^i \sigma_s) dW_s = \gamma^i x_{T_k},$$

where we define the driver

$$x_t := \int_0^t \sigma_s \, dW_s \, . \tag{10}$$

Furthermore, notice that for each T_k we could fix the value of x_{T_k} and compute the approximated drift terms $H_{T_k}^i$ appearing in (8) inductively using the Brownian bridge drift approximation. By induction, each $H_{T_k}^i$ will be a function of the previously determined values of $L_{T_k}^{i+1}, \ldots, L_{T_k}^n$. In turn, the forward LIBOR $L_{T_k}^i$ will be a function of $H_{T_k}^i$ and x_{T_k} (since $X_{T_k}^i = \gamma^i x_{T_k}$). Therefore, implicitly each forward LIBOR will be a function of x_{T_k} only.

The combination of the use of a drift approximation and the specification of a separable volatility structure results in a model under which all (drift-approximated) forward LIBORs are known functions of the one-dimensional driving Markov process x. This permits the application of efficient computational methods such as numerical integration or finite differences in the calculation of derivative prices. Note that any other drift approximation could be substituted for the Brownian bridge approximation in this approach.

We shall subsequently refer to the market model with such a separable volatility structure above as the market model with 'driving process' x. Applying a drift approximation means that we only need compute values of LIBORs at each exercise date and we may view these as a function of the one-dimensional Markov process x. Note that given a parameterisation of x, the specification of this LIBOR market model is complete once the constants γ^i have been determined by, for example, matching vanilla caplet prices. This final calibration step is discussed in the exposition of our numerical study below.

Theoretically, the use of any drift approximation will of course introduce arbitrage. Pelsser et al. [2004] show that in pricing short maturity Bermudan swaptions (8Y), these effects are relatively small and the drift approximation model yields reasonably similar prices to those computed using the least-squares simulation-based methodology introduced by Longstaff & Schwartz [2001]. However, we shall see that the presence of arbitrage in the drift approximation model becomes noticeable for long maturities and in unusual market conditions.

The swap-based market model

In this section the drift approximation approach introduced above in the context of LIBOR-based market models is applied to the analogous swap-based market models. These results extend those described in Pelsser et al. [2004].

We assume a set of co-terminal forward par swap rates, denoted by y^i for $i = 1, \dots, n$. The ith forward par swap rate y^i sets on date T_i with coupon payments on dates $T_{i+1}, \dots, T_{n+1}$ and satisfies the relationship

$$y_t^i = \frac{D_{tT_i} - D_{tT_{n+1}}}{P_t^i}\,, \tag{11}$$

where

$$P_t^i := \sum_{j=i}^{n} \alpha_j D_{tT_{j+1}}.$$

Here the term P_t^i is commonly referred to as the present value of a basis point (PVBP), or annuity factor.

Following Jamshidian [1997], the one-factor swap market model is specified by assuming these forward par swap rates satisfy the usual lognormal dynamics

$$dy_t^i = \bar{\mu}_t^i y_t^i \, dt + \bar{\sigma}_t^i y_t^i \, dW_t, \tag{12}$$

where W is a standard Brownian motion, $\bar{\sigma}_t^i$ denotes the deterministic instantaneous volatility of the ith swap rate and $\bar{\mu}_t^i$ is the drift. For this model, it may be shown that under the terminal measure $\mathbb{F}$ the drift restriction imposed by no-arbitrage is given by

$$\bar{\mu}_t^i = - \sum_{j=i+1}^{n} \left(\prod_{k=i}^{j-1} (1 + \alpha_k y_t^{k+1}) \right) \frac{\hat{P}_t^j}{\hat{P}_t^i} \left(\frac{\alpha_{j-1} y_t^j}{1 + \alpha_{j-1} y_t^j} \right) \bar{\sigma}_t^i \bar{\sigma}_t^j, \quad 1 \leq i < n,$$

where

$$\hat{P}_t^i := \frac{P_t^i}{D_{tT_{n+1}}}$$

denotes the numéraire-rebased PVBP (see, for example, Hunt & Kennedy [2000]). Formally, the solution to the SDE (12) at the time $t = T_k$ may be written

$$y_{T_k}^i = y_0^i \exp \left[\bar{H}_{T_k}^i - \int_0^{T_k} \frac{1}{2} \left(\bar{\sigma}_s^i \right)^2 ds + \bar{X}_{T_k}^i \right], \tag{13}$$

where $\bar{X}_{T_k}^i$ is given by

$$\bar{X}_{T_k}^i := \int_0^{T_k} \bar{\sigma}_s^i \, dW_s$$

and the corresponding integrated drift term is

$$\bar{H}_{T_k}^i := \int_0^{T_k} - \sum_{j=i+1}^{n} \left(\prod_{k=i}^{j-1} (1 + \alpha_k y_s^{k+1}) \right) \frac{\hat{P}_s^j}{\hat{P}_s^i} \left(\frac{\alpha_{j-1} y_s^j}{1 + \alpha_{j-1} y_s^j} \right) \bar{\sigma}_s^i \bar{\sigma}_s^j \, ds \,. \tag{14}$$

In an analogous procedure to that in the LIBOR case, a swap drift approximation model may be constructed as follows. The nth forward par swap rate y^n has zero drift under $\mathbb{F}$ so its value at time T_k is immediate given the value of $\bar{\sigma}^n_{T_k}$. Working back recursively down to the first swap rate, suppose that we wish to estimate the value of $y^i_{T_k}$ given the value of $\bar{X}^i_{T_k}$, assuming the values of $y^j_{T_k}$ have already been approximated for all $j > i$.

The value of $\bar{H}^i_{T_k}$ may be estimated by approximating the values of y^j_s and $\hat{P}^j_s$ for $j = i+1, \ldots, n$ at intermediate times $s \in (0, T_k)$ using a similar Brownian-bridge approach to that described in the LIBOR case. Specifically, the value of y^j_s for $s \in (0, T_k)$ may be approximated by the mean at time s of the generalised geometric Brownian bridge that joins y^j_0 and $y^j_{T_k}$, given by

$$\bar{m}^j_s = y^j_0 \left(\frac{y^j_{T_k}}{y^j_0} \right)^{(\bar{\xi}^j_s / \bar{\xi}^j_{T_k})} \exp \left\{ \left(\frac{\bar{\xi}^j_s}{2\bar{\xi}^j_{T_k}} \right) (\bar{\xi}^j_{T_k} - \bar{\xi}^j_s) \right\}, \qquad 0 \leq s \leq T_k \,,$$

where

$$\bar{\xi}^j_s := \int_0^s (\bar{\sigma}^j_u)^2 \, du \,.$$

Once the y^j_s have been approximated by the $\bar{m}^j_s$, the values of $\hat{P}^j_s$ for $j = i+1, \ldots, n$ may be approximated by terms p^j_s by observing the recurrence relationship satisfied by $\hat{P}^j_s$:

$$\hat{P}^j_s = \alpha_j + (1 + \alpha_j y^{j+1}_s) \hat{P}^{j+1}_s \,.$$

That is, we set $p^n_s = \hat{P}^n_s = \alpha_n$ and recover the remaining p^j_s inductively using

$$p^j_s = \alpha_j + (1 + \alpha_j \bar{m}^{j+1}_s) p^{j+1}_s \,, \qquad j = i+1, \ldots, n-1.$$

The value of $\bar{H}^i_{T_k}$ may be estimated by substituting $\bar{m}^j_s$ for y^j_s and p^j_s for $\hat{P}^j_s$ in the integrand of (14) and evaluating the integral numerically. The estimate of $y^i_{T_k}$ is immediate from (13), given the value of $\bar{X}^i_{T_k}$.

If the instantaneous volatility structure is separable, one may write

$$\bar{\sigma}_t^i = \gamma^i \bar{\sigma}_t$$

for some common instantaneous volatility function $\bar{\sigma}_t$ and constants γ^i. The resulting swap drift approximation model is then driven by the one-dimensional Markov process

$$\bar{x}_t := \int_0^t \bar{\sigma}_s \, dW_s$$

and approximates the dynamics of the original swap market model.

3 Markov-functional models

Basic assumptions of Markov-functional models

We now turn our attention to the specification of Markov-functional models that are analogous to the market models of the previous section. The defining characteristic of Markov-functional models is that pure discount bond prices are at any time a function of some low-dimensional process x which is Markovian in some martingale measure. Implementation of these models is efficient as it is only necessary to track the driving Markov process (c.f. market models which suffer from high dimensionality). The functional forms are chosen so that calibration to relevant market prices and market skew is achieved, a property not possessed by short rate models, and so that the model is arbitrage free. Note that in the Markov-functional approach we are not restricted to fitting Black's formula for caplets (or swaptions) but our discussion will focus on this case here as we interested in studying the relationship of this approach to market models. A general discussion of Markov-functional models can be found in Hunt & Kennedy [2000].

To set up the Markov-functional model to match the LIBOR market model introduced earlier we assume the same tenor structure $T_1, \ldots, T_{n+1}$ and work with the terminal discount bond $D_{\cdot T_{n+1}}$ as numéraire. The driving Markov process x is of the form given in Equation (10). The model will actually only be defined on a grid. That is, we specify the functional forms $D_{T_iT_j}(x_{T_i})$ for $1 \leq i < j \leq n+1$, since this is (typically) all that is needed in practice. Further, note that since the model is arbitrage-free, we need only define the functional forms associated with the numéraire bond $D_{T_iT_{n+1}}$, $i = 1, \ldots, n$. This follows because the remaining functional forms can be recovered using the martingale property for numéraire-rebased assets under $\mathbb{F}$: for $0 \leq t \leq T \leq T_{n+1}$,

$$\begin{aligned} D_{tT}(x_t) &= D_{tT_{n+1}}(x_t)\mathbb{E}_{\mathbb{F}}\left[\left(D_{T\,T_{n+1}}(x_T)\right)^{-1}|\mathcal{F}_t\right] \\ &= D_{tT_{n+1}}(x_t)\int_{-\infty}^{\infty}\left(D_{T\,T_{n+1}}(u)\right)^{-1}\phi_{x_T|x_t}(u)\,du\,, \end{aligned} \tag{15}$$

where $\phi_{x_T|x_t}$ denotes the density of x_T given x_t and $\{\mathcal{F}_t\}$ is the augmented Brownian filtration associated with the driving process x. Note from (10) that $\phi_{x_T|x_t}$ is a Normal density function with mean x_t and variance $\int_t^T(\sigma_u)^2\,du$.

In the next section, we show how to determine the functional form of the numéraire discount bond by fitting it to the prices of caplets as given by Black's formula. This leads to a LIBOR Markov-functional model which, as we shall see, is closely related to the LIBOR market model of the last section. Later in our analysis of swap-based models we shall calibrate a Markov-functional model to Black's swaption prices instead to obtain a swap model.

