Stochastic evolution as a quasiclassical limit of a boundary value problem for Schrödinger equations

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Abstract

We develop systematically a new unifying approach to the analysis of linear stochastic, quantum stochastic and even deterministic equations in Banach spaces. Solutions to a wide class of these equations (in particular those decribing the processes of continuous quantum measurements) are proved to coincide with the interaction representations of the solutions to certain Dirac type equations with boundary conditions in pseudo Fock spaces. The latter are presented as the semi-classical limit of an appropriately dressed unitary evolutions corresponding to a boundary-value problem for rather general Schrödinger equations with bounded below Hamiltonians.

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Content

1 Introduction

Stochastic evolution models in a Hilbert space have recently found interesting applications in quantum measurement theory, see e.g. the reviews in [4], [8]. Here we are going to show that the solutions to a wide class of stochastic and quantum stochastic equations describing these models can be obtained from a positive (relativistic or non-relativistic) Hamiltonian with singular interaction as a strong limit of the input flow of quantum particles with asymptotically infinite momentum but a constant velosity. Thus the problem of stochastic approximation is reduced to a sort of quasiclassical asymptotics of a quantum mechanical boundary value problem in extra dimension.

There exists a broad literature on the stochastic limit in quantum physics in which quantum stochastics is derived from a nonsingular interaction following the approach suggested in [1] (see also [18] and the monograph [2], and references therein). Here we follow a different approach recently outlined in [10], [11]: instead of rescaling the interaction potentials we treat the singular interaction δ-potentials rigorously as the boundary conditions, and obtain the stochastic limit as an ultra-relativistic limit of a Schrödinger boundary value problem in a Hilbert space of infinite number of particles.

We start with a short Section 2 fixing some general notations that are used throughout the paper. In Section 3, we discuss a toy model with "unphysical" Hamiltonian $\varepsilon(p) = -p$. The toy model is discussed from other point of view in [15], [16], where it was shown that the toy Hamiltonian model in Fock space with a discontinuity condition is equivalent to the Hudson-Parthasararthy quantum stochastic evolution models [20] in the case of commuting operator-valued coefficients or in the case of one dimensional noise (for multidimensional extension in this direction see [19]). Our approach is different and is based on the observation that working in multiple pseudo Fock spaces allows for a representation of stochastic and quantum stochastic evolutions that preserves the number of particles (though changes their colour), and consequently reduces a study of general quantum stochastic flows to the study of Poisson driven evolutions in coloured simplices. Moreover, unlike [15] and [10] we systematycally consider the evolutions in general Banach spaces (and, in particular, non-unitary boundary conditions), which are important for applications to general stochatic equations, in particular those decribing the models of continuous quantum measurements. At the end of Section 3, we discuss "more physical" representation for the toy model which enjoy the symmetry with respect to the inverse of time combined with compex conjugation. In this representation, the toy model is described by a two-dimensional Dirac equation with additional (internal) degrees of freedom. In Sections 4, on a simple example of a single-kick stochastic evolution in a Banach space, we show the basic idea on the connection of boundary value problems with stochastic evolutions.

In Section 5, we show how the solutions to a class of linear stochastic (non-unitary) differential equations driven by a compound Poisson process can be obtained as the interaction picture representation for the boundary- value problem for toy Hamiltonians. This class of equations includes the Belavkin quantum filtering equations describing the aposterior dynamics of a quantum system under continuous non-demolition measurement of counting type [7]. For completeness, we give in Appendix a short deduction of these equations from the von Neumann model of instanteneous measurements. Since the stochastic equations driven by a Wiener process can be obtained as the limits of linear equations driven by compound Poisson process (in Appendix, we show how it is done concretely for the case of quantum filtering equations), the results of Section 5 allows the representation of the solutions to linear diffusion equations in Banach spaces to be represented as the limits of certain (deterministic) boundary value problems.

The main results of the paper are obtained in Sections 6 and 7. In Section 6 we are going to deal with diffusion equations (and more general continuous noises) directly, without a limiting procedure. To this end, we develop a theory of boundary value problems for shifts in "coloured simplices", which describe the restrictions of the shifts in multiple and/or pseudo Fock spaces to the finite dimensional invariant spaces, which correspond, in a physical language, to the states with a fixed number of particles. The boundary value problem for these shifts is shown to be equivalent to the linear stochastic evolution in multiple and/or pseudo Fock spaces driven by Poisson processes. Equivalently, it can be presented as the evolution described by certain (secondly quantised) operators in multiple Fock spaces. Due to the theory developed in [12], [13], these results lead to the representation of the solutions to general stochastic and quantum stochastic evolutions in terms of boundary value problems. This is explained at the end of Section 6.

In Section 7 we show how the toy model Hamiltonians and the corresponding boundary value problems can be obtained by a sort of semiclassical limit $h \to 0$ (which is however quite different from the usual semiclassical limit for stochastic equations [24], [31], and which generalises the ultra-relativistic limit of [10]) from rather general Schrödinger problems with a bounded below Hamiltonian.

2 Main notations

(i) *General notations.* For a function ϕ on $\mathbb{R}$ we shall denote by $\phi(z_-)$ (respectively $\phi(z_+)$) the left (respectively the right) limit of $\phi(t)$ as $t \to z$ (when it exists, of course). As usual, $\delta(t) = \delta_0(t)$ denotes the standard Dirac δ-function, and $\delta_z(t) = \delta(t - z)$.

For a subset $M \subset \mathbb{R}^n$ we shall denote by $\chi_M(z)$ the indicator function of M that equals 1 or 0 respectively when $z \in M$ or $z \in \mathbb{R}^n \setminus M$.

For a (usually complex) Banach space B with the norm $\| \|_B$ and for $M \subset \mathbb{R}^n$, we denote by $\mathcal{L}(B)$ the space of bounded operators in B. If $p \geq 1$, we denote by $L^p_B(M)$ the Banach space of B-valued functions $\psi: M \to B$ with the norm

$$\|\psi\| = \left(\int_M \|\psi(z)\|^p_B \, dz \right)^{1/p}.$$

If B is a Hilbert space H, then $L^2_H(M)$ is clearly a Hilbert space as well, which can be identified with the Hilbert tensor product $H \otimes L^2(M)$.

We shall also consider the local L^p spaces, namely the locally convex topological spaces $L^{p,loc}_B(M)$ of measurable functions $\psi : M \mapsto B$ (more precisely, equivalence classes of such functions) defined by the countable set of norms

$$\|\psi\|_N = \left(\int_{M \cap \{z:|z| \leq N\}} \|\psi(z)\|^p_B \, dz \right)^{1/p},$$

with N being an arbitrary positive integer.

The notation L^p_B (respectively $L^{p,loc}_B$) is reserved for the space $L^p_B(\mathbb{R})$ (respectively $L^{p,loc}_B(\mathbb{R})$).

We shall denote by $C_B(M)$ (resp. $C^{comp}_B(M)$) the space of bounded continuous functions $M \mapsto B$ (respectively with a compact support) equipped with the usual sup-norm

$$\|\psi\| = \sup_z \|\psi(z)\|_B. \tag{1}$$

We shall usually denote by the same letter, A say, a function $z \mapsto A(z)$ on $M \subset \mathbb{R}^n$ with values in (perhaps unbounded) linear operators in B and the corresponding multiplication operator $\varphi \mapsto A\varphi$ in $L^p_B(M)$ defined as $(A\varphi)(z) = A(z)\varphi(z)$ for $\varphi \in L^p_B(M)$. The domain of such an operator consists of those φ for which $\varphi(z)$ belongs to the domain of $A(z)$ for almost all z and the integral $\int_M \|A(z)\varphi(z)\|^p \, dz$ is finite.

(ii) *Shifts and reflection on the line.* We shall denote by R the reflection operator in L^2_B given by the formula

$$R\varphi(z) = \varphi(-z). \tag{2}$$

Next, it is well known that absolutely continuous functions $\varphi(z)$ on the line are almost everywhere differentiable with locally integrable derivatives. We shall denote by $D_1 = D_1(p, B)$ the dense subspace of L^p_B consisting of absolutely continuous functions whose derivatives belong to L^p_B. The operator $\partial = \partial_z$ defined on the domain $D_1(p, B)$ generates the shift $T(t)$ in L^p_B, which is a continuous group of norm preserving transformations:

$$(T(t)\varphi)(z) = (\exp\{t\partial_z\}\varphi)(z) = \varphi(z + t).$$

In particular, in the most important case of the Hilbert space L^2_H, the operator $i\partial$ is clearly self-adjoint and the shift $T(t)$ is a unitary group.

