Notes taken from the first problem session — not guaranteed to be free from errors.

Gabor Pete

We consider bootstrap percolation on Cayley graphs (reference: Balogh, Peres, Pete).

Definition: A group is *amenable* if $\exists$ (or, equivalently, $\forall$) Cayley graph with $F_1 \subseteq F_2 \subseteq \cdots \subseteq V(G)$ such that $\bigcup_{n\geq 1} F_n = V(G)$, $|F_n| < \infty$ for all $n \geq 1$, and

$$|\partial F_n|/|F_n| \to 0,$$

where ∂F_n is the set of vertices in F_n with a neighbor outside F_n.

For example $\mathbb{Z}^d$ is amenable. The Diestel-Leader graph $\mathrm{DL}(2,2)$, a Cayley graph of the lamplighter group, is also amenable. Exercise: $\mathrm{DL}(2,2)$ is amenable.

Question: prove or disprove: a group is amenable if and only if for every generating set S and for every k-rule bootstrap percolation,

$$p_c(\mathrm{Cay}(\Gamma, S), k) \in \{0, 1\}.$$

For $\mathbb{Z}^d$ (amenable) and any group containing a free group with two generators (very non-amenable), the result holds. But for $\mathrm{DL}(2,2)$, it is unknown. The Heisenberg group may also be good to look at.

James Martin

Close sites on $\mathbb{Z}^d$ independently with probability p. Consider a two-player game on this site-percolated board, where players alternate moving the location of a token in some specified set of directions D (to a site which are not closed), never repeating a location. A player loses when they can no longer make a move. If play goes on forever, then the game is a draw. What is the probability of a draw?

When p is so large that there are no infinite components, the game is forced to end, so there is no draw. When p is sufficiently small, is there some positive probability of a draw?

Consider the case $D = \{e_1, \ldots, e_d\}$, the standard basis (in the positive direction(s)). For $d = 2$, it is known that the probability of a draw is 0 for all p: label each site with whether the first or second player wins when the token starts at that site. If we know the labels of the diagonal $x + y = c$, then we can determine the labels of the diagonal $x + y = c - 1$. This looks like a 1-dimensional cellular automata which has a unique stationary distribution.

Conjecture: For $d \geq 3$ and $D = \{e_1, \ldots, e_d\}$, there is a positive probability of a draw for sufficiently small p.

We can prove it for $D = \{e_1 \pm e_i : i \geq 2\}$ and $d \geq 3$. In this case, the game in dimension d is related to a hard-core model (and the uniqueness of stationary measure) on dimension $d - 1$.

Conjecture: For $D = \{\pm e_i : i \in [d]\}$, the probability of a draw is 0 for $d = 2$ but there is a posiive probability of a draw for $d = 3$ and sufficiently small p.

This problem is somewhat related to bootstrap percolation (the Froböse model) because enough closed sites 'force' other sites to be closed, as a player would never move to the site if it is a losing position for them.

How much advantage does one player have over the other? The probability of the first player (in dimension 2) is related to a markov chain computation on a hard-core model. How does closing only even sites of $\mathbb{Z}^d$ shift this advantage?

Consider playing this game on finite graphs (without the site percolation, but, as before, the token may never repeat a site). It is an exercise that the first player wins if and only if the token starts at a vertex which is contained in every maximum matching. To transfer this to the infinite volume, we ask: are the vertices in every maximum matching sensitive to boundary conditions?

Ivailo Hartarsky

Conjecture: for any subcritical $\mathcal{U}$-bootstrap percolation process (i.e. one with $p_c > 0$), we have for all $p > p_c$ that there exists $c > 0$ such that $\mathbb{P}_p(\tau \geq n) \leq e^{-cn}$.

This is known to hold if $\mathcal{U}$ is "oriented" in the sense that there exists a half space $\mathbb{H}$ through the origin containing all of the rules $U \in \mathcal{U}$.

One open case is the directed triangle bootstrap percolation.

Maksim Zhukovskii

We consider graph bootstrap percolation. The weak saturation number $\text{wsat}(G, F)$ is the minimum number of edges in a spanning subgraph H of G (comprising activated edges) which percolates to G: as soon as an edge completes a copy of F, then add it to H.

Example: $\text{wsat}(K_n, K_s) = \binom{n}{2} - \binom{n-s+2}{2}$. The upper bound is contructive: take an $(s-2)$-clique in K_n and all edges incident to the clique. There are other constructions. In general, $\text{wsat}(K_n, F)$ is linear in n, i.e. $\text{wsat}(K_n, F) = (c_F + o(1))n$.

It is known that for constant p, $\text{wsat}(G(n,p), K_s) = \text{wsat}(K_n, K_s)$ w.h.p. (due to Korándi and Sudakov, 2017).

Conjecture 1: for all F and constant p, $\text{wsat}(G(n,p), F) = \text{wsat}(K_n, F)$ w.h.p.

We know for some graphs F for which Question 1 is true, but we do not know $\text{wsat}(K_n, F)$ (in particular, for unbalanced complete bipartite graphs, though it is known up to an additive constant, proved by Kalinicheko, Zhukovskii, 2023).

There exists some p_s such that if $p \gg p_s$ then $\text{wsat}(G(n,p), K_s) = \text{wsat}(K_n, K_s)$ w.h.p.; and if $p \ll p_s$, then they are not equal (Bidgoli, Mohammadian, Tayfeh-Rezaie, Zhukovskii, 2024).

Question 2: Find p_s.

For the second question, we just know

$$n^{-g(s)} < p_s < n^{-f(s)}.$$

For triangles, Peled and Zhukovskii proved $p_3 = n^{-1/3+o(1)}$ (unpublished).