The LIBOR Markov-functional model

As in the LIBOR market model, we assume a set of contiguous forward LIBORS denoted by L^i for $i = 1, \ldots, n$ with tenor structure $T_1, \ldots, T_{n+1}$. The market prices for the caplets on these LIBOR rates are assumed to be given by Black's formula with volatility $\tilde{\sigma}^i$. We make one further assumption in setting up the model; that is that the ith forward LIBOR rate at time T_i, $L^i_{T_i}$, is a monotonic increasing function of the variable x_{T_i}.

Initially the functional form of $D_{T_n T_{n+1}}$ is determined by observing that

$$(D_{T_n T_{n+1}})^{-1} = 1 + \alpha_n L^n_{T_n} \, .$$

Now, the assumption that the final caplet price is given by Black's formula with implied volatility $\tilde{\sigma}^n$ means that $\log(L^n_{T_n})$ has a Normal distribution under $\mathbb{F}$ with mean $(\log(L^n_0) - \frac{1}{2}(\tilde{\sigma}^n)^2 T_n)$ and variance $(\tilde{\sigma}^n)^2 T_n$. Using (10) we can express $L^n_{T_n}$ explicitly in terms of the Markov process x at time T_n :

$$L^n_{T_n}(x_{T_n}) = L^n_0 \exp\left(-\frac{1}{2}(\tilde{\sigma}^n)^2 T_n + \sqrt{\frac{(\tilde{\sigma}^n)^2 T_n}{\int_0^{T_n} (\sigma_u)^2 \, du}} \, x_{T_n} \right) , \qquad (16)$$

and thus

$$(D_{T_n T_{n+1}})^{-1}(x_{T_n}) = 1 + \alpha_n L^n_0 \exp\left(-\frac{1}{2}(\tilde{\sigma}^n)^2 T_n + \sqrt{\frac{(\tilde{\sigma}^n)^2 T_n}{\int_0^{T_n} (\sigma_u)^2 \, du}} \, x_{T_n} \right) .$$

Note that $L^n_{T_n}$ is a monotonic increasing function of x_{T_n}.

We now show how market prices of the calibrating vanilla caplets can be used to imply, numerically at least, the functional forms $D_{T_i T_n}$ for $i < n$. Since we are assuming these caplet prices are given by Black's formula, it is equivalent to calibrate to the inferred market prices of digital caplets. If the market price of the ith vanilla caplet is given by Black's formula with volatility $\tilde{\sigma}^i$, the price at time zero for the corresponding digital caplet is

$$\widetilde{V}_0^i(K) = D_{0T_{i+1}}(x_0)\Phi\left(\frac{\log(L_0^i/K)}{\tilde{\sigma}^i\sqrt{T_i}} - \frac{1}{2}\tilde{\sigma}^i\sqrt{T_i}\right),$$

where Φ denotes the standard cumulative Normal distribution function. Also working with the terminal measure $\mathbb{F}$ and applying the usual valuation formula this digital caplet value can be expressed as

$$\widetilde{V}_0^i(K) = D_{0T_{n+1}}(x_0)\mathbb{E}_{\mathbb{F}}\left[\hat{D}_{T_iT_{i+1}}(x_{T_i})\mathbf{1}_{\{L^i_{T_i}(x_{T_i})>K\}}\right], \tag{17}$$

where $\hat{D}_{tT}(x_t)$ denotes the numéraire-rebased discount bond defined in Equation (3).

To determine the functional forms for the numéraire $D_{T_iT_{n+1}}$ for $i < n$ we work back iteratively from the terminal time T_n. Consider the ith step in this procedure and suppose that $D_{T_jT_{n+1}}$ has already been determined for $j = i+1, \ldots, n$. As at time T_n, first the functional form of the LIBOR rate $L^i_{T_i}$ is determined, from which the functional form of $D_{T_iT_{n+1}}$ may be recovered.

Suppose we choose some $x^* \in \mathbb{R}$. Define

$$\begin{aligned} J_0^i(x^*) &= D_{0T_{n+1}}(x_0)\mathbb{E}_{\mathbb{F}}\left[\hat{D}_{T_iT_{i+1}}(x_{T_i})\mathbf{1}_{\{x_{T_i}>x^*\}}\right] && (18)\\ &= D_{0T_{n+1}}(x_0)\mathbb{E}_{\mathbb{F}}\left[\mathbb{E}_{\mathbb{F}}\left[\hat{D}_{T_{i+1}T_{i+1}}(x_{T_{i+1}})\,|\mathcal{F}_{T_i}\right]\mathbf{1}_{\{x_{T_i}>x^*\}}\right] \\ &= D_{0T_{n+1}}(x_0)\int_{x^*}^{\infty}\left[\int_{-\infty}^{\infty}\frac{1}{D_{T_{i+1}T_{n+1}}(u)}\phi_{x_{T_{i+1}}|x_{T_i}}(u)\,du\right]\phi_{x_{T_i}}(v)\,dv && (19) \end{aligned}$$

where $\phi_{x_{T_i}}$ denotes the transition density function of x_{T_i} and $\phi_{x_{T_{i+1}}|x_{T_i}}$ the density of $x_{T_{i+1}}$ given x_{T_i}. Note that the integrand in (19) only depends on $D_{T_{i+1}T_{n+1}}(x_{T_{i+1}})$ which has already been determined in the previous iteration at T_{i+1}. Thus at time T_i, $J_0^i(x^*)$ may be evaluated numerically for different values of x^*. Furthermore, using market prices it is possible to find the value of K such that

$$J_0^i(x^*) = \widetilde{V}_0^i(K)\,. \tag{20}$$

Comparing (17) and (18) it is clear that the value of K satisfying (20) is precisely $L^i_{T_i}(x^*)$, since $L^i_{T_i}(x_{T_i})$ is monotonically increasing in x_{T_i} by assumption. If market prices are taken to be given by Black's formula, this means that

$$L^i_{T_i}(x^*) = L^i_0 \exp\left[-\frac{1}{2}(\tilde{\sigma}^i)^2 T_i - \tilde{\sigma}^i\sqrt{T_i}\ \Phi^{-1}\left(\frac{J_0^i(x^*)}{D_{0T_{i+1}}(x_0)}\right)\right].$$

Finally, to obtain the value of $D_{T_iT_{n+1}}(x^*)$ we observe using (4) that

$$D_{T_iT_{n+1}}(x^*) = \Big(\big(1+\alpha_i L^i_{T_i}(x^*)\big)\,\hat{D}_{T_iT_{i+1}}(x^*)\Big)^{-1},$$

noting that the numéraire-rebased discount factor on the right-hand side may be evaluated using the martingale property (15).

A uniqueness result

The following result is crucial in making sense of the numerical results to follow.

Theorem (Uniqueness)**.** *Consider a LIBOR Markov-functional model based on the tenor structure $T_1, \ldots, T_{n+1}$ which satisfies the following conditions:*

(i) The driving Markov process x is a deterministic time change of a Brownian motion and satisfies (10) *under the equivalent martingale measure $\mathbb{F}$ corresponding to the numéraire $D_{\cdot T_{n+1}}$;*

(ii) The pure discount bond prices are of the form

$$D_{tT} = D_{tT}(x_t), \qquad 0 \le t \le T \le T_{n+1}\,,$$

and satisfy the martingale property (15)*;*

(iii) The i*th forward LIBOR at time* T_i*,* $L^i_{T_i}$*, is a monotonic increasing function of the variable* x_{T_i}*;*

(iv) The model is calibrated to vanilla caplet prices corresponding to the rates $L^1, L^2, \ldots, L^n$ *setting at dates* $T_1, T_2, \ldots, T_n$*.*

If such a model exists then it is unique as far as its determination on grid points is concerned. That is, the functional forms $D_{T_iT_j}(x_{T_i}) : 1 \leq i < j \leq n+1$ *are uniquely determined.*

Proof. This follows immediately from the construction of the Markov-functional model discussed in the last section. □

The above result, though a trivial observation mathematically, has significant implications in practice. *Any approximation to a one-factor separable LIBOR market model that is designed to be approximately arbitrage-free but reduces to a function of the one-dimensional process* x *is, in effect, also an approximation (on grid points) to the unique arbitrage-free Markov-functional model that calibrates to Black's formula for pricing the corresponding vanilla caplets.* We take up this discussion again in the following section.

4 Numerical comparison of LIBOR Markov-functional and LIBOR market models

It is natural to study the structure of the drift-approximation and Markov-functional models by regarding the values of LIBORs L^i at a given time as functions of the driving process x, or equivalently as functions of the terminal LIBOR L^n. In our analysis we explore these functional relationships under a number of realistic implied volatility and initial LIBOR curve scenarios for a particular parameterisation of x described below. The uniqueness theorem indicates that, provided the use of drift approximations does not introduce arbitrage that is practically significant, the drift approximation model must be similar to the arbitrage-free Markov-functional model. However, it is not clear from this result how these models compare numerically. For our choice of x the functional forms under each model are found to be very close for realistic values of initial LIBORs and implied volatilities.

The corresponding separable LIBOR market model (with the same driving process) is also investigated by approximating the SDE (5) using a log-Euler discretisation. A scatter plot of the ith vs the nth LIBOR at time t gives us some indication of the relationship between these random variables under the true market model. These results are suggestive only since it is not possible to compute an exact functional relationship under the LIBOR market model. However the scatter plot may be overlaid on the graph of the functional forms implied by the Markov-functional or drift approximation models, thus enabling comparison between models.

For reasonable parameter values, our results give a strong indication that both the Markov-functional model and drift approximation model are very close to the separable LIBOR market model. The relationship between the logarithms of the ith and nth LIBORs is found to be approximately linear (thus the ith LIBOR is approximately lognormal under the terminal measure $\mathbb{F}$). This linear relationship is a general feature of all three models under consideration, certainly at 10 years and, to a lesser degree, even as far as 30 years. Slopes and intercepts for different models are virtually indistinguishable for tenor structures associated with shorter maturities such as 10Y with a small bias for longer maturities above 30Y. The trends observed in our results may be explained using a heuristic argument based on an approximate log-linear model, presented as part of our scenario analysis below.

During our investigations we have also explored the implied distributions of certain rates not explicitly fixed by the calibration procedure in both LIBOR and swap-based models. Our scenario analysis concludes with a description of the implied functional forms of co-terminal swap rates under the LIBOR Markov-functional model. Subsequently, as part of our numerical study of swap-based models, the implied functional forms of LIBORs under the swap-based Markov-functional model are discussed.

Toward the end of this section, we compare Bermudan swaption prices under the LIBOR Markov-functional and separable LIBOR market models. Prices under the latter are computed using both the drift approximation model and the least-squares method of Longstaff & Schwartz [2001]. As it is important to determine for what range of parameter values and maturities the LIBOR Markov-functional model is numerically close to the corresponding separable LIBOR market model and its associated drift approximation model, we close this section with a discussion of the comparative behaviour of these models under stress-testing.