(iii) *Dressing.* Let $-iE$ be a (closed and densely defined) generator of a continuous group $\exp\{-itE\}$ in B, and let A be an operator in $L^p_B(M)$. For a real-valued continuous function $f(z)$ on M we can define an operator in $L^p_B(M)$ by the formula

$$A_{Ef(z)}\varphi(z) = e^{iEf(z)}Ae^{-iEf(z)}\varphi(z). \tag{3}$$

Clearly, if E is a self-adjoint operator in a Hilbert space H, and if A is self-adjoint in $L^2_H(M)$, then $A_{Ef(z)}$ is also self-adjoint in $L^2_H(M)$.

(iv) *Operators describing jumps.*

For a function $A: M \mapsto \mathcal{L}(B)$ and a Borel subset $s \subset M$ we define a bounded operator A^s in $L^p_B(M)$ by the formula

$$(A^s\varphi)(z) = \chi_s(z)A(z)\varphi(z) + (1 - \chi_s(z))\varphi(z) \tag{4}$$

(more correct, but more heavy notation for A^s would be of course A^{χ_s}). Clearly, the operator A^s remains the same if s is changed on a set of Lebesgue measure zero. If $A(z)$ is a unitary operator in a Hilbert space H for all z, then clearly A^s is unitary in $L^2_H(M)$.

The following two simple formulas are often used in what follows:

$$T(t)A^s = A^{s-t}T(t)$$

and

$$(A^s)_{Ef(z)} = (A_{Ef(z)})^s.$$

In particular, the last formula implies that the notation $A^s_{Ef(z)}$ is not ambiguous.

3 Boundary-value problems for shifts on the line and δ-potentials

Here we are going to discuss the properties and various representations of the solutions to the Cauchy problem of the following differential equation with boundary conditions

$$i\partial_t\varphi = (i\partial_z + E)\varphi, \quad \varphi(0_-) = \sigma\varphi(0_+), \quad \varphi \in L^p_B, \tag{5}$$

where $\sigma \in \mathcal{L}(B)$ and $-iE$ is a generator of a continuous group $\exp\{-itE\}$ in a Banach space B.

The main example is, of course, when B is a Hilbert space H, E is a self-adjoint operator in H and $p = 2$. It turns out however that even for the analysis of the diffusion type stochastic equations in Hilbert spaces one needs the boundary value problem in Banach spaces L^p with $p \neq 2$.

We start with the simplest case $E = 0$. Let $D_\sigma = D_\sigma(p, B)$ denote the dense subspace of L^p_B consisting of the functions $\varphi: \mathbb{R} \to B$ such that the restrictions $\varphi|_{(-\infty,0]}$ and $\varphi|_{[0,\infty)}$ can be chosen to be absolutely continuous, the restrictions of the derivatives $\varphi'|_{(-\infty,0]}$ and $\varphi'|_{[0,\infty)}$ (which exist almost everywhere for any absolutely continuous function) belong to $L^p_B(\mathbb{R}_-)$ and $L^p_B(\mathbb{R}_+)$ respectively, and $\varphi(0_-) = \sigma\varphi(0_+)$. Notice that the value $\varphi(0)$ itself is irrelevant, since we are working in L^p_B. Let us denote by $\partial^\sigma = \partial^\sigma_z$ the operator of differentiation restricted to D_σ. This notation is clearly consistent with our previous notation D_1 for the domain of the standard differentiation operator $\partial_z = \partial^1_z$.

Proposition 1 *For any $p \geq 1$, the operator ∂^σ in L^p_B is the generator of the continuous semigroup $T^\sigma(t) = \exp\{t\partial^\sigma)\}$, $t \geq 0$, in L^p_B, which acts by the formula*

$$(T^\sigma(t)\varphi)(z) = (T(t)\sigma^{[0,t]}\varphi)(z) = \begin{cases} \varphi(z+t), & z \notin [-t,0) \\ \sigma\varphi(z+t), & z \in [-t,0) \end{cases}. \tag{6}$$

Proof. Clearly the operators (6) define a continuous semigroup, and the domain D_σ is invariant under its action. To prove the statement, we need to show that for $\varphi \in D_\sigma$

$$\int_{-\infty}^{\infty} \|\frac{1}{t}((T^\sigma(t)-1)\varphi)(z) - \partial^\sigma_z\varphi(z)\|^p \, dt \to 0$$

as $t \to 0$. We decompose this integral into the sum of three integrals over the domains $(-\infty,-t)$, $[-t,0]$, and $(0,\infty)$ respectively. Due to the properties of the "free" (i.e. without boundary conditions) operator ∂_z, the first and the third integrals tend to zero as $t \to 0$, because the functions under these integrals do not cross the boundary at the origin. For $z \in [-t,0]$ the lower line in (6) is applicable. Hence, it remains to prove that

$$\int_{-t}^{0} \|\frac{1}{t}(\sigma\varphi(z+t) - \varphi(z)) - \varphi'(z)\|^p \, dt \to 0 \tag{7}$$

as $t \to 0$. To this end let us define a new function $\tilde{\varphi}(z)$ that equals $\sigma\varphi(z)$ (respectively $\varphi(z)$) for $z > 0$ (respectively $z < 0$). The integral in (7) equals

$$\int_{-t}^{0} \|\frac{1}{t}(\tilde{\varphi}(z+t) - \tilde{\varphi}(z)) - \tilde{\varphi}'(z)\|^p \, dt,$$

which tends to zero as $t \to 0$, because $\tilde{\varphi}$ is absolutely continuous (due to the definition of D_σ), and therefore one can again apply the corresponding well known property of the "free" operator ∂_z.

As usually one says that the semigroup $T^\sigma(t)$ solves the Cauchy problem for equation (5) with $E = 0$ in the sense that if $\varphi \in D_\sigma$ then $T^\sigma(t)\varphi \in D_\sigma$ for all $t \geq 0$ and the function $(T^\sigma(t)\varphi)(z)$ satisfies (5) (with $E = 0$) and has the initial condition φ at $t = 0$.

Remark. One can not directly extend Propostion 1 to the case of the Banach space L^∞_B of bounded functions $\mathbb{R} \mapsto B$ with the norm (1) (or to the corresponding space of equivalence classes up to Lebesgue measure zero), because the standard shift $T(t)$ is not continuous in this spaces. On the other hand, though this shift is continuous in (the completion of) $C_B^{comp}(M)$, the jumps can not be described in this space. In order to overcome this obstacle, we can define a new space depending explicitly on σ, namely, the Banach space $C_{B,\sigma}^{comp}(M)$, which differs from $C_B^{comp}(M)$ by the property that the functions $\varphi \in C_{B,\sigma}^{comp}(M)$ are not continuous anymore at the origin, but instead they have there right and left limits such that $\varphi(0_-) = \sigma\varphi(0_+)$. One easily sees that the results of Proposition 1 are valid in the completion of the space $C_{B,\sigma}^{comp}(M)$.

Since formally

$$e^{\pm iEz}(i\partial_z)e^{\mp iEz} = i\partial_z \pm E,$$

one can consider the dressed operator

$$(i\partial_z^\sigma)_{Ez} = e^{\pm iEz}(i\partial_z^\sigma)e^{\mp iEz}$$

defined on the domain $D_\sigma^{\pm E} = e^{\pm iEz}D_\sigma(p,B)$, as a rigorous version (or extension) of the operator $i\partial_z^\sigma \pm E$. From this observation one easily obtains the following properties of the operator $i\partial_z^\sigma + E$.

Proposition 2 *(i) For an arbitrary $p \geq 1$, the operator $\partial_z^\sigma - iE$ generates the continuous semigroup*

$$T_{Ez}^\sigma(t) = e^{iEz}T^\sigma(t)e^{-iEz} = T_{Ez}(t)(\sigma_{Ez})^{[0,t]} \tag{8}$$

in L_B^p, which solves the Cauchy problem for equation (5). *Moreover,*

$$(T(t)^{-1}T_{Ez}^\sigma(t) = e^{-iEt}\sigma_{Ez}^{[0,t]}. \tag{9}$$

(ii) The operator $\partial_z^\sigma - iE$ generates a semigroup of invertible operators if and only if σ has an inverse $\sigma^{-1} \in \mathcal{L}(H)$. In the latter case, $\sigma_{Ez}^{(-\infty,0]}D_1^E = D_\sigma^E$ and

$$T_{Ez}^\sigma(t) = \sigma_{Ez}^{(-\infty,0]}T_{Ez}(t)(\sigma_{Ez}^{(-\infty,0]})^{-1}. \tag{10}$$

In particular, all ∂^σ with invertible σ are similar.

(iii) If B is a Hilbert space H and E is a self-adjoint operator in H, then $i\partial_z^\sigma + E$ is a self-adjoint operator in L_H^2.

(iv) The operators (8) *form a continuous semigroup also in the locally convex spaces $L_B^{p,loc}$, where both formulas* (9), (10) *remain valid.*

Proof. Formula (8) follows directly from Proposition 1. Equation (9) follows from (8) and a simple observation that $T(t)^{-1}T_{Ez}(t) = e^{-iEt}$. Formula (9) implies the continuity of all operators $T_{Ez}^\sigma(t)$. Other statements are straightforward for $E = 0$ and for general case are obtained by dressing.