Tibor Szabó

Given two graphs H and G_0, the H-bootstrap percolation process starting with G_0 involves iteratively setting G_i to consist of G_{i-1} together with any edges that produce a copy of H. We let $\langle G_0 \rangle$ be the graph obtained at the end of the process. We define $M_H(n)$ to be the maximum time it takes for the H-boostrap percolation on some n-vertex graph G_0 with $\langle G_0 \rangle = K_n$ to terminate. There are many open questions, here are a few:

Question: Is $M_{K_5}(n) = o(n^2)$. Note that we have a lower bound of $n^{2-o(1)}$ and it's known that for larger cliques we have $M_{K_r}(n) = \Theta(n^2)$.

Question: Is $M_T(n) \leq e(T)$ for all n sufficiently large. Would be tight for the star. Current best known bound is $O(e(T)^2)$. It's theoretically possible that one might be able to prove a bound of the form $O(\Delta(T))$.

Question: do there exist H with a degree 1 vertex v such that $M_H)n) = o(M_{H-v}(n))$ and $M_H(n) = \omega(1)$.

Conjecture: if $M_H(n) = o(n)$, then $M_H(n) = \Theta(\log n)$ or $M_H(n) = O(1)$. Note that cycles and trees show that either of these behaviors can happen.

Conjecture: H having tree width 2 implies $M_H(n) = O(n)$.

Question: Does there exist H_1, H_2 such that $M_{H_1 \cup H_2}(n) = \omega(M_{H_1}(n) + M_{H_2}(n))$.

Janko Gravner

Consider bootstrap percolation on the Hamming graph of dimension 2, say on $\mathbb{Z}_+^2$. The process is parameterized by a Young diagram $\mathcal{Z}$ called the *zero set*. For a given point p, count the number of occupied sites in p's column and the number of occupied sites in p's rows; if this ordered pair is outside the Young diagram, we occupy p, but otherwise, we do nothing. Let $\gamma(\mathcal{Z})$ be the size of the smallest set that percolates to the plane.

Theorem: when $\mathcal{Z}$ is a rectangle, then $\gamma(\mathcal{Z}) = |\mathcal{Z}|$, the area of the rectangle.

It is know that $\frac{1}{4}|\mathcal{Z}| \leq \gamma(\mathcal{Z}) \leq |\mathcal{Z}|$. (The upper bound is just using the initial set $\mathcal{Z}$.)

Question: Is there some algorithm that computes $\gamma(\mathcal{Z})$ (efficiently)? Can you approximate $\gamma(\mathcal{Z})$?

Suspicion: $\gamma(\mathcal{Z}) \geq \frac{1}{2}|\mathcal{Z}|$. (It is known that this can be realized.)

Omer Angel

The goal is to understand if certain models of bootstrap percolation are local.

Suppose you are given some sequence of functions $f_n : \{0,1\}^V \to \{0,1\}$ where each function determines an update rule for a process on a vertex transitive graph. The vague question is when do we have

$$p_c(f_n) \to p_c(f_\infty), \tag{1}$$

where f_∞ is some suitably defined limit object (that the reader may choose). One might require that the sequence (f_n) is increasing. Perhaps the answer depends on whether the models are subcritical, critical, or supercritical.

More concretely, suppose a sequence of graphs G_n tends to a graph G in the Benjamini-Schramm sense and pick your favorite bootstrap rule.

Question: Is it true that $p_c(G_n) \to p_c(G)$.

Sam Spiro

The zero forcing process for a graph G starts with some initial set of activated vertices B_0. Iterativelyif there exists $v \in B_i$ such that there exists a unique neighbor $u \in N(v)$ which is not in B_i, then u gets added to B_{i+1} together with all previous vertices of B_i. We write $B_0 \in ZFS(G)$ if $B_\infty = V(G)$. Note that this property is monotone in that if $B \in ZFS(G)$ then so is any superset of B.

There is a lot of literature studying deterministic B_0, but the case when we start with a p-random set B_p has only been studied very recently. There are many questions to explore here, the main one being the following:

Conjecture: if G is an n-vertex graph and $p \in [0,1]$, then $\Pr[B_p \in ZFS(G)] \leq \Pr[B_p \in ZFS(P_n)]$. That is, the path is the easiest graph to completely activate with a p-random set of vertices. This is known to hold if G has a hamiltonian path via a simple coupling argument (which fails for general graphs), and is also known to hold for trees of sufficiently large order.

Bob Krueger

The firefighting (single-player) game on a (infinite) graph G is the following: a fire breaks out at a vertex v. You may protect (with a 'firefighter') k non-burning vertices of G each turn as "unburnable" for the rest of the game (protecting a vertex is equivalent to deleting it from the graph). Between your turns, the fire spreads along the edges of the graph, and a vertex which catches fire burns forever. What is the minimum k needed so that the fire eventually stops spreading? (See a 2009 survey of Finbow and MacGillivray.)

On $\mathbb{Z}^2$, it is an easy exercise to show that 1 firefighter per turn is not enough, but 2 is.

Conjecture: For the infinite triangular grid, 2 firefighters per turn is not enough (but 3 clearly are).

Conjecture: For the infinite hexagonal grid, 1 firefighter per turn is not enough.

It is known that if you are allowed 1 firefighter per turn, but at on some turn you are given an extra firefighter, then it is possible to contain the fire on the hexagonal grid. There is some relationship between strategies on the triangular grid and strategies on the hexagonal grid.

It is somewhat constraining to allow the same number of firefighters on every turn. You could instead have a (deterministic or random) sequence that tells you how many firefighters you can use. There are many natural variations.