Choice of correlation structure for numerical results

In comparing the Markov-functional and market models we assume the same correlation structure for both, that is, the driving Markov process x (see Equation (10)) of the LIBOR Markov-functional model is the same as that of the separable LIBOR market model. Although we present numerical results only for a particular parameterisation of this driving process, our uniqueness result leads us to believe that our findings hold for any parameterisation. We have found this to be true for an alternative parameterisation of the driving process, *mean reversion,* where $\sigma_s = \exp(-as)$ for some $a > 0$. This parameterisation is used by Pelsser et al. [2004] in their study of the drift approximation model.

Our choice of the process x is motivated by the Hull-White model, a model which has been popular in the market for many years because of its tractability. Under a LIBOR model, the variances $\xi_{T_i} := \mathrm{var}(x_{T_i}) = \int_0^{T_i} \sigma_u^2 du$ of x at times $T_i,\ i = 1, \ldots, n$, are taken to be

$$\xi_{T_i} = \left(\frac{\alpha_i L_0^i}{(1 + \alpha_i L_0^i)(\psi_{T_i} - \psi_{T_{i+1}})} \right)^2 (\tilde{\sigma}^i)^2 T_i \,, \tag{21}$$

where

$$\psi_t := \frac{1}{a}(1 - e^{-at}) \,.$$

This approximation is arrived at by considering a Hull-White model calibrated to at-the-money caplet prices. The mean reversion parameter a appearing in this approximation is a user input that, as in the usual Hull-White model, must be hedged. The details of the derivation of this approximation may be found in Exhibit 15. Note that the variance of the process x at the times $T_i,\ i = 1, \ldots n$, is all that is necessary for a practical implementation of the Markov-functional model as this fixes the conditional distributions of the x_{T_i}'s.

To complete the specification of the corresponding separable LIBOR market model it is necessary to extend this definition for general t. We choose a simple interpolation that is smooth in t (see Exhibit 15, Equation (26)). We find that our results are insensitive to this particular choice; for reasonable parameter values performing simple linear interpolation has a negligible effect on the numerical distributions of LIBORs at each exercise date. This observation is anticipated, at least for values of parameters and tenor structure where the drift approximation model is an accurate approximation, since it is only the values of integrals of the instantaneous volatility over intervals $[0, T_k]$

that appear in the integrated drift term of each LIBOR (refer to Equations (8) and (9)).

Calibration of both the separable LIBOR market model and the corresponding LIBOR Markov-functional model to caplet implied volatilities is straightforward given the driving process x. The calibration of the LIBOR market model with separable volatility structure is completed by determining the constants γ^i from caplet prices as follows. If $\xi_t = \int_0^t \sigma_u^2 du$ is known for times $t = T_1, \ldots, T_n$, then for $i = 1, \ldots, n$,

$$(\gamma^i)^2 \xi_{T_i} = (\tilde{\sigma}^i)^2 T_i \,, \tag{22}$$

where $\tilde{\sigma}^i$ is the implied volatility of the caplet on the ith forward rate. Hence γ^i is immediate. Since we are assuming caplet prices are given by Black's formula, calibration of the LIBOR Markov-functional model to the implied volatility of the terminal forward rate L^n is also immediate (see Equation (16)). The remaining caplet volatilities are fitted indirectly (for all strikes) when determining the functional forms of asset prices numerically at each step of the algorithm. Note that a separable LIBOR market model may be calibrated in various ways. However this is done, we may construct an analogous Markov-functional model by calibrating to caplet volatilities calculated via Equation (22).

Scenario analysis

In this subsection, we present our numerical comparison of the LIBOR models under a number of typical market data scenarios. Recall the values of the LIBORs under the LIBOR Markov-functional model are only determined at the exercise dates $T_1, \ldots, T_n$, since that is all that is typically required in practice. Therefore it is only necessary to compute the functional relationship

$$L_t^i = g(L_t^n)$$

at times $t = T_k$ (for all LIBORs $i \geq k$ which have not yet expired). These functional relationships can be directly compared with those computed under the drift approximation model. In presenting our results we plot the functional form of $\log(L_{T_k}^i)$ against $\log(L_{T_k}^n)$ under these models. This is equivalent to examining the functional relationship with the driving process x_{T_k} since $\log(L_{T_k}^n)$ is just a linear transformation of x_{T_k}.[6] A scatter plot of these variables simulated under the separable LIBOR market model may be overlaid for comparison.

The market scenarios considered in our numerical study are detailed in Exhibit 1.[7] The tenor structure is taken to be annual with $n = 29$, thus $T_1 = 1,\ T_2 = 2, \ldots, T_n = 29$, with final maturity $T_{n+1} = 30$. In our specification of the common driving process the mean reversion parameter a is taken to be 5%. We have also examined results for other values of $a \in (0, 20\%)$ and find they are consistent with our conclusions.

We first present our results under Scenario A, where initial LIBORs and caplet implied volatilities are flat, since these are typical of the results across all scenarios. The lines shown on Exhibit 2 display the functional relationship between a selection of LIBORs L^i and the terminal LIBOR L^n under the LIBOR Markov-functional model at T_{15}. The drift approximation model could not be distinguished from the Markov-functional model on this plot and so is not shown. It is striking that the scatter plot overlaid of the corresponding market model simulation exhibits very little dispersion. We observe the plots are very close to a straight line (on a log-log scale) under both the Markov-functional and market models, for all exercise dates and scenarios. As an approximate measure of the linearity of these plots we consider the value of the statistic R^2 computed using a large number of points; for this exercise date all plots have an R^2 of at least 0.999 (indicating they are extremely close to straight lines).[8]

In general under Scenario A there is a close match between slopes and intercepts corresponding to the Markov-functional model and those of the least squares linear regression computed from the separable LIBOR market model simulation (results for T_{15} are shown in Exhibit 3). For a given exercise date T_k, the slopes corresponding to the Markov-functional model (MF) tend to be slightly higher than for the market model (BGM); the greatest difference generally occurs for LIBORs L^i where i lies midway between k and n (at T_{15} this occurs for L^{23}). Note that under all models the relationship between $L^k_{T_k}$ and $L^n_{T_k}$ under $\mathbb{F}$ is constrained to some extent by fitting to the kth Black's caplet price. In addition, the terminal LIBOR L^n is exactly lognormal under $\mathbb{F}$. Therefore, if the market model exhibits little dispersion it is only for LIBORs L^i with i between k and n that we would expect any significant differences between models.

The drift approximation (DA) model is very close to both the Markov-functional and market models (in terms of slopes and intercepts). In general we observe that the Markov-functional model appears to be slightly closer to the market model for LIBORs i close to k (at T_{15} this holds for L^{15}, L^{16} and L^{17}) and the drift approximation is closer for the remainder.

Any small differences between slopes and intercepts increases with the maturity of the tenor structure under consideration. These slopes and intercepts match to at least 3 s.f. for a maturity of 10Y, whereas we begin to observe small numerical differences for longer maturities (matching only to 2 s.f. at 20Y). These small differences may lead to minor differences in derivative prices calculated under each model; these are discussed with reference to the example of the standard Bermudan swaption.

This analysis of the relationship between LIBORs for various times T_k has been repeated under all scenarios given in Exhibit 1. The qualitative observations detailed above are found to hold under all scenarios. The same conclusions are also reached under a scenario corresponding to typical USD market data.[9]

The linearity of the market model's scatter plot is perhaps surprising, as one might reasonably expect the model to produce more dispersion because the drift term is stochastic for LIBORs $i < n$. These plots indicate that the stochastic component of the drift remains small, hence although the market model is theoretically Markovian only in n dimensions, it generally resembles a one-dimensional model for all practical purposes. We take up this discussion again when we perform stress-testing on these models. Here we observe that for high volatilities and long maturities this is no longer the case and the market model plot exhibits much greater scatter.

As a means of understanding the trends in slopes of the three models it is convenient to contrast their behaviour with the following *log-linear* model. Since we have observed that the relationship between $\log(L_t^i)$ and $\log(L_t^n)$ is close to linear, it follows that L_t^i is approximately lognormal under $\mathbb{F}$. Therefore, suppose

$$\log(L_t^i) \approx \eta_t^i + c^i x_t = \eta_t^i + c^i \int_0^t \sigma_u dW_u$$

under $\mathbb{F}$ for some constant c^i and a deterministic function of time η_t^i. Note that this model will admit arbitrage since otherwise we would require η_t^i to be stochastic. Now $\mathrm{Var}(\log(L_t^i)) \approx (c^i)^2\xi_t$, hence $(c^i)^2\xi_{T_i} \approx (\tilde{\sigma}^i)^2 T_i$ by matching terminal variances (since we are calibrating our model to caplet prices). Comparing with the separable volatility structure of the analogous LIBOR market model, $c^i \approx \gamma^i$. Thus,

$$\log(L_t^i) \approx \left(\frac{\gamma^i}{\gamma^n}\right) \log(L_t^n) + \hat{\eta_t}^i$$

for some deterministic $\hat{\eta_t}^i$. This is a coarse approximation to the LIBOR market model and the corresponding LIBOR Markov-functional model but the slopes of this log-linear model are certainly comparable with the actual slopes observed under these models (matching to at least 1 s.f.; see Exhibit 3). The approximation provides a good guide to trends expected in slopes of the log-log plots. For example, when ξ_{T_i} is specified according to our Hull-White approximation (21), then γ^i is decreasing with i for flat caplet volatilities (see Equation (22)). Therefore, it is not surprising that we see decreasing slopes on the associated log-log plots (see Exhibit 2).

We now consider the functional forms of the co-terminal forward par swap rates $y_{T_i}^i$ (corresponding to swaps with fixed maturity T_{n+1}) implied by the one-factor LIBOR Markov-functional model. Subsequently, we perform a similar examination of the functional forms of LIBORs $L_{T_i}^i$ under the swap-based Markov-functional model.

Functional relationships between $\log(y_{T_i}^i)$ and $\log(y_{T_i}^n)$ under Scenario A are displayed in Exhibit 4 for a selection of forward rates i.[10] These functional forms are typical in that the numerical relationship appears to be close to linear, with slight positive convexity. This convexity is anticipated since par swap rates are a linear combination of lognormal forward rates, hence cannot also be lognormal.

Example application: Pricing a Bermudan swaption

It is clear from the numerical results above that for typical market data the LIBOR Markov-functional model is very close to the separable LIBOR market model with the same driving process, especially for short maturity tenor structures. Therefore we would also expect prices of exotic derivatives under the two models to be similar because these prices are effectively summary statistics. We demonstrate this with the example of a standard Bermudan swaption.

In common with most exotic derivatives with early exercise features, it is very difficult to price a standard Bermudan swaption directly using a simulation of the market model. It is necessary to introduce further approximations to determine the optimal exercise boundary. In theory, simulation-based methods such as the least-squares approach suggested by Longstaff & Schwartz [2001] can be used to compute the exercise boundary to any required accuracy but considerations of computation time must be taken into account. In contrast with the market model, the arbitrage-free Markov-functional model permits an efficient implementation as it stands, without the need for approximation.