Remark. Using the physical language, one can say that formula (9) gives the solutions of equation (5) in the interaction representation with respect to the "free" shift $T(t)$.

We conclude that imposing boundary conditions to the dressed shift semigroup $T_{Ez}(t)$ is equivalent to its dressing by means of the operators $\sigma_{Ez}^{(-\infty,0]}$. This is a specific feature of the first order operator ∂_z, which does not hold for other differential operators. Hence, if one wants to present the evolution $T_{Ez}^{\sigma}(t)$ as a limit of the evolutions defined by other (more physical) pseudo-differential generators, one needs first to approximate $i\partial_z$ itself and then to dress the corresponding evolution. This will be done in Section 7.

Remark. Let us discuss shortly the connection with the theory of singular interactions. Suppose one wants to give a rigorous meaning to the formal (singular) symmetric operator

$$i\partial_z + L\delta(z), \tag{11}$$

where L is a self-adjoint operator in H. There are several ways to tackle this problem (see e.g. the books [AGHK], [Kosh] for a general theory in case $H = \mathbb{C}$). One of them is based on the observation that the operator (11) must coincide with $i\partial$ on the domain $\tilde{D} = \{\varphi \in D_1 : \quad \varphi(0) = 0\}$. Therefore, in order to define the operators of form (11) one must look for the possible extensions of the symmetric operator $i\partial$ defined on $\tilde{D}$. It is not difficult to prove that all self-adjoint extentions of this operator are given by the operators ∂_z^{σ} defined above with all possible unitary operators σ in H. One can show [10] that the "right" extension corresponding to the formal expression (11) is given by $\sigma = e^{-iL}$.

Let us describe another representation for the evolution $T(t)^{\sigma}$, which is more physical in the sense that it is symmetric with respect to the inverse of time combined with complex conjugation. To this end, let us note that an element $\varphi \in L_B^p$ can be uniquely described by a pair of functions (φ^-, φ^+) (input and output) on the half-line $\mathbb{R}^+$, defined for $z > 0$ by the formulas $\varphi^{\pm}(z) = \varphi(\mp z)$. Clearly the problem (5) is equivalent to the problem given by the system

$$\begin{cases} i\partial_t \varphi^- = (E + i\partial_z)\varphi^- \\ i\partial_t \varphi^+ = (E - i\partial_z)\varphi^+ \end{cases} \tag{12}$$

combined with the boundary condition $\varphi^+(0) = \sigma\varphi^-(0)$. This representation already enjoys the required symmetry, namely, if E is real (here we suppose that B itself is given as a certain Banach space of complex functions over a measurable space), then inverting the time $t \to -t$ and taking complex conjugation will transform the first equation of (12) into the second one and vice versa. However, we like to have a representation which allows the input and output wave functions to propogate freely in both directons of $\mathbb{R}$ through the origin $z = 0$.

For this discussion, let us restrict ourselves to the most important example of B being a Hilbert space H and σ being a unitary operator. Let

$$H_\sigma = \{(\psi^-, \psi^+) \in L_H^2 \oplus L_H^2 \colon \quad \psi^+(-z) = (\sigma_{Ez})\psi^-(z)\}$$

where notation (2) was used. The map $U_\sigma\colon L^2_H \mapsto H_\sigma$ which takes $\varphi \in L^2_H$ to the pair (ψ^-, ψ^+) given by

$$\begin{cases} \psi^-(z) = (\sigma^{-1}_{Ez})^{(-\infty,0]}\varphi(z) \\ \psi^+(-z) = (\sigma_{Ez})^{[0,\infty)}\varphi(z) \end{cases}. \tag{13}$$

is an isometric isomorphism. The following statement can be checked directly.

Proposition 3 *The isometric operator U_σ takes the evolution $T^\sigma_{Ez}(t)$ into the free evolution*

$$(\psi^-, \psi^+)(z) \mapsto (T_{Ez}(t)\psi^-(z), T_{-Ez}(-t)\psi^+(z)) \tag{14}$$

(which gives the solution to the Cauchy problem for system (12) *without boundary conditions), restricted to H_σ. The restrictions of $\psi^\mp$ on the positive half-line coincide with the evolution given by* (12) *on the pair (φ^-, φ^+).*

4 Single-kick equation

For a given $z > 0$, let us consider the single kick equation

$$d\eta + iE\eta\, dt = (\sigma - 1)\eta\, d\chi_{(z,\infty)}(t), \quad \eta \in B, \quad t \geq 0, \tag{15}$$

where $\eta_0 = \eta|_{t=0}$ is a given vector from the Banach space B and the operator $-iE$, as usual, is a generator of a continuous group of linear operators in B. As in the case of stochastic equations, this equation should be understood rigorously as the corresponding integral equation, where for any function $f(t)$ having everywhere right and left limits the integral with respect to $d\chi_{(z,\infty)}(t)$ is defined by the formula

$$\int_0^t f(\tau)\, d\chi_{(z,\infty)}(\tau) = \begin{cases} 0, & t \leq z, \\ f(z_-), & t > z \end{cases} \tag{16}$$

Here we have chosen to consider the left contionuous version of the solution. We note however that since we are interested in the solutions in L^p-sense, the difference between the left and right continuous versions is not essential for our purposes. For continuous $f(t)$ the integral (16) is just a standard Stieltjez integral.

One can consider equation (15) as a rigorous version of the evolutionary equation with singular in time non-homogeneous potential

$$\frac{\partial \eta}{\partial t} = -iE\eta + \delta_z(t)(\sigma - 1)\eta.$$

This form of equation (15) can also be made rigorous by using the left and right δ-functions as discussed in [18].

If z is a (positive) random variable $z = z(\omega)$ on a certain probability space Ω, equation (15) can be written in the form of a simplest Ito's type stochastic equation. Namely, the formula $1_t = 1_t(z) = \chi_{(z,\infty)}(t)$ defines a stochastic process on Ω (which describes a jump at a random time z and which is clearly a left continuous process with trajectories of a finite variation), and equation (15) can be written in the form

$$d\eta + iE\eta\, dt = (\sigma - 1)\eta\, d1_t, \quad \eta \in B, \quad t \geq 0. \tag{17}$$

For $t \neq z$, the evolution (15) coincides with the evolution given by the deterministic equation

$$\frac{\partial \eta}{\partial t} = -iE\eta, \tag{18}$$

and at the time $t = z$ the wave function jumps $\eta \mapsto \eta + (\sigma - 1)\eta = \sigma\eta$. Therefore, for $\eta_0 \in B$, the solution $V(t, z)\eta_0$ of equation (15) is given by the formula

$$V(t, z)\eta_0 = \begin{cases} \exp\{-iEt\}\eta_0, & t \leq z \\ \exp\{-iE(t - z)\}\sigma \exp\{-iEz\}\eta_0, & t > z. \end{cases} \tag{19}$$

Comparing this formula with (9) gives the following result.

Theorem 1 *Solution* (19) *to the Cauchy problem of equation* (15) *or* (17) *can be written in the form*

$$V(t, z)\eta_0 = ((T(t)^{-1}T^{\sigma}_{Ez}(t)\varphi)(z), \tag{20}$$

where $\varphi(z) = \eta_0$ *for all* $z > 0$ *(the values of* φ *for negative* z *are easily seen to be irrelevant, one can put* $\varphi(z) = 0$ *for* $z \leq 0$*, say).*

Thus the solutions to the simplest stochastic equation (15) are given as the interaction representation for the solutions of the simplest boundary value problem (5).

Remark. The function $\varphi(z)$ which equals to a constant vector η_0 for all positive z does not belong to L^p_B, but only to $L^{p,loc}_B$, which was the main reason for introducing these spaces. However, if one wants to work in genuine L^p-spaces, one can consider the evolution $V(t, z)$ only till a certain (arbitrary large) time t_0. Then one can take $\varphi(z)$ such that it equals η_0 for $z \in [0, t_0]$ and vanishes otherwise. More generally, one can take the initial function φ to be $\eta_0\rho(z)$ for a certain cut-off function ρ on the line and then make a corresponding simple change in (20), as it was done in [11].