Suppose we wished to price Bermudan swaptions in a model in which LIBORs are lognormal (this may be to mirror the behaviour of the LIBOR market model or to avoid negative LIBOR rates, for instance). In practice, we would need to choose the driving process x and market model parameters γ^i carefully to reflect the appropriate joint distributions of rates (for example, we may wish to calibrate to a particular set of swaptions). The analogous Markov-functional model could then be constructed. Here we will use those parameters chosen previously for consistency in comparing the models. The correlation structure of all models is as described above with mean reversion parameter $a = 5\%$.

In Exhibits 5-7, we display a summary of prices of annual Bermudan swaptions (7% payers swaptions) with various maturities under the one-factor LIBOR Markov-functional (MF) model and the corresponding drift approximation (DA) model. Also included are Longstaff-Schwartz (LS) prices computed by direct simulation of the separable LIBOR market (SLM) model. A single explanatory variable (the current swap net present value) was used in the LS algorithm to determine the exercise boundary of the Bermudan swaption (via a simple linear regression across all in-the-money sample paths at each exercise date). Including further explanatory variables, which should theoretically improve the approximation to the exercise boundary, was not found to increase prices significantly. This observation may also be found in Pelsser et al. [2004] and Pelsser & Pietersz [2005]. The prices shown correspond to flat initial LIBORs and flat implied volatilities (Scenario A), however the results are found to be very similar over all scenarios.

Although the MF and SLM models are specified in very different ways, the prices of Bermudan swaptions are extremely close under both models at 10Y. The differences between Bermudan prices computed under the MF model and those computed using the LS approximation to the SLM price are all much less than the standard errors in the LS prices. The MF vegas, which are a good proxy for the margins currently charged on such trades, are much greater than these LS standard errors. Therefore, we conclude the prices are virtually indistinguishable from a practical perspective (any differences are certainly not statistically significant). One would not necessarily anticipate such close numerical similarities simply by observing that they are both one-factor models calibrated to the same (Black) caplet prices.

At 20Y, slight price differences are observed between the (arbitrage-free) SLM and MF models in all scenarios. The MF model gives consistently higher prices especially for out-of-the-money options. Recall from section 4 that the slopes and intercepts of the log-LIBOR plots do not match to such high accuracy at 20Y, though it is clear from the distributional study that the models remain very similar qualitatively. It is arguable that in practice these price differences would not be considered large (they are consistently well below the MF vega). Prices under the DA model are reasonably close to LS but are systematically lower. This may be of concern since the LS price is theoretically a lower bound for the true Bermudan price under the SLM model (since the exercise strategy may theoretically be improved).

At 30Y, the price differences increase across all scenarios. Again DA prices are observed to be below LS prices, which in turn lie below MF prices. The numerical error between LS and DA prices is still small in comparison with MF vega; the maximum difference is approximately half the vega. From a practitioner's viewpoint, it is arguable that this model error is still acceptable, being within what would be taken in profit, though it is clear the observed differences could represent a large proportion of that profit.

Numerical accuracy is important in determining the LS price. To achieve convergence to the desired accuracy 100,000 paths were required (50,000 plus 50,000 antithetic), each with 100 time steps between each exercise date. Using fewer time steps introduces discretisation error that may affect the Bermudan price at this accuracy.[11] As the MF model remains qualitatively similar to the SLM model its efficient implementation would appear to be preferable.

It is anticipated that including the smile in implied volatilities (in this one-factor setting) will have a much larger impact on prices than these model differences, since this will change the functional forms significantly. This is illustrated by Pelsser & Pietersz [2005], who note similarities in Bermudan prices between the MF and SLM models that both exhibit displaced diffusion dynamics. A version of the uniqueness result can still be formulated in this situation; this would certainly help explain the similarity between the models in the one-factor case.

Stress testing

In this subsection the three LIBOR models are compared under more unusual market conditions. We find that it is the presence of high volatilities that has a significant impact on the match between models. Results of the distributional study are presented only for a maturity of 30Y because for shorter maturities the effect is far less noticeable. The effects of stressing the values of initial LIBORs for reasonable volatility levels have also been examined but the consequences are relatively insignificant.

The impact of high volatilities is clearly illustrated in Exhibit 8, where we plot $\log(L^i_{T_{15}})$ against $\log(L^n_{T_{15}})$ for extremely high implied volatilities of 50% (as before we assume a mean reversion parameter of $a = 5\%$ and initial LIBORs of 7%). Under the market model, the linear relationship previously observed between $\log(L^i_{T_k})$ and $\log(L^n_{T_k})$ breaks down. Also the points of the scatter plot are more widely spread out, hence the market model can no longer be well represented by a single functional form.

On initial inspection, the drift approximation appears to provide a better match to the market model than the Markov-functional model in these unusual market conditions. Indeed, under this scenario, the Markov-functional model and the drift approximation model may give rise to very different functional forms even when the market model exhibits little dispersion at a given exercise date. This can be seen for example by looking at the plots for $L^{28}_{T_{15}}$ in Exhibit 8 and is further illustrated below by increasing mean reversion. Note that under the conditions of Exhibit 8 the drift approximation model begins to exhibit significant arbitrage and the effects of this are not immediately clear (see discussion below).

Exhibit 9 displays the same results as given in Exhibit 8 for a higher value of the mean reversion parameter ($a = 15\%$). Consider the plots of $L^{21}_{T_{15}}$ under each model. The presence of high mean reversion means that the common instantaneous volatility function σ increases steeply over successive time intervals. This results in the constants γ^i, chosen via Equation (22), decreasing dramatically as i increases. Therefore, under the market model the stochastic component of the integrated drift terms appearing in the expression for $L^{21}_{T_{15}}$, which contains a $(\gamma^i)^2$ term, will dominate the non-stochastic component of the drift, which only contains terms $\gamma^i\gamma^j,\ j > i$ (see equations (8) and (9)). Thus, the scatter plot of the market model simulation exhibits little dispersion at T_{15}. For the same reason, the standard application of the Brownian bridge drift approximation to this market model gives a functional form that lies close to the scatter plot. In contrast, the functional form of $L^{21}_{T_{21}}$ under the Markov-functional model is typically very close to the corresponding market model plot at T_{21} but may differ at earlier times; we observe significant differences at T_{15}. As we explain below, this is because these functional forms are computed iteratively, backwards through time, by applying the martingale property (15).

The explanation for the observed disparity between the Markov-functional and drift approximation models is that these plots mask the presence of noticeable arbitrage in the drift approximation model. In order to ensure the implementation of any model is arbitrage-free in practice, we require that the martingale property of numéraire-rebased discount factors is numerically sufficiently accurate at all times. This is far from true for the drift approximation model under these unusual circumstances, as we show below. Accuracy of the martingale property is essential for pricing Bermudan-style derivatives since it is implicitly assumed when computing the time value of a derivative (the value of continuation) at a given exercise date (for a Bermudan swaption this is the maximum of the expectation under $\mathbb{F}$ of the payoff at the subsequent exercise date and the value of immediate exercise).

A practical implementation of the drift approximation model may of course be constructed by assuming the functional forms of $L^i_{T_i}$ are taken to be those given by the usual drift approximation model for $1 \leq i \leq n$ and recovering the remaining functional forms of $L^i_{T_j}$ at exercise dates $T_j < T_i$ using the martingale property of numéraire-rebased discount factors (15).[12] In our example, the terminal LIBOR L^{29} is a known analytic function of x at all times. If $L^{28}_{T_{28}}$ is assumed to be given by the drift approximation as usual, then $L^{28}_{T_{27}}$ may be recovered by applying the martingale property.

Exhibit 10 allows us to compare the functional forms of $\log L^{28}_{T_{28}}$ under both the Markov-functional model and the drift approximation model constructed using the martingale property. In displaying these functional forms, for each value of the terminal LIBOR $L^{29}_{T_{28}}$ we have simulated the market model conditional on this value and subtracted the mean value of $\log L^{28}_{T_{28}}$ under this model from each of the functional forms. Confidence intervals under the market model for the value of $\log L^{28}_{T_{28}}$ conditional on the value of $L^{29}_{T_{28}}$ are also provided. It appears that the functional form of $\log L^{28}_{T_{28}}$ under the Markov-functional model is closer to the mean value of $\log L^{28}_{T_{28}}$ under the market model (given $L^{29}_{T_{28}}$) than under the drift approximation model.

Given this observation, it is reasonable to expect that when applying martingale property to compute the values of L^{28} at earlier exercise dates the Markov-functional model will be closer to the market model than the drift approximation model. This is confirmed by Exhibit 11, which shows a typical LIBOR functional form computed by applying the martingale property to the drift approximated LIBORs $L^{28}_{T_{28}}$ to compute the values of $L^{28}_{T_{27}}$. This plot demonstrates that if our pricing model is forced to remain arbitrage-free under these extreme circumstances, then in general it is the Markov-functional model that appears to be closer to the market model than the drift approximation model.

Note that since the functional forms for the original drift approximation model are very different to those calculated by using the martingale property, as is the case with the Markov-functional model, we must have introduced a significant arbitrage into the drift approximation model. The effects of this arbitrage may be magnified in the pricing algorithm, as errors are compounded when we compute the time value of the option iteratively down from the last exercise date.

We conclude this section with a brief discussion of Bermudan swaption prices under this unusual scenario (see Exhibit 12). Generally, the standard error in the Longstaff-Schwartz price is very large; this is not surprising because simulated Bermudan prices are likely to be much more spread out if implied volatilities are very high. For 30Y, the standard error is so large (approx 220 bps for 100,000 paths), it renders the method practically useless without much more sophisticated variance reduction methods. For all maturities, including a second explanatory variable (the current LIBOR) in the least-squares regression at each step increases the Bermudan price significantly (these prices are denoted by 'LS2 Price' in Exhibit 12). This is because under these market conditions the separable LIBOR market model is no longer well represented by a one-dimensional model (with the corresponding one-dimensional exercise boundary). It is possible that including further explanatory variables in the regression may increase the price still further. The MF prices consistently remain very close to the centre of the 95% LS2 confidence interval, whereas the DA price is typically below the lower 95% confidence limit. This example illustrates how any approximation to the LIBOR market model may break down in unusual circumstances even if it performs well in the majority of situations.

5 Numerical comparison of swap models

In this section we report the results of a similar numerical study of the analogous relationships between rates under the swap market model,[13] the associated swap drift approximation model and the corresponding swap Markov-functional model with the same driving process.

The construction of a swap Markov-functional model that closely matches the swap market model described above is analogous to that for the LIBOR case. As in the swap market model we assume a set of co-terminal forward par swap rates, denoted by y^i for $i = 1, .., n$. The ith forward par swap rate y^i sets on date T_i with coupon payments on dates $T_{i+1}, \ldots, T_{n+1}$ and satisfies (11). We assume that the market prices for the vanilla swaptions on the ith swap rate are given by Black's formula. The driving Markov process and the choice of numéraire are exactly as in the LIBOR case but now it is the ith forward par swap rate at time T_i, $y^i_{T_i}$, which is assumed to be a monotonic increasing function of the variable x_{T_i}.

The numéraire bond at time T_n, $D_{T_n T_{n+1}}(x_{T_n})$, is chosen exactly as for the LIBOR model. However the functional form for the numéraire $D_{\cdot T_{n+1}}$ at times T_i, $i = 1, \ldots, n-1$, needs to be determined. The reader is referred to Hunt & Kennedy [2000] for the full details of the calibration step, this time carried out using synthetic PVBP-digital swaptions as the calibrating instruments. The algebra involved in these intermediate steps is no more onerous than for the LIBOR-based Markov-functional model (whereas the drift term of the swap market model is found to be more complex than in the LIBOR market model). The reader will note that a similar uniqueness statement to that given for the LIBOR Markov-functional model can be formulated for the swap Markov-functional model.

In the following, the driving process x of the swap Markov-functional model is taken to be of the same form as for the LIBOR-based model but the variances of x at each T_i are now chosen by considering a Hull-White model calibrated to at-the-money European swaption prices (see Exhibit 15). Linear interpolation is used to complete the specification of the swap-based market model. The mean reversion parameter a is taken to be 5%. The tenor structure under consideration is taken to be the same as for the LIBOR case.

Our conclusions are very similar to those for the analogous LIBOR-based models for the scenarios in Exhibit 1. We observe that $\log(y^i_{T_k})$ is approximately linear in $\log(y^n_{T_k})$ for all models and that the slopes and intercepts agree to high accuracy (see Exhibit 13 for the case of flat initial LIBORs and implied caplet volatilities (Scenario A)). Note that the accuracy of approximations suggested in Pelsser & Pietersz [2005] for the calibration of a swap Markov-functional model to a swap correlation matrix (either market-implied or historically estimated) is easily explained by the linearity of this relationship, since this means the Taylor expansion of $\log(y^i_{T_k})$ about x_{T_k} to order one is almost exact. Approximations along the same lines could be derived to aid calibration of the LIBOR Markov-functional model by observing the linearity of the corresponding relationship between log LIBORs and the driving process under the LIBOR model.

In exploring the functional forms of the forward LIBORs $L^i_{T_i}$ implied by the swap-based Markov-functional model we find that these may be unrealistic for long maturities above twenty years (see discussion below). This is also the case for the analogous swap market model. Recall from our scenario analysis of the LIBOR model above that under the LIBOR model the functional forms of par swap rates behave as expected. Therefore, although the one-factor swap Markov-functional model may be considered an adequate choice for pricing a Bermudan swaption, a LIBOR-based model may be a more appropriate choice in other applications.

Typical functional forms of LIBORs under the swap Markov-functional model are displayed in Exhibit 14. These particular results correspond to flat zero curves and flat swaption volatilities but results are very similar in all scenarios. Notice there is significant non-linearity, which is far more pronounced than the relationship between forward swap rates under the LIBOR-based model. This non-linearity is easily explained by observing that forward swap rates are a linear combination of LIBORs. Thus a change in the distribution of a single LIBOR will have a marginal effect on the distribution of the forward swap rate, which is an average, but a similar change in the distribution of a single forward par swap rate has far more significant impact on the distribution of the LIBORs, which are effectively obtained by differencing. We note that using typical market data this non-linearity is minor for maturities up to twenty years. However, for longer maturities such as thirty years these effects become more apparent and we may also observe negative LIBORs. Functional forms are truncated for negative values of $L^n_{T_i}$ in the graph shown.

6 Conclusion

In this article, we have explored the relationship between LIBORs under the one-factor LIBOR market model with separable volatility structure and the corresponding one-factor Markov-functional model. We have observed that for short maturities (10Y) these models are numerically equivalent for all practical purposes under a wide range of market conditions. For longer maturities, slight differences are observed in our distributional study, however the models remain qualitatively similar. Therefore, much of the intuition of the familiar SDE formulation of the separable market model may be applied in the specification and calibration of the Markov-functional model. As expected given the close match between models at 10Y, the prices of exotic derivatives such as Bermudan swaptions under these models are practically identical. For longer maturities, it is possible to distinguish between prices, however it is arguable that the difference is not material from a practical perspective. In this case, the straightforward efficient implementation LIBOR Markov-functional model may be preferable to any time-consuming simulation-based implementation of the LIBOR market model. It is also preferable to the drift-approximation model because it is guaranteed to be arbitrage-free.

Under scenarios corresponding to long maturities and high volatilities, the market model is no longer well approximated by a one-dimensional model and the relationship between each LIBOR and the terminal LIBOR cannot be approximated by a single functional form. We have demonstrated that the drift approximation model now exhibits noticeable arbitrage and consequently it may lead to inaccurate derivative prices. In contrast, the LIBOR Markov-functional model remains qualitatively similar to the LIBOR market model and may therefore be considered a more appropriate choice of pricing model. Considering again the example of the Bermudan swaption, it appears that prices under these two models remain consistent under this extreme scenario, whereas the drift approximation model tends to lead to a significant underpricing. Our results highlight the dangers of using an approximation to an arbitrage-free model where the limitations of the approximation are not fully understood.

In a separate line of discussion, the behaviour of functional forms of forward LIBORs under the swap-based Markov-functional model are found to be somewhat unrealistic for long maturities (where in some cases LIBORs may become negative). This is an artefact common to all one-factor swap rate based models. In contrast, the behaviour of forward par swap rates under the LIBOR Markov-functional is found to be as expected.

Notes

[1]For example, Andersen & Andreasen [2000] consider one and two-factor LIBOR market models and find a suitably parameterised one-factor model is sufficient for pricing Bermudan swaptions in practice. Recent work by Pelsser & Pietersz [2005] comparing single factor Markov-functional and multi-factor market models also supports the claim that Bermudan swaptions can be adequately priced and risk managed with single factor models. For a discussion of when a low dimensional model is appropriate see Hunt & Kennedy [2005]. Pelsser et al. [2004] state the view that separability is a non-restrictive assumption.

[2]We have not necessarily written fully optimized code in either case, however, and we are working to higher precision than would typically be required in practice.

[3]This standard construction may also be found in, for example, Hunt & Kennedy [2000].

[4]Pelsser et al. [2004] state that separability is a "non-restrictive requirement on the form of the volatility function." In this thesis we choose not to discuss the justification of this assumption in respect of any particular pricing problem.

[5]This definition extends to d-dimensional volatility specifications, see Pelsser et al. [2004].

[6]For the market model and drift approximation model this is immediate since L^n has zero drift (see Equation (7)). Under the Markov-functional model this holds by definition at T_n (see Equation (16)). At earlier times we may recover the relationship between L^n and x by applying the martingale property to L^n (the relationship is the same as under the market model).

[7]Under the market model and associated drift approximation model we require values of initial LIBORs and implied volatilities at times other than $T_1, \ldots, T_{n+1}$ in the computation of ξ_t (see Exhibit 15); these are obtained by linear interpolation. The scenario for decreasing rates and implied volatilities has been adjusted to ensure that the approximation ξ_t is strictly increasing for all t.

[8]That is, the proportion of the variance in observations explained by a

linear relationship is at least 99.9%.

[9]Market quotes taken at the close of 14 Feb 2001.

[10]Note that the terminal par swap rate y^n is simply the terminal LIBOR L^n.

[11]This could be remedied by, for example, applying a predictor-corrector approximation over slightly larger time steps.

[12]Note that this may be considered to be a different approximation model to that given in Pelsser et al. [2004], where all functional forms are determined using the drift approximation and the martingale property is not used in the construction of the model.

[13]Some authors refer to these models as "Swap-rate based LIBOR market models."

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Exhibit 1.
Scenarios for initial LIBORs L_0^i and caplet implied volatilities $\tilde{\sigma}^i$ considered in our numerical study.

Scenario	Description	L_0^i	$\tilde{\sigma}^i$
A	Flat LIBORs and vols	7%	15%
B	Increasing LIBORs, flat vols	$5\% + 5\%(T_i/30)$	15%
C	Decreasing LIBORs, flat vols	$9\% - 5\%(T_i/30)$	15%
D	Flat LIBORs, increasing vols	7%	$10\% + 10\%(T_i/30)$
E	Increasing LIBORs, increasing vols	$5\% + 5\%(T_i/30)$	$10\% + 10\%(T_i/30)$
F	Decreasing LIBORs, increasing vols	$9\% - 5\%(T_i/30)$	$10\% + 10\%(T_i/30)$
G	Flat LIBORs, decreasing vols	7%	$20\% - 10\%(T_i/30)$
H	Increasing LIBORs, decreasing vols	$5\% + 5\%(T_i/30)$	$20\% - 10\%(T_i/30)$
K	Decreasing LIBORs, decreasing vols	$8\% - 2\%(T_i/30)$	$17\% - 4\%(T_i/30)$

Exhibit 2.

Graph of $\log(L^i_{T_{15}})$ vs. the terminal LIBOR, $\log(L^{29}_{T_{15}})$, for a selection of forward rates i, assuming flat initial LIBORs and implied volatilities (Scenario A).