5 Stochastic equations driven by a Poisson noise as the interaction representations for boundary value problems

Here we shall generalise the results of the previous section to more general equations containing an arbitrary number of kicks. We shall denote by Σ_n the infinite simplex:

$$\Sigma_n = \{z = (z_1, ..., z_n) \in \mathbb{R}^n : \quad z_1 < z_2 < ... < z_n\},$$

equipped with Lebesgue measure. Clearly this simplex can be decomposed into the union of $n+1$ cells Σ_n^k:

$$\begin{aligned} \Sigma_n^0 &= \{z \in \Sigma_n : \quad z_1 \geq 0\}, \\ \Sigma_n^k &= \{z \in \Sigma_n : \quad z_k \leq 0 \leq z_{k+1}\}, \quad k = 1, ..., n-1, \\ \Sigma_n^n &= \{z \in \Sigma_n : \quad z_n \leq 0\}. \end{aligned} \tag{21}$$

Vectors $z \in \Sigma_n$ are usually identified with the subsets $\zeta = \zeta(z) = \{z_1, ..., z_n\} \subset \mathbb{R}$ of the real line of cardinality $|\zeta| = n$. The representation of the points of Σ_n by the subsets of $\mathbb{R}$ (respectively by n-dimensional vectors with ordered coordinates) is more natural for defining stochastic processes (respectively, boundary value problems) we are dealing with.

let $\sigma = \{\sigma_1, ..., \sigma_n\}$ be an arbitrary (ordered) family of operators from $\mathcal{L}(B)$. For a given $z \in \Sigma_n^0$, let us consider the following multiple kick generalisation of equation (15)

$$d\eta + iE\eta\, dt = \sum_{j=1}^{n} (\sigma_j - 1)\eta\, d\chi_{(z_j,\infty)}, \quad \eta \in B. \tag{22}$$

If $z \in \Sigma_n^0$ is a random variable on a probability space Ω, one can (as in the case of a single kick equation (15)) to rewrite equation (22) as the stochastic equation

$$d\eta + iE\eta\, dt = (\sigma_{n_t} - 1)\eta\, dn_t, \quad \eta \in B, \quad t \geq 0, \tag{23}$$

driven by the counting process $n_t(\zeta) = n_t(\zeta(z)) = |\zeta \cap [0, t)|$.

As in the case of a single kick, it follows that $\eta(t)$ satisfying (22) evolves according to the free equation (18) between the jump-times z_k, and at the times $t = z_k$ the wave function experiences the jump $\eta \mapsto \sigma(z_k)\eta$. This proves the following statement.

Proposition 4 *For any $z \in \Sigma_n^0$, the operator $V(t, z)$, which gives the solution $V(t, z)\eta_0$ to the Cauchy problem for equation* (22) *with the initial function η_0, belongs to $\mathcal{L}(B)$ and has the following explicit form: for $z_k \leq t < z_{k+1}$*

$$V(t, z) = \exp\{-iE(t - z_k)\}\sigma_k \exp\{-iE(z_k - z_{k-1})\}\sigma_{k-1} ... \sigma_1 \exp\{-iEz_1\}. \tag{24}$$

If B is a Hilbert space H and if all σ_j are unitary, then $V(t, z)$ is also a unitary operator.

In order to give a representation for this operator similar to that given to equation (15) we need to generalise slightly the results of Section 3. Let us write shortly $L_B^p(n)$ for the Banach space $L_B^p(\Sigma_n))$ of B-valued functions on Σ_n. By

$T_n(t) = T_{n,p,B}(t)$ we shall denote the shift in $L^p_B(n)$ which is generated by the operator $\partial_{z_1} + ... + \partial_{z_n} = \partial_z$ and which takes a function $\varphi \in L^p_B(n)$ to the function $(T_n(t)\varphi)(z_1, ..., z_n) = \varphi(z_1 + t, ..., z_n + t)$.

To find explicitly the domain of the generator of the shift $T_n(t)$, let us introduce the new coordinates $x = (x_1, ..., x_n)$ in Σ_n by the formula $x_1 = z_1 + ... + z_n$, $x_2 = z_2 - z_1$, ... $x_n = z_n - z_{n-1}$. Then

$$\Sigma_n = \{x : \quad x_j > 0,\, j = 2, ..., n\} = \mathbb{R} \times (\mathbb{R}_+)^{n-1}.$$

Hence $L^p(\Sigma_n) = L^p(\mathbb{R}) \otimes L^p(\mathbb{R}^{n-1}_+)$ and $L^p_B(n) = L^p_{\tilde{B}}$ with $\tilde{B} = L^p_B(\mathbb{R}^{n-1}_+)$. In this representation the operator $\partial_z = \partial_{z_1} + ... + \partial_{z_n}$ takes the form $n\partial_{x_1}$ in $L^p_{\tilde{B}}$. Consequently, the domain of this operator is given by absolutely continuous functions $\varphi: \mathbb{R} \mapsto \tilde{B}$ from $L^p_B(n)$ such that $\partial_{x_1}\varphi \in L^p_B(n)$.

The multi-dimensional analogue of the problem (5) is the equation

$$i\partial_t\varphi = (i\partial_z + E)\varphi = (i(\partial_{z_1} + \partial_{z_2} + ... + \partial_{z_n}) + E)\varphi, \quad \varphi \in L^p_B(n), \qquad (25)$$

combined with the boundary conditions

$$\varphi(z_1, ..., z_{k-1}, 0_-, z_{k+1}, ..., z_n) = \sigma_k\varphi(z_1, ..., z_{k-1}, 0_+, z_{k+1}, ..., z_n), \quad k = 1, ..., n. \qquad (26)$$

Let $D_\sigma = D_{\sigma_1,...,\sigma_n}(p, B)$ denote the dense subspace of functions $\varphi \in L^p_B(n)$ with the properties:

(i) for each $k = 0, ..., n$ the restriction $\varphi|_{\Sigma^k_n}$ has a continuous version such that on all lines parallel to the vector $(1, ..., 1)$ this restiction is absolutely continuous and

$$n\partial_{x_1}\varphi|_{\Sigma^k_n} = (\partial_{z_1} + ... + \partial_{z_n})\varphi|_{\Sigma^k_n} \in L^p_B(n).$$

(ii) the boundary conditions (26) are satisfied.

Let us define the operator

$$\partial^\sigma_z = \partial^{\sigma_1}_{z_1} + ... + \partial^{\sigma_n}_{z_n} = n\partial^{\sigma_1,...\sigma_n}_{x_1}$$

as the closure of the differentiation operator $n\partial_{x_1}$ defined on the domain D_σ. The following result is a direct generalisation of Proposition 1.

Proposition 5 *For an arbitrary $p \geq 1$, the operators $T^\sigma(t)$ defined by the formulas*

$$T^\sigma(t) = \sigma_n^{\{z_n\in[-t,0]\}}...\sigma_1^{\{z_1\in[-t,0]\}}T_n(t) = T_n(t)\sigma_n^{\{z_n\in[0,t]\}}...\sigma_1^{\{z_1\in[0,t]\}} \qquad (27)$$

form a continuous semigroup of operators in $L^p_B(n)$ with the generator $\partial^{\sigma_1}_{z_1} + ... + \partial^{\sigma_n}_{z_n}$. The subspace D_σ is invariant under the action of the semigroup.

However, unlike the one-dimensional case, we can not define (apart from the trivial case of commuting E and σ) the operator $i\partial^\sigma_z + E$ by dressing the "free" operator $i\partial_z^\sigma$, because such a dresing would inevitably destroy our boundary conditions (26). Instead, we shall define this operator directly as follows. Clearly the domain

$$D_\sigma^E = D_\sigma \cap \{\varphi : E\varphi \in L_B^p(n)\}$$

is a dense subspace in $L_B^p(n)$ for any p, and the operator

$$i\partial_z^\sigma + E = i(\partial_{z_1}^{\sigma_1} + ... + \partial_{z_n}^{\sigma_n}) + E \tag{28}$$

is defined on D_σ^E in the obvious way. By usual abuse of notations, we shall denote by the same symbol $i\partial_z^\sigma + E$ the closure of this operator (defined originally on D_σ^E).

Proposition 6 *(i) The operator* (28) *generates a continuous semigroup*

$$U_E^\sigma(t) = (\sigma_n^{\{z_n \in [-t,0]\}})_{Ez_n} ... (\sigma_1^{\{z_1 \in [-t,0]\}})_{Ez_1} \exp\{-iEt\} T_n(t), \tag{29}$$

(where we used notations introduced in (3)*) which solves the Cauchy problem for equations* (25), (26). *Moreover,*

$$(T_n(t))^{-1} U_E^\sigma(t) = e^{-iEt} (\sigma_n^{\{z_n \in [0,t]\}})_{Ez_n} ... (\sigma_1^{\{z_1 \in [0,t]\}})_{Ez_1}. \tag{30}$$

(ii) The operators (29) *are invertible for all* $t \geq 0$ *if and only if all* σ_k, $k = 1, ..., n$, *are invertible. Operators* (28) *with invertible* σ_k *are similar. More precisely, if* σ_k *has a continuous inverse* σ_k^{-1}, $k = 1, ..., n$, *then*

$$U_E^\sigma(t) = (\sigma_n^{\{z_n<0\}})_{Ez_n} ... (\sigma_1^{\{z_1<0\}})_{Ez_1}$$

$$\times \exp\{-iEt\} T_n(t) \left((\sigma_1^{\{z_1<0\}})_{Ez_1} \right)^{-1} ... \left((\sigma_n^{\{z_n<0\}})_{Ez_n} \right)^{-1}. \tag{31}$$

(iii) If B *is a Hilbert space* H *and all* σ_k *are unitary, the operator* (28) *is self-adjoint in* $L_H^2(n)$.

(iv) All statements of the Proposition remain valid in the spaces $L_B^{p,loc}(n)$.

Proof. Follows by a straightforward verification.

Let us stress again that in the case $n = 1$ the evolution U_E^σ coincides with the evolution T_{Ez}^σ obtained from T^σ by dressing. But this is not the case in general.

Comparing formulas (24) and (30) yields the following multiple-kick version of Theorem 1.

Theorem 2 *Solution* (24) *to the Cauchy problem of equation* (22) *can be written in the form*

$$V(t, z)\eta_0 = ((T_n(t)^{-1} U_E^\sigma(t)\varphi)(z), \tag{32}$$

where $\varphi(z) = \eta_0$ *for* $z \in \Sigma_n^0$ *and vanishes otherwise. The Remark given after Theorem 1 applies here as well.*

Notice that stochastic linear equations driven by a compound Poisson noise, in particular, equations of type (81) (see Appendix A) describing the aposterior dynamics of quantum states under continuous observations, can be reduced to equation of type (22) or (23) pathwise, because a Poisson process has almost surely a finite number of jumps on each bounded time interval. Therefore, the solutions to these stochastic equations can thus be obtained as the interaction representation of the solutions of problem (25), (26) with respect to the "free" shift T_n. In Section 7 it is shown that the model (25), (26) in its turn can be obtained as a semiclassical limit of Schrödinger evolutions with a bounded below Hamiltonian. Moreover, as linear stochastic equations driven by Wiener process can be obtained as a limit of equations driven by Poisson process (see Appendix A for a concrete case of quantum filtering equation), the solutions to these equations can be obtained as a limit of the solutions of the boundary value problems considered above. In the same way one can consider the equations driven by more general Lévy noises, as indicated in the Remark below.

In order to obtain the solution of stochastic equations driven by Wiener process (or more general stochastic or quantum stochastic noise) directly in terms of an appropriate boundary value problem (without any limiting procedure), we shall generalise, in the next Section, the theory developed so far for Σ_n to the case of "coloured" simplices.

Remark. Let us sketch here a generalisation of representations (24), (32) for equation of type (81), if the underlying noise is a more general pure jump Lévy process with possibly infinite number of jumps. Namely, let $X(t)$ be a pure jump Lévy process constructed from the Lévy measure ν on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n \setminus \{0\}} \min(1, |x|)\nu(dx) < \infty.$$

From this condition it follows that

$$X(t) = \sum_{0<z\leq t} (\Delta X_z)$$

where the summation is carried out over all moment of jumps $z \leq t$, $y_z = \Delta X_z$ are the sizes of these jumps, and the sum is (almost surely) absolutely convergent. Let a process P_λ have form (80) with $G(y) = 1 + O(y^2)$. Then the analogue of formula (24) with infinite number of jumps holds, because the sum $\sum |y_z|^2$ is known to be convergent almost surely for any Lévy process (actually for any semimartingale).

6 General stochastic evolutions and boundary value problems for shifts in coloured simplices and multiple Fock spaces

The aim of this section is two-folds. First, we shall generalise the results of the previous one to the case of evolutions in coloured simplices. Roughly speaking, the difference with the situation considered above will consists in the assumption that the points of a simplex have an additional label (a colour), and that the jumps may not only change the value of a function in a point, but also a colour of this point. Secondly, we shall describe the combinatorics of multiple Fock spaces, which allows one to conclude that the interaction representation for the evolutions given by the boundary value problem for shifts in coloured simplices (considered at the beginning) is given by the secondly quantised operators in multiple Fock spaces, which in their turn, present the solutions to the pure jump stochastic equations (driven by a compound Poisson process).

By a coloured simplex of n particle having m colours we understand the set

$$CS_{n,m} = \cup_{n_1+...+n_m=n} \Sigma_{n_1} \times ... \times \Sigma_{n_m}, \tag{33}$$

where the (disjoint) union is taken over all partitions of the integer number n in the sum of m non-negative numbers (the order is relevant), and where it is assumed that the product is over all non-vanishing n_j. The points of $CS_{n,m}$ can be parametrised either by ordered chains of labeled variables

$$z = \{z^\alpha\} = \{z_1^{\alpha(1)} < ... < z_n^{\alpha(n)}\}, \tag{34}$$

with α being functions $\alpha : \{1,...,n\} \mapsto \{1,...,m\}$ (that label the variables in a standard simplex Σ_n), or by the families of m vector variables

$$\zeta = \{\zeta^1 = (z_1^1,...,z_{n_1}^1),...,\zeta^m = (z_1^m,...,z_{n_m}^m)\}, \tag{35}$$

where the entries of each ζ^j are ordered: $z_1^j < ... < z_{n_j}^j$, each ζ^j can be thus considered either as a vector in Σ_{n_j} or as a subset of $\mathbb{R}$ of cardinality $|\zeta^j| = n_j$, and where the subsets ζ^j are disjoint. We shall use both representations (35) and (34) for the variables parametrising $CS_{n,m}$.

There is a natural projection from $CS_{n,m}$ to the standard (uncoloured) simplex Σ_n, which simply "forgets" the colour. We shall denote by $pr(z)$ (or $pr(\zeta)$) the image of the point (34) (or (35)) under this projection. In particular, $pr(\zeta)$ is just a subset of $\mathbb{R}$ of the cardinality n.

Instead of just a Banach space valued functions (as in the previous sections) we shall now consider the functions on $CS_{n,m}$ with values in a certain complex Banach algebra $\mathcal{A}$ with the unit 1, which, in applications we have in mind, will be the Banach algebra of continuous linear operators in a Banach space B. The topology we shall introduce on these functions looks a bit ugly, but we need, for applications to stochastic calculus, to have different L^p norms for different colours. Let $p = \{p_1 \leq ... \leq p_m\}$ be a non-decreasing set of positive numbers, where $p_1 \geq 1$ and p_m is allowed to be $+\infty$. Let us define a norm on the space

$$C_{\mathcal{A}}^{com}(\Sigma_{n_1} \times ... \times \Sigma_{n_m}) \tag{36}$$

by the formula

$$\|\varphi\|_p = \left(\int_{\Sigma_{n_1}} \left(... \left(\int_{\Sigma_{n_m}} |\varphi(\zeta^1, ..., \zeta^m)|^{p_m} \, d\zeta^m \right)^{p_{m-1}/p_m} ... \right)^{p_1/p_2} d\zeta^1 \right)^{1/p_1}, \tag{37}$$

if p_m is finite, and by the formula

$$\|\varphi\|_p = \left(\int_{\Sigma_{n_1}} \left(... \left(\max_{\Sigma_{n_m}} |\varphi(\zeta^1, ..., \zeta^m)| \right)^{p_{m-1}} ... \right)^{p_1/p_2} d\zeta^1 \right)^{1/p_1}, \tag{38}$$

if $p_m = +\infty$.

The most important case for application is when $m \geq 3$, and $p_1 = 1$, $p_2 = p_3 = ... = p_{m-1} = 2$, $p_m = \infty$. In this case, it is convenient to index the variables ζ as $\zeta^-, \zeta^0, \zeta^+$ which are connected with the previous notations ζ^j by the formulas: $\zeta^- = \zeta^1$, $\zeta^0 = (\zeta^{0,1}, ..., \zeta^{0,m-1})$ with $\zeta^{0,j} = \zeta^{j+1}$, and $\zeta^+ = \zeta^m$. In this case,

$$\|\varphi\|_p = \|\varphi\|_{\{1,2,...,2,\infty\}} = \int_{\Sigma_{n_-}} \left(\int_{\times_{j=1}^{m-1} \Sigma_{n_0^j}} \max_{\Sigma_{n_+}} |\varphi(\zeta^-, \zeta^0, \zeta^+)|^2 \, d\zeta^0 \right)^{1/2} d\zeta^-. \tag{39}$$

We shall denote by $L^p_{\mathcal{A}}(\Sigma_{n_1} \times ... \times \Sigma_{n_m})$ the completion of the space (36) with respect to the norm (37) or (38) and by $L^p_{\mathcal{A}}(CS_{n,m})$ the direct sum of these spaces over all partition $n = n_1 + ... + n_m$ of n. By $L^{p,loc}_{\mathcal{A}}(CS_{n,m})$ we shall denote the corresponding locally convex space defined by the countable set of norms parametrised by the positive integers N and defined by (37) or (38) with all integrations performed not over the whole infinite simplices but over their intersections with the balls of radius N.

We shall use the same notation $T_n(t) = T_{n,p,\mathcal{A}}(t)$ as before for the shift in $L^p_{\mathcal{A}}(CS_{n,m})$ or $L^{p,loc}_{\mathcal{A}}(CS_{n,m})$ that shifts all variables independently of their colours. Clearly, as in the case of the standard simplex (without colours) these operators form a continuous group, whose generator we shall again denote by $\partial = \partial_z = \partial_{z_1} + ... + \partial_{z_n}$. Thus

$$(T_n(t)\varphi)(\{z_1^{\alpha(1)} < ... < z_n^{\alpha(n)}\})$$

$$= (\exp\{t\partial_z\}\varphi)(\{z_1^{\alpha(1)} < ... < z_n^{\alpha(n)}\}) = \varphi(\{(z_1 + t)^{\alpha(1)} < ... < (z_n + t)^{\alpha(n)}\}). \tag{40}$$

Let

$$S = \{S(k)\} = \{(S_{\mu,\nu})(k), \quad k = 1, ..., n\}, \tag{41}$$

where $\mu, \nu = 1, ..., m$, be a family of n block upper triangular $m \times m$-matrices (i.e. $S_{\mu,\nu}$ is allowed not to vanish only if either (i) $\mu \le \nu$ or (ii) $\mu > \nu$ but $p_\mu = p_\nu$) with entries from $\mathcal{A}$, which define the family σ of linear operators σ_k in the space (36) by the formula

$$(\sigma_k \varphi)(z_1^{\alpha(1)}, ..., z_n^{\alpha(n)})$$

$$= \sum_{\nu=\alpha(k)}^{m} S_{\alpha(k),\nu}(k) \varphi(z_1^{\alpha(1)}, ..., z_{k-1}^{\alpha(k-1)}, z_k^{\nu}, z_{k+1}^{\alpha(k+1)}, ..., z_n^{\alpha(n)}). \tag{42}$$

These operators may not be continuous in the spaces $L^p_{\mathcal{A}}(CS_{n,m})$. However, since the matrices $S(k)$ are triangular and since for any $p_1 \le p_2$, the standard L^{p_1} norm of any function on a compact set can be estimated by its L^{p_2} norm, the following statement holds.

Proposition 7 *The operators σ_k are continuous in $L^{p,loc}_{\mathcal{A}}(CS_{n,m})$ for all p.*

Generalising the boundary value problem (25), (26), we are going to consider the equation (25) in $L^p_{\mathcal{A}}(CS_{n,m})$ combined with the boundary conditions

$$\varphi(z_1^{\alpha(1)}, ..., z_{k-1}^{\alpha(k-1)}, 0_-^{\alpha(k)}, z_{k+1}^{\alpha(k+1)}, ..., z_n^{\alpha(n)})$$

$$= (\sigma_k \varphi)(z_1^{\alpha(1)}, ..., z_{k-1}^{\alpha(k-1)}, 0_+^{\alpha(k)}, z_{k+1}^{\alpha(k+1)}, ..., z_n^{\alpha(n)}), \quad k = 1, ..., n. \tag{43}$$

To deal with this problem in the same way as with the problem (25), (26), let us decompose the coloured simplex $CS_{n,m}$ into the union of $n+1$ cells $CS^k_{n,m}$ using the decomposition (21) of the underlying uncoloured simplex Σ_n:

$$CS^k_{n,m} = \{z \in CS_{n,m} : pr(z) \in \Sigma^k_n\}, \quad k = 0, ..., n, \tag{44}$$

and then define the subspaces $D_S = D_S(p, \mathcal{A})$ (respectively D^{loc}_S) of functions $\varphi(z)$ from $L^p_{\mathcal{A}}(CS_{n,m})$ (respectively from $L^{p,loc}_{\mathcal{A}}(CS_{n,m})$) with the properties:

(i) for each $k = 0, ..., n$ and each particion $n = n_1 + ... + n_m$ the restriction of φ on

$$CS^k_{n,m} \cap \Sigma_{n_1} \times ... \times \Sigma_{n_m}$$

has a continuous version,

(ii) for any $k = 0, ..., n$ and z of form (34), the restriction of the function (40) on the cell $CS^k_{n,m}$ is absolutely continuous as a function of t and such that

$$(\partial_z \varphi)(z) = ((\partial_{z_1} + ... + \partial_{z_n})\varphi)(z) = n\partial_t (T_n(t)\varphi)(z)$$

belongs to $L^p_{\mathcal{A}}(CS_{n,m})$ (respectively $L^{p,loc}_{\mathcal{A}}(CS_{n,m})$),

(iii) the boundary conditions (44) are satisfied.

Let us use the same notation ∂_z^S for the closures of the operator ∂_z defined on the domains D_S or D_S^{loc}. We introduced the notations for coloured simplices in such a way that the main formulas of the previous section still make sense in this new framework. It remains only to assume that the use of the operator-valued functions of the variables z without a colour means that the colour is preserved. For example, the action of the operator $\exp\{-iEz_j\}$, say, is given by the formula

$$(\exp\{-iEz_j\}\varphi)(\{z_1^{\alpha(1)} < ... < z_n^{\alpha(n)}\}) = \exp\{-iEz_j^{\alpha(j)}\}\varphi(\{z_1^{\alpha(1)} < ... < z_n^{\alpha(n)}\}).$$

At last, we can now define the operator $i\partial_z^S + E$ quite similarly to the case without colours. Moreover, due to Proposition 7, we get the following

Proposition 8 *Propositions 5 and 6 remain valid for spaces $L_{\mathcal{A}}^{p,loc}(CS_{n,m})$ for all p with finite p_m. If $p_m = \infty$, the same holds under an additional assumption that all elements $S_{mm}(j)$ are units of the algebra $\mathcal{A}$ for all j.*

Remark. The last assumption was necessary, because as we noted earlier the shift is not continuous in the space $L^\infty(\mathbb{R})$. The asumption $S_{mm}(j) = 1$ ensures that there will be no discontinuity in the variables ζ^m. Using the Remark after Proposition 1, one can weaken this assumption, but we take it in its simplest form, which turns out to be natural in applications to stochastic analysis.

At the end of this section, we shall write down the corresponding stochastic equation driven by a Poisson noise which generalises equation (22) and which gives the interaction representation for the shifts in coloured simplices considered above. However, before this, we shall descibe the combinatorics of Fock spaces which allows one to represent our shifts as secondly quantised operators in multiple and/or pseudo Fock spaces. Since these secondly quantised operators solve rather general stochastic (and even quantum stochastic) differential equations (as was discovered in [12], [13], see short comments in the Appendix B), this leads to the representation of general stochastic evolutions in terms of the solutions to the boundary value problems.

We start with the algebraic aspects of the theory. To this end let us choose a certain space l^m of functions from $\mathbb{R}$ to $\mathbb{C}^m$. As the main example of the space l^m, one can have in mind the space $C_{\mathbb{C}^m}^{comp}(\mathbb{R})$ of continuous functions with compact support. Choosing a basis $\{e_j\}$, $j = 1, ..., m$, in $\mathbb{C}^m$ gives the standard isomorphism (algebraic) of l^m and the sum $l^1 \oplus ... \oplus l^1$ of m copies of l^1 by presentating any function $f \in l^m$ as the sum $f = \sum f_j e_j$ with all $f_j \in l^1$. As usual, the tensor product $(\mathbb{C}^m)^{\otimes n}$ is defined as a mn-dimensional vector space with the basis $e_{\alpha(1)} \otimes ... \otimes e_{\alpha(n)}$, parametrised by arbitrary functions $\alpha: \{1, ..., n\} \mapsto \{1, ..., m\}$. The (algebraic) symmetric tensor product $l_n^m = (l^m)_{sym}^{\otimes n}$ can be defined as the space of functions $\Sigma_n \mapsto (\mathbb{C}^m)^{\otimes n}$ generated by the monomials

$$f_1(z_1)...f_n(z_n)e_{\alpha(1)} \otimes ... \otimes e_{\alpha(n)}, \quad f_j \in l^1. \tag{45}$$

It is convenient to get rid of tensors by transfering the index from the basis to the variables and to encode the element (45) by the function $f_1(z_1^{\alpha(1)})...f_n(z_n^{\alpha(n)})$ of n ordered labeled variables. Thus the symmetric tensor product $l_n^m = (l^m)^{\otimes n}_{sym}$ is represented as a space of functions of the variables (34), or, in other words, as a space of functions on the coloured simplex $CS_{n,m}$. We shall call this representation the functional representation for the tensor product l_n^m.

By definition, the infinite (algebraic) direct sum $\oplus_{n=0}^{\infty} l_n^m$ is the (algebraic) symmetric Fock space over l^m, and the space l_n^m is called the n-particle subspace in this context.

If $l^m = L^2_{\mathbb{C}^m}$, then taking the Hilbert direct sum (instead of the algebraic) one obtains the standard symmetric Fock space over l^m. If $l^m = C^{comp}_{\mathbb{C}^m}(\mathbb{R})$, then the completion of l_n^m with respect to the norm (37) or (38) clearly coincides with the space $L^p_{\mathbb{C}}(CS_{n,m})$ considered above. Thus the spaces $L^p_{\mathcal{A}}(CS_{n,m})$ can be considered as n-particle subspaces in the (multiple) Fock space

$$\mathcal{F}^{p_1,\dots,p_m} = \mathbb{C} \oplus L^p_{\mathcal{A}}(CS_{1,m}) \oplus L^p_{\mathcal{A}}(CS_{2,m}) \oplus ..., \tag{46}$$

which is a Banach version of the Fock space over the Banach space $L^{p_1}_{\mathcal{A}}(\mathbb{R}) \oplus ... \oplus L^{p_m}_{\mathcal{A}}(\mathbb{R})$ (where the last term in this sum must be replaced by the completion (with respect to the sup-norm) of the space $C^{comp}(\mathbb{R})$ whenever $p_m = +\infty$). In particular, the shifts $T^{\sigma_1,\dots,\sigma_n}(t)$ can be considered as the restrictions (to the n-particle subspaces) of the corresponding shifts in the Fock space $\mathcal{F}$.

Clearly, each matrix $S(k)$ from a family of the type (41) defines an operator $S(k)\colon l^m \mapsto \mathcal{A} \otimes l^m$ which takes the function $f(z) = \sum f_j(z) e_j$ to the function

$$(S(k)f)(z) = \sum_{\mu,\nu} S_{\mu,\nu}(k) f_\nu(z) e_\mu.$$

We are interested in the tensor product $S(n) \otimes ... \otimes S(1)$, which is defined as the operator in l_n^m that takes the element (45) to the element

$$\sum_{\mu_1,\dots,\mu_n} S_{\mu_n,\alpha(n)}(n)...S_{\mu_1,\alpha(1)}(1) f_1(z_1)...f_n(z_n) e_{\mu_1} \otimes ... \otimes e_{\mu_n}. \tag{47}$$

In particular, if $S(j) = S$ does not depend on j, then $S^{\otimes n}$ is the restriction on the n-particle subspace of the second quantization of the operator S.

Clearly, each monomial in the sum (47) has the form

$$S_{\mu_n,\alpha(n)}(n)...S_{\mu_1,\alpha(1)}(1) f_1(z_1^{\mu_1})...f_n(z_n^{\mu_n}), \tag{48}$$

in the functional representation. Since the monomials (45) form the basis for the space l_n^m, we obtain the following functional representation for the operator $S(n) \otimes ... \otimes S(1)$:

Proposition 9

$$(S(n)\otimes...\otimes S(1)f)(z_1^{\alpha(1)},...,z_n^{\alpha(n)}) = \sum_\beta \prod_k S_{\alpha(k)\beta(k)}(k) f(z_1^{\beta(1)},...,z_n^{\beta(n)}), \quad (49)$$

where the sum is taken over all functions $\beta: \{1,...,n\} \mapsto \{1,...,m\}$ *(since* S *is supposed to be upper triangular, only those* β *must be taken into consideration for which* $\beta(k) \geq \alpha(k)$ *for all* k*), and the* $\prod$ *means the ordered product, where the index* k *decreases from the left to the right.*

Let us write down this formula more explicitly in the two particular cases which are important for applications to stochastic analysis (see Appendix B).

(i) Let $m = 2$ and S has the form $S = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ with a certain $s \in \mathcal{A}$. Using the representation (35), where the two group of variables are denoted by ζ^- and ζ^+, we deduce from (49) that

$$(S^{\otimes n} f)(\zeta^-, \zeta^+) = \sum_{\zeta \subset \zeta^-} \prod_{z \in \zeta} s(z) f(\zeta, \zeta^+ \cup (\zeta^- \setminus \zeta)), \quad (50)$$

where the sum is taken over all subsets ζ of ζ^-.

(ii) If $m = 3$, it is convenient to denote the three groups of variables in the representation (35) by $\zeta^-, \zeta^0, \zeta^+$. Let S has the form

$$S(k) = \begin{pmatrix} 1 & S_0^- & S_+^- \\ 0 & S_0^0 & S_+^0 \\ 0 & 0 & 1 \end{pmatrix} (k). \quad (51)$$

In this case, formula (49) can be written in the form

$$(S^{\otimes n} f)(\zeta^-, \zeta^0, \zeta^+)$$

$$= \sum \prod S_0^-(k_0) S_+^-(k_+) S_0^0(j_0) S_+^0(j_+) f(\zeta_-^-, \zeta_0^- \cup \zeta_0^0, \zeta_+^- \cup \zeta_+^0 \cup \zeta^+), \quad (52)$$

where the sum is taken over all partitions $\zeta^- = \zeta_-^- \cup \zeta_0^- \cup \zeta_+^-$ (here ζ_-^- stands for the part of ζ^- which preserves the colour under the change $\alpha(k) \mapsto \beta(k)$, ζ_0^- stands for the part of ζ^- which is transformed to ζ^0, and ζ_+^- stands for the part of ζ^- which is transformed to ζ^+) and $\zeta^0 = \zeta_0^0 \cup \zeta_+^0$ (here ζ_0^0 stands for the part of ζ^0 which preserves the colour and ζ_+^0 stands for the part of ζ^0 which is transformed to ζ^+) and the ordered product is taken over all $k_0 \in \zeta_0^-$, $k_+ \in \zeta_+^-$, $j_0 \in \zeta_0^0$, $j_+ \in \zeta_+^0$.

(iii) in case $m \geq 3$, $p = 1, 2, ..., 2, \infty$, the formula for $S^{\otimes n}$ is similar to (52) but a bit more lengthy.

Comparing formula (49) with the formula for the boundary value problem for shifts described in Proposition 8 (see, in particular, (27) with σ_k defined in (42)) and by straightforward generalisation of Proposition 4 one obtains the following result, which connects shifts in coloured simplices with pure jump stochastic equations and with secondly quantised operators in multiple (Banach) Fock spaces.

Theorem 3 *Let us introduce a time dependent version of the operator* (49) *which acts only "till the time t", i.e. the operator*

$$(S_t^{\otimes}\varphi)(z_1^{\alpha(1)},...,z_n^{\alpha(n)}) = (S(k(t)) \otimes ... \otimes S(1)\varphi)(z_1^{\alpha(1)},...,z_n^{\alpha(n)}), \tag{53}$$

where $k(t)$ is the largest k such that $pr(z)_k \leq t$. Then

$$((T_n(t))^{-1}T^{\sigma_1,...,\sigma_n}(t)\varphi)(z_1^{\alpha(1)},...,z_n^{\alpha(n)}) = (S_t^{\otimes}\varphi)(z_1^{\alpha(1)},...,z_n^{\alpha(n)}). \tag{54}$$

Moreover, if $\mathcal{A} = \mathcal{L}(\mathcal{B})$, the r.h.s. of (54) *gives the solution to the Cauchy problem for a "coloured" version of the multiple-kick equation* (22) *with $E = 0$, i.e. to the equation*

$$d\varphi = \sum_{j=1}^{n}(\sigma_j - 1)\varphi\, d\chi_{(pr(z)_j,\infty)}, \quad \varphi \in L_B^{p,loc}(CS_{n,m}). \tag{55}$$

A more general case with a non-vanishing E, i.e. the equation

$$d\varphi + iE\varphi\, dt = \sum_{j=1}^{n}(\sigma_j - 1)\varphi\, d\chi_{(pr(z)_j,\infty)}, \quad \varphi \in L_B^{p,loc}(CS_{n,m}). \tag{56}$$

can be considered similarly. As in the previous section, equation (56) can be written as a stochastic equation driven by Poisson noise, if the times $pr(\zeta)$ of jumps are random variables. In fact, introducing, as in the previous section, the counting process $n_t = |pr(\zeta) \cap [0,t)|$ one can rewrite equation (56) in the stochastic form

$$d\varphi + iE\varphi\, dt = (\sigma_{n_t} - 1)\varphi\, dn_t, \quad \varphi \in L_B^{p,loc}(CS_{n,m}). \tag{57}$$

In particular, since the number of jumps of a Poisson process is almost surely finite on each finite interval of time, one can consider the process n_t in (57) to be a standard Poisson process.

Theorem 3 expresses the solutions to pure jump stochastic equations in multiple Fock spaces in terms of the boundary value problems for shifts. As was proven in [12], [13], the general stochastic and even quantum stochastic linear equations can be obtained as the epimorphic projection of such pure jump stochastic equations. Let us recall now how this projection is constructed. First, we discuss the simlest example particular case, which allows one (as can be checked straightforwardly) to repersent a non-stochastic (say, Schrödinger type) evolution by means of a pure jump equation in pseudo Fock space, and secondly, we shall write down the general reduction formula refering to [12] for proofs and details.

In our simplest example, we need the Fock space $\mathcal{F}^{1,\infty}$ defined by (46) with $m = 2$ and $p = 1,\infty$. We shall use the notations for the variables in this Fock space which were already exploited in formula (50). The main role in the reduction is played by the operators $J: H \mapsto H \otimes \mathcal{F}^{1,\infty}$ and $J^\star: H \otimes \mathcal{F}^{1,\infty} \mapsto H$ defined by the formulas

$$(J(\eta))(\zeta^-, \zeta^+) = \eta \delta_\emptyset(\zeta^-) 1(\zeta^+),$$

$$J^\star \varphi = \int \varphi(\zeta^-, \emptyset)\, d\zeta^-,$$

where $\delta_\emptyset(\zeta^-)$ is the indicator function of the vacuum (i.e. it equals one if ζ^- is empty and vanishes otherwise), and $1(\zeta^+)$ is the constant function which equals one for all ζ^+. The integral over ζ^- means (as is usually assumed in calculations with Fock spaces) the sum of the integrals over all finite dimensional simplices Σ_n.

From formula (50) it follows by simple manipulations that for an arbitrary self-adjoint operator E in a Hilbert space H and a vector $\eta \in H$ (η can be arbitrary, if E is bounded and η must be an analytic vector for E in general situation) one has

$$J^\star S_t^\otimes J \eta = e^{-iEt}\eta, \tag{58}$$

where $S_t^\otimes$ is defined by (53) and (50) with $s = -iE$.

The justification of the notations J, $J^\star$ is based on the following observation. If one introduces in the space $L^1_H \oplus C^{comp}_H$ a pseudo scalar product

$$((f^-, f^+)|(g^-, g^+)) = \int ((f^-, g^+)_H(z) + (f^+, g^-)_H(z))\, dz,$$

and then lifts this product to the corresponding multiple Fock space $\mathcal{F}^{1,\infty}$ (equipping it with the structure of a pseudo Fock space), the operators J and $J^\star$ become adjoint with respect to this product.

The procedure which allows the same reduction for general stochastic equations is similar but a bit more involved. First of all, the general reduction uses the Fock space $H \otimes \mathcal{F}^p$, where $p = (p_1, ..., p_m)$ with $p_1 = 1$, $p_2 = ... = p_{m-1} = 2$, $p_m = \infty$, constructed over the Banach space

$$H \otimes L^{1,2,\ldots,2,\infty} = H \otimes (L^1 \oplus L^2 \oplus ... \oplus L^2 \oplus C(\mathbb{R}))$$

The pseudo-scalar product in this space is defined by the formula

$$((f^-, f^0, f^+)|(g^-, g^0, g^+)) = \int ((f^-, g^+)_H(z) + \sum_{j=1}^{m-1} (f^0_j, g^0_j)_H(z) + (f^+, g^-)_H(z))\, dz,$$

which is then naturally lifted to the Fock space $H \otimes \mathcal{F}^p$.

The linear (pseudo) isometry operator $J : H \otimes \mathcal{F}^2 \mapsto H \otimes \mathcal{F}^{1,2,\infty}$ and its (psedo) adjoint are defined now by the formula

$$(J(\psi))(\zeta^-,\zeta^0,\zeta^+) = \delta_\emptyset(\zeta^-)\psi(\zeta^0)1(\zeta^+), \quad J^\star\psi(\zeta) = \int \psi(\zeta^-,\zeta,\emptyset)\,d\zeta^-. \quad (59)$$

It turns out (see [12]) that with these J, $J^\star$ the l.h.s. of (58) solves the linear quantum stochastic equation (see [20])

$$d\eta + S_+^-\eta\,dt = S_0^-\eta\,dA^+(t) + S_+^0\eta\,dA^- + S_0^0\eta\,d\Lambda, \quad (60)$$

where $A^\pm = A_j^{pm}$ with $j = 1,...,m-1$ are the creation and annigilation quantum martingales respectively and Λ is the gauge process.

Therefore, the following result holds.

Theorem 4 *The solution operator for equation* (60) *is given by the formula* $J^\star S_t^\otimes J$ *with* J, $J^\star$ *defined in* (59) *and* S_t *defined by* (52), (53),(54).

Thus the solution to a general quantum stochastic equation is expressed in terms of the boundary value problem in a coloured (pseudo) Fock space. Moreover, as was proved in [5],[30] (see also [27]), any Lévy process can be represented in a Fock space and thus any stochastic equation driven by such a process can be written in the form of a quantum stochastic equation given above.

7 The stochastic dynamics as a semi-classical limit

The aim of this section is to show that the evolutions defined by the boundary value problems for shifts can be obtained as a sort of semiclassical limit of the evolutions defined by a boundary value problem for rather general Schrödinger equations. This completes the description of stochastic evolutions as certain limits of boundary value problems for the standard (deterministic) quantum mechanical equations with physical (real and bounded below) Hamiltonians.

We begin with the notations describing the vector-valued Hardy classes and the pseudo-differential operators (ΨDO) with operator-valued symbols.

By $\mathcal{H}_H^+$ (respectively $\mathcal{H}_H^-$) we denote the H-valued Hardy spaces defined as the spaces of the Fourier transforms of functions from $L_H^2(\mathbb{R}^+)$ (respectively $L_H^2(\mathbb{R}^-)$. In other words,

$$\mathcal{H}_H^\mp = \{\varphi \in L_H^2 : \varphi(z) = \int_{\mathbb{R}^\mp} e^{ikz} f(k)\,dk, \quad f \in L_H^2(\mathbb{R}^\mp)\}.$$

We shall need also the corresponding shifted Hardy classes. Namely, if E is a selfadjoint operator in H, we define

$$\mathcal{H}_H^\mp(E) = \{\varphi \in L_H^2 : \varphi(z) = e^{iEz}\psi(z), \quad \psi \in \mathcal{H}_H^\mp\}. \quad (61)$$

In particular, if ξ is a positive number (which can be considered as the operator of multiplication by ξ in H), then clearly $\mathcal{H}_H^+(\xi)$ (respectively $\mathcal{H}_H^-(\xi)$) coincides with the subspace of functions from L_H^2 which are Fourier transforms of functions with support in $[-\xi, \infty)$ (respectively $(-\infty, \xi]$).

If γ is a measurable function on $\mathbb{R}$ with values in linear operators in H, we shall denote by $\gamma(-i\partial_z)$ the corresponding ΨDO in L_H^2, which (by definition) acts as

$$(\gamma(-i\partial_z)\varphi)(z) = \int_{-\infty}^{\infty} e^{ikz}\gamma(k)f(k)\,dk \tag{62}$$

on the functions φ given by their Fourier transforms as

$$\varphi(z) = \int_{-\infty}^{\infty} e^{ikz} f(k)\,dk, \quad f \in L_H^2. \tag{63}$$

The domain of the operator $\gamma(-i\partial)$ consists of the functions φ of form (63) such that the corresponding f belongs to the domain of the operator of multiplication by $\gamma(k)$. The function $\gamma = \gamma(p)$ is called the symbol of the ΨDO $\gamma(-i\partial_z)$. Choosing a positive parameter h, we shall denote by $\hat{\gamma} = \hat{\gamma}(h)$ the operator

$$\hat{\gamma} = \hat{\gamma}(h) = h^{-1}\gamma(-ih\partial_z). \tag{64}$$

If the opreators $\gamma(p)$ are selfadjoint for all p, then clearly $\hat{\gamma}$ is a selfadjoint operator (on a properly defined domain), since it generates the (obviously unitary) evolution given by the explicit formula

$$\exp\{-it\hat{\gamma}\}\varphi(z) = \int_{\mathbb{R}} e^{ikz}\exp\{-it\gamma(hk)/h\}f(k)\,dk \tag{65}$$

on the functions φ of form (63). This unitary evolution defines the solution to the Cauchy problem of the Schrödinger equation

$$ih\partial_t\varphi = \gamma(-ih\partial_z)\varphi, \quad \varphi \in L_H^2. \tag{66}$$

It is well known (and easy to see) that the dressing of a ΨDO is equivalent to the shift in its symbol. More precisely, if the operators $\gamma(k)$ form a commuting family and if E is a selfadjoint operator in H commuting with all $\gamma(k)$, then

$$(\gamma(-i\partial_z))_{Ez} = e^{iEz}\gamma(-i\partial_z)e^{-iEz} = \gamma(-i\partial_z - E). \tag{67}$$

From now on, let $\varepsilon(p)$ be an even function on $\mathbb{R}$ with values in a set of commuting non-negative selfadjoint operators in H defined on the same dense domain $D \subset H$. Suppose also