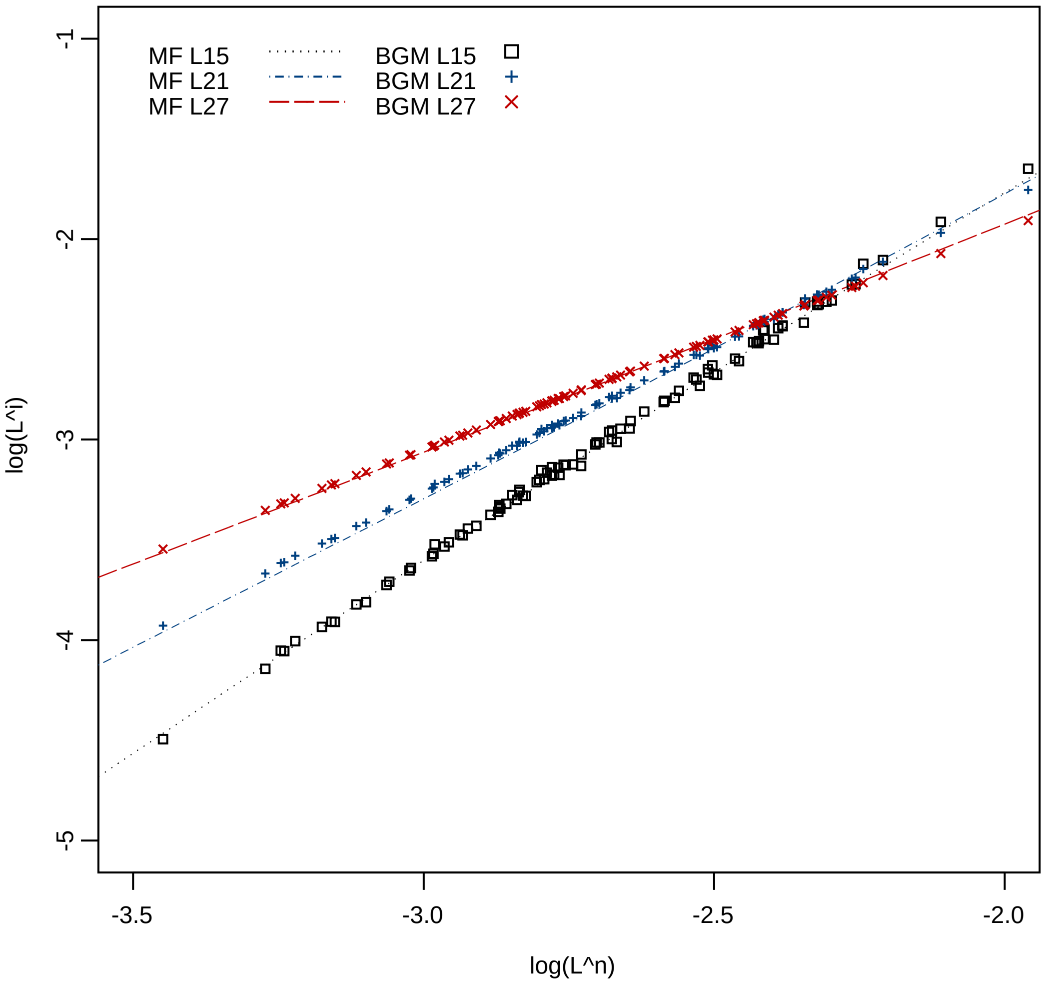

Exhibit 3.
Slopes and intercepts of the functional forms of (log) LIBORs shown in Exhibit 2.

	LIBOR	Log-linear	MF	BGM	DA
Slopes	L^{15}	2.01	1.84	1.88	1.84
	L^{21}	1.49	1.51	1.45	1.44
	L^{27}	1.11	1.14	1.10	1.10
Intercepts	L^{15}		1.9	2.0	1.9
	L^{21}		1.3	1.1	1.1
	L^{27}		0.4	0.2	0.2

Exhibit 4.
Graph of a selection of co-terminal forward par swap rates $\log(y^i_{T_i})$ vs. the terminal forward rate $\log(y^{29}_{T_i})$ under the LIBOR MF model.

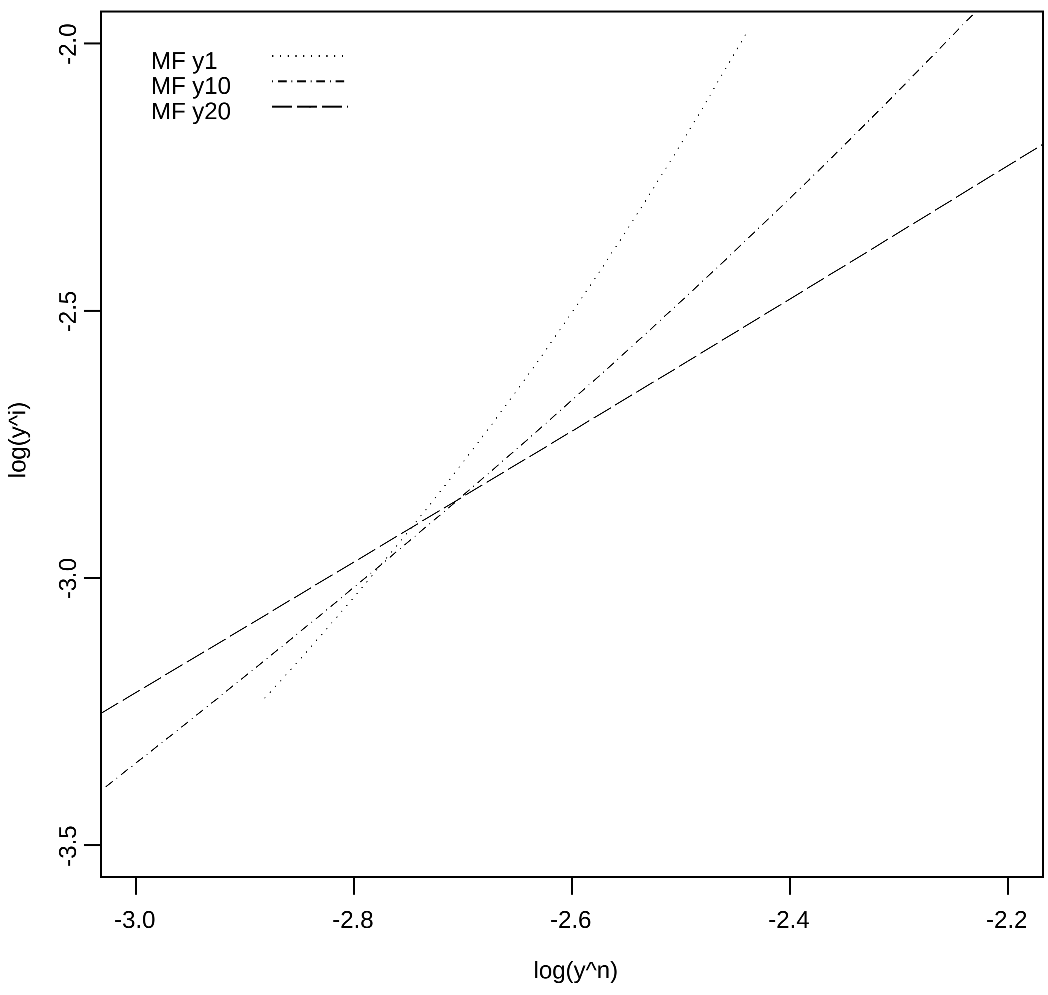

Exhibit 5.
10Y annual Bermudan swaption prices (in basis points).

Strike	MF Price	DA Price	LS Price	LS s.e.	MF vega
5%	123.0	123.0	123.0	0.19	0.6
6%	73.1	72.9	73.0	0.18	2.0
7%	41.4	41.2	41.3	0.16	2.8
8%	24.0	23.9	24.0	0.13	2.7
9%	14.4	14.3	14.4	0.10	2.2

Exhibit 6.
20Y annual Bermudan swaption prices.

Strike	MF Price	DA Price	LS Price	LS s.e.	MF vega
5%	197.0	196.6	196.6	0.30	1.7
6%	124.8	123.5	123.8	0.33	4.5
7%	80.6	79.0	79.4	0.32	5.7
8%	54.4	53.0	53.3	0.29	5.7
9%	38.1	36.8	37.3	0.25	5.2

Exhibit 7.
30Y annual Bermudan swaption prices.

Strike	MF Price	DA Price	LS Price	LS s.e.	MF vega
5%	235.8	234.5	234.9	0.38	2.7
6%	154.9	151.3	152.7	0.44	6.5
7%	105.9	101.4	103.2	0.44	8.0
8%	76.0	71.5	73.6	0.42	8.0
9%	56.5	52.4	54.5	0.38	7.6

Exhibit 8.
Plot of $\log(L^i_{T_{15}})$ vs. $\log(L^n_{T_{15}})$ for implied caplet volatilities of 50%.

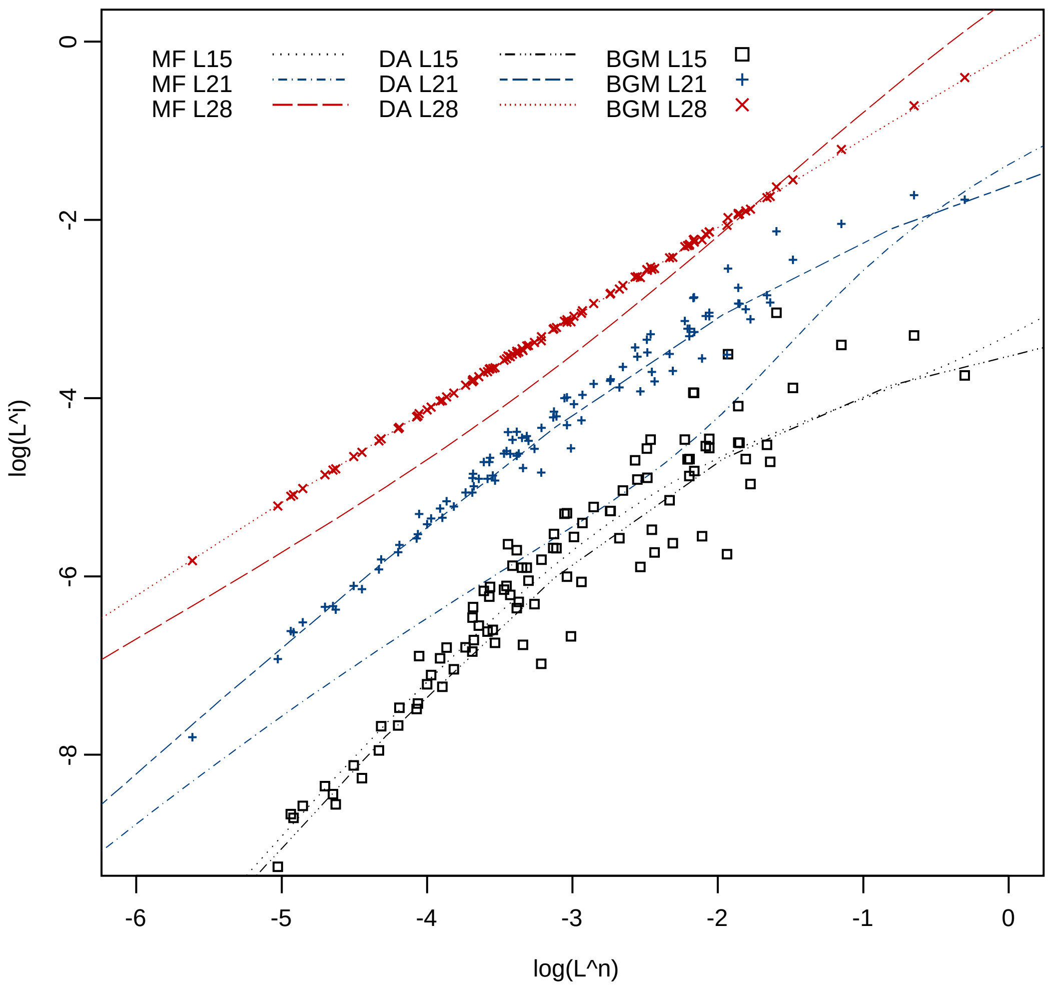

Exhibit 9.
The same set of results for high volatilities as displayed in Exhibit 8 but with mean reversion parameter $a = 15\%$.

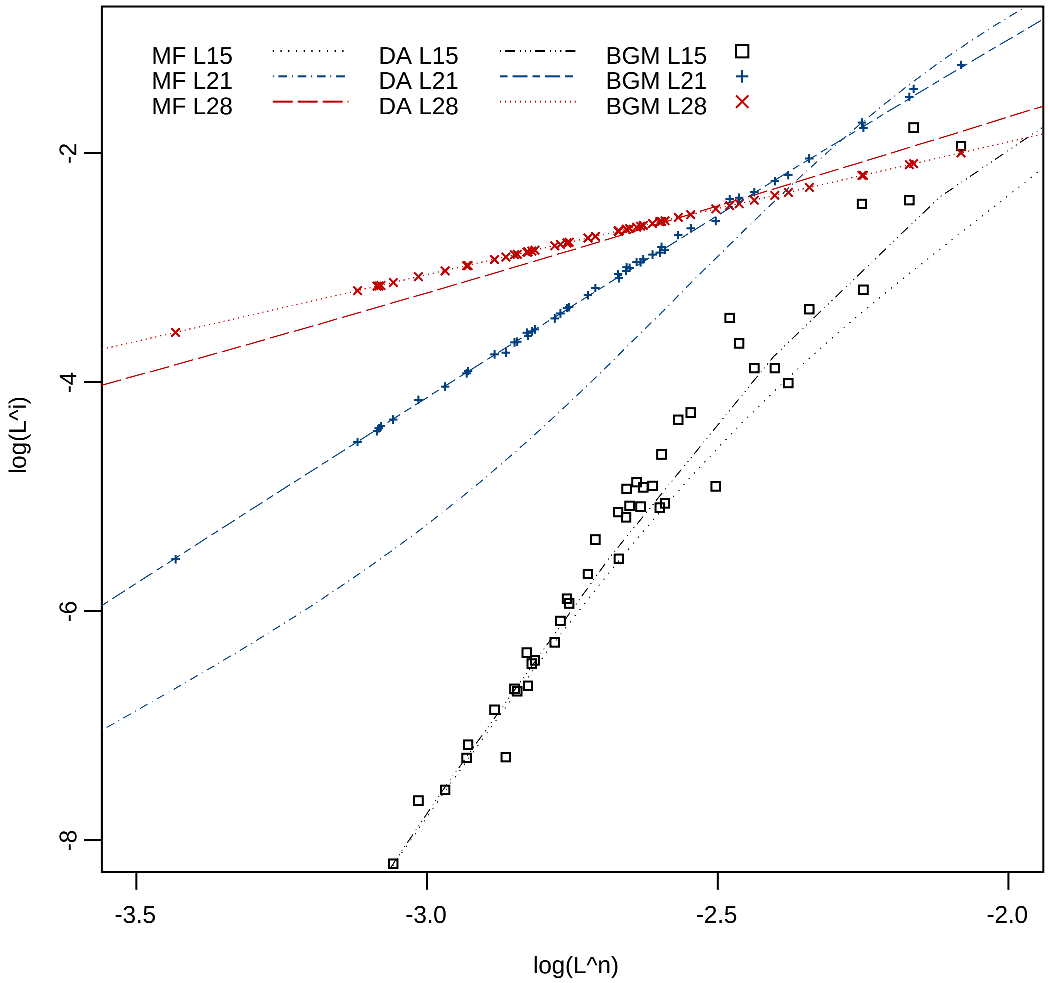

Exhibit 10.

Plot of $\log(L^{28}_{T_{28}}) - \mathbb{E}[\log(L^{28}_{T_{28}})]$ conditional on the value of $L^{29}_{T_{28}}$ under the SLM model, against the terminal LIBOR, $\log(L^{29}_{T_{28}})$.

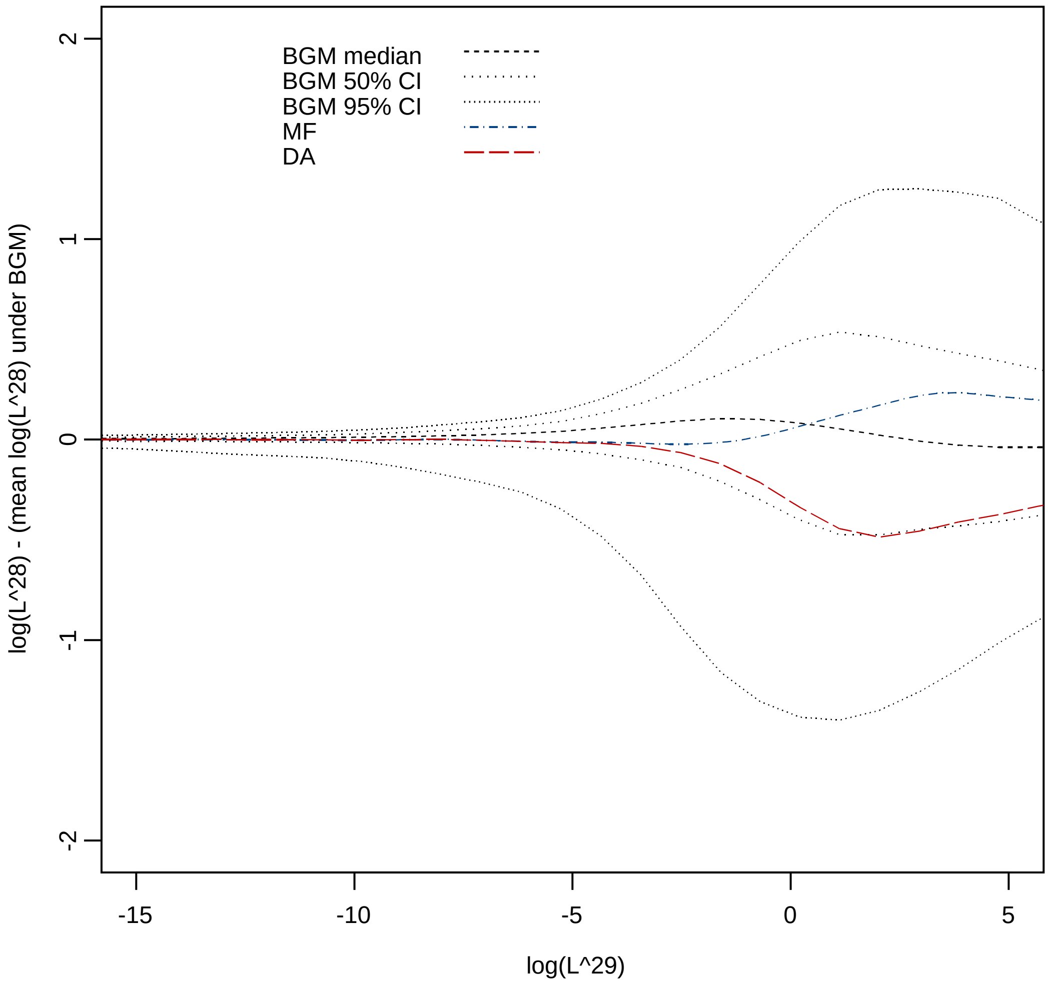

Exhibit 11.
Plot of $\log(L^{28}_{T_{27}})$ vs. the terminal LIBOR, $\log(L^{29}_{T_{27}})$.

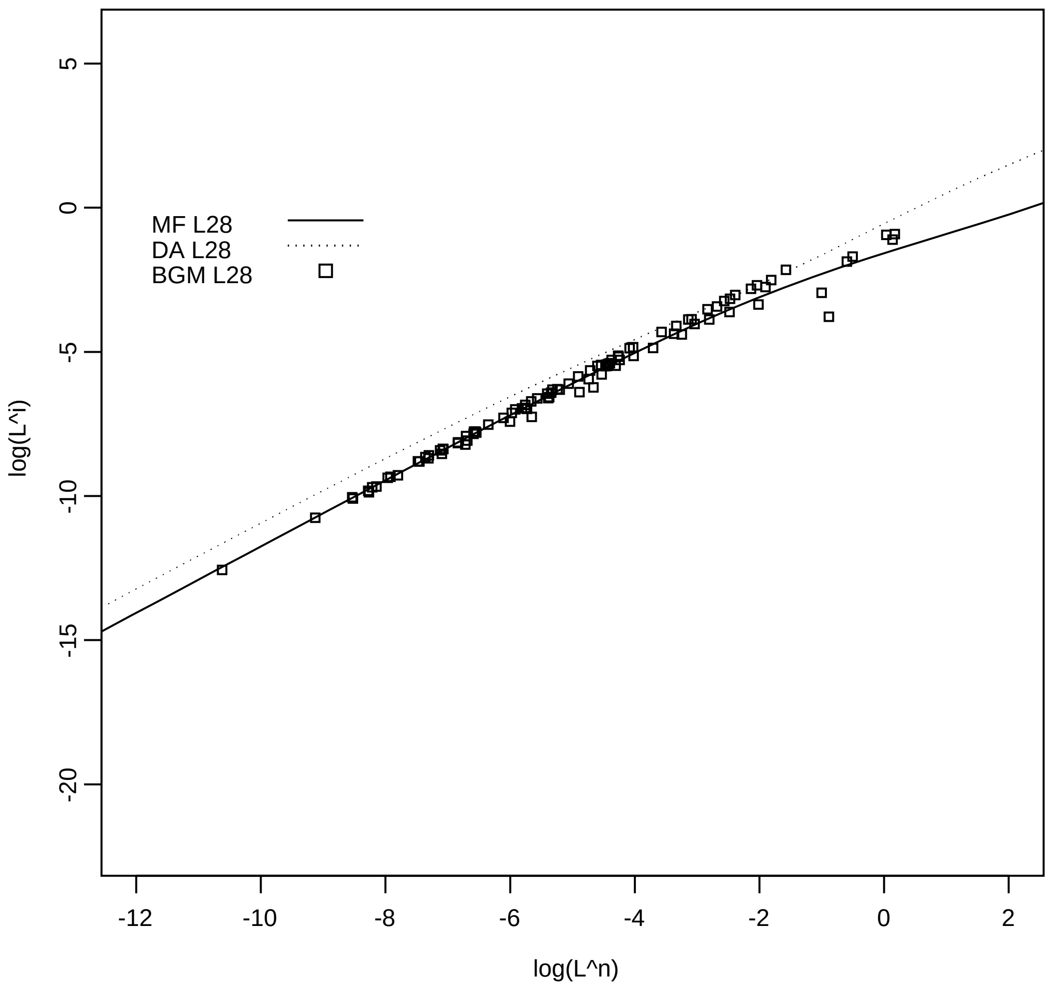

Exhibit 12.
10Y annual Bermudan swaption prices for implied volatilities of 50%.

Strike	MF Price	DA Price	LS Price	LS s.e.	LS2 Price	LS2 s.e.	MF vega
5%	183.8	176.1	166.8	0.7	181.5	2.8	2.3
6%	160.2	151.0	139.0	0.7	157.1	2.9	2.7
7%	141.7	131.6	119.5	2.8	138.5	2.9	2.9
8%	127.0	116.5	100.7	2.8	126.3	4.3	3.1
9%	115.0	104.3	88.9	2.9	113.8	4.2	3.2

Exhibit 13.
Slopes and intercepts of functional forms of a selection of (log) forward par swap rates at T_{15}.

	Swap rate	Log-linear	MF	BGM	DA
Slopes	y^{15}	1.45	1.32	1.36	1.41
	y^{21}	1.23	1.22	1.20	1.21
	y^{27}	1.05	1.07	1.05	1.05
Intercepts	y^{15}		0.54	0.74	0.90
	y^{21}		0.51	0.43	0.48
	y^{27}		0.17	0.11	0.11

Exhibit 14.
Graph of $\log(L^i_{T_i})$ vs. $\log(L^n_{T_i})$ under the swap MF model.

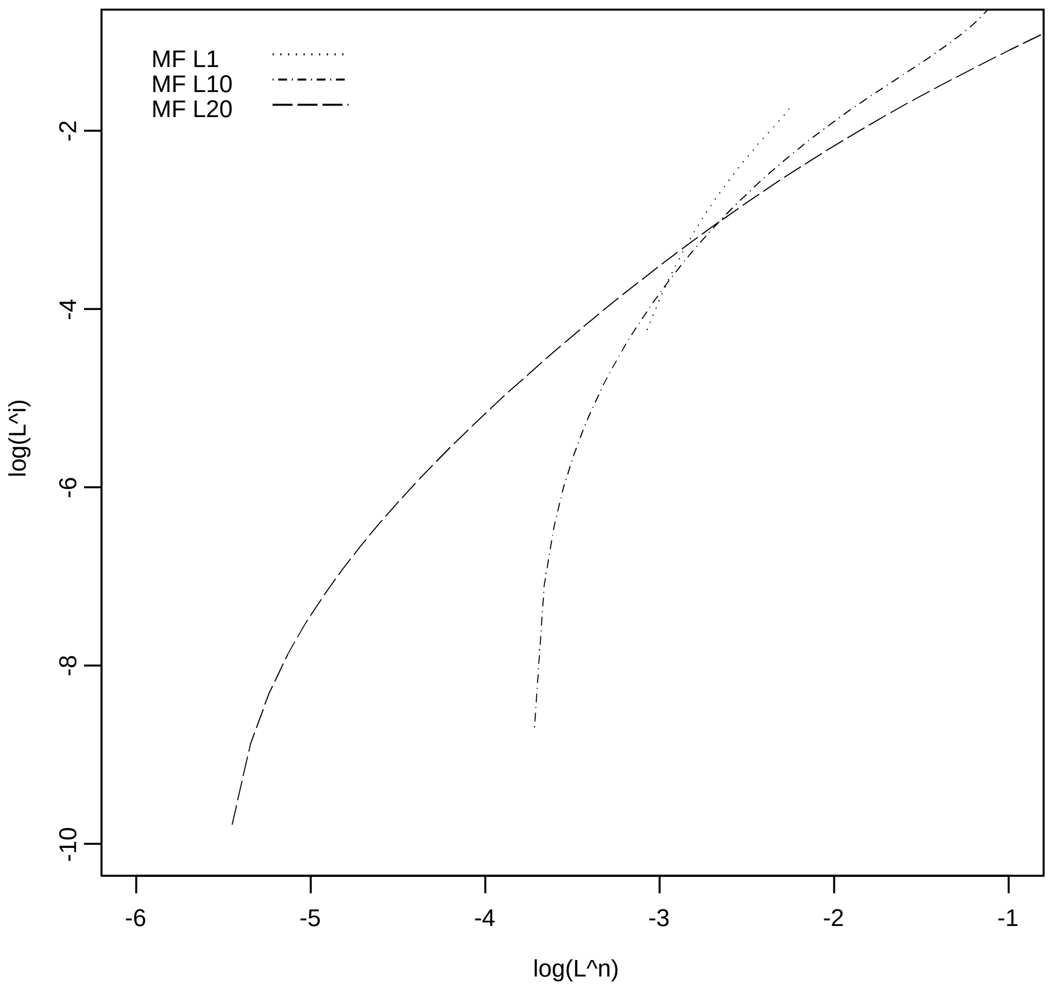

Exhibit 15.
In this Exhibit we specify the driving process x by deriving an approximate expression for the variance

$$\xi_{T_i} := \mathrm{var}(x_{T_i})$$

of x at times $T_i,\ i = 1, \ldots, n$. This approximation is arrived at by considering a Vasicek-Hull-White model calibrated to at-the-money caplet prices in the LIBOR case and at-the-money European swaption prices in the swap case.

Consider a Hull-White model in which the short-rate process r solves the SDE

$$dr_t = (\theta_t - ar_t)dt + \hat{\sigma}_t d\widehat{W}_t\,,$$

where a is a constant, θ and $\hat{\sigma}$ are deterministic functions of t and $\widehat{W}$ is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$. For $0 \le t \le T_{n+1}$ the measures $\mathbb{F}$ and $\mathbb{Q}$ are related by

$$\left.\frac{d\mathbb{F}}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \exp\left(-\int_0^t r_u du\right)\frac{D_{tT_{n+1}}}{D_{0T_{n+1}}}\,.$$

Let x be defined as in Equation (10) and define $\hat{\sigma}_t = e^{-at}\sigma_t$. Working in the measure $\mathbb{F}$ it is straight forward to derive an analytical expression for the functional forms $\hat{D}_{tT_i}(x_t),\ i = 1, \ldots, n$. We find

$$\hat{D}_{tT_i} = \hat{D}_{0T_i}\exp\left((\psi_{T_{n+1}} - \psi_{T_i})x_t - \frac{1}{2}(\psi_{T_{n+1}} - \psi_{T_i})^2\xi_t\right)\,, \tag{23}$$

where

$$\psi_t := \frac{1}{a}(1 - e^{-at})\,,$$

and

$$\xi_t := \int_0^t e^{2au}\hat{\sigma}_u^2 du\,.$$

Suppose we wish to use the above Hull-White model to price a product where the relevant calibrating instruments have cash flows restricted to the times $T_i,\ i = 1, \ldots, n+1$, and suppose that the parameter a has been chosen. From the above Equation we can easily see that in order to specify completely the Markov-functional implementation of the Hull-White model only the variances $\xi_{T_i} = \mathrm{var}(x_{T_i}),\ i = 1, \ldots, n$, are required. In practice this could be done numerically by calibrating directly to an appropriate choice of cap or swaption prices.

We now derive a crude approximation to the ξ_{T_i}'s in the case where the Hull-White model is calibrated to caplets on the forward LIBORS L^i. The market prices of these caplets are assumed to be given by Black's formula with implied volatilities

$\tilde{\sigma}^i$. The formula obtained is used as the basis for the choice of the driving process used in the numerical comparison of all LIBOR models discussed in this article.

Observe that approximately

$$(D_{T_iT_{i+1}})^{-1} = (1 + \alpha_i L^i_{T_i})\,. \tag{24}$$

Note that this approximation is exact if the $L^j_{T_i},\ j \geq i$, are equal. Writing

$$\exp(Z^i_t) := 1 + \alpha_i L^i_t\,,$$

by Itô's formula

$$\exp(Z^i_t)dZ^i_t + \frac{1}{2}\exp(Z^i_t)d[Z^i]_t = \alpha_i dL^i_t$$

and so

$$dZ^i_t = \alpha_i(1 + \alpha_i L^i_t)^{-1}dL^i_t + \text{f.v.}\,,$$

where f.v. denotes terms having finite variation. Thus

$$d[Z^i]_t = \alpha_i^2(1 + \alpha_i L^i_t)^{-2}d[L^i]_t\,.$$

Setting $t = T_i$ in Equation (23) we can obtain an expression for

$$\frac{\hat{D}_{T_iT_i}}{\hat{D}_{T_iT_{i+1}}} = (D_{T_iT_{i+1}})^{-1}\,.$$

Comparing the quadratic variation of the exponential term for this expression with that in Equation (24), the following approximate relationship is obtained:

$$(\psi_{T_i} - \psi_{T_{i+1}})^2\xi_{T_i} = \alpha_i^2(1 + \alpha_i L^i_{T_i})^{-2}[L^i]_{T_i}\,. \tag{25}$$

Further, assuming the market prices of the caplets are given by Black's formula we see that approximately

$$[L^i]_{T_i} = (\tilde{\sigma}^i)^2(L^i_0)^2T_i\,,$$

where $\tilde{\sigma}^i$ denotes the implied volatility of the ith caplet. Substituting this in (25), approximating $L^i_{T_i}$ by L^i_0 and solving for ξ_{T_i} yields Equation (21):

$$\xi_{T_i} = \left(\frac{\alpha_i L^i_0}{(1 + \alpha_i L^i_0)(\psi_{T_i} - \psi_{T_{i+1}})}\right)^2(\tilde{\sigma}^i)^2T_i\,.$$

Note that here we have proposed a correlation structure that is linked explicitly to market volatilities.

For a constant tenor structure $\alpha_i = \alpha$ this formula may be extended for general t by taking

$$\xi_t = \left(\frac{\alpha L_0(t)}{(1 + \alpha L_0(t))(\psi_t - \psi_{t+\alpha})} \right)^2 (\tilde{\sigma}(t))^2 t \,. \tag{26}$$

where $L_0(t) = L_0[t, t+\alpha]$ is the initial forward LIBOR corresponding to time t with tenor α and $\tilde{\sigma}(t)$ is the implied volatility of the caplet associated with this LIBOR. Note that linear interpolation of the ξ_{T_i}'s is equally viable since we observe in our numerical comparison that this leads to indistinguishable results. To complete the specification of the LIBOR market model with this correlation structure, observe that

$$x_t = \hat{W}_{\xi_t} \,,$$

where $\hat{W}$ is a Brownian motion under $\mathbb{F}$. Therefore, the instantaneous volatility of the driving process in the log-Euler discretisation of the market model SDE may be approximated with

$$\sigma_t = \sqrt{\frac{\xi_{t+h} - \xi_t}{h}} \,,$$

where h is the step-size of the discretisation.

For the case when the Hull-White model is calibrated to Black's swaption prices an argument similar to the above yields the approximation

$$\xi_{T_i} = \left(\frac{T_{n+1} - T_i}{(1 + \alpha_i y_0^i)(\psi_{T_{n+1}} - \psi_{T_i})} \right)^2 (\tilde{\sigma}^i)^2 T_i \,, \tag{27}$$

where $\tilde{\sigma}^i$ now denotes the implied volatility of the ith co-terminal European swaption. In this case we use linear interpolation to complete the specification of the swap-based market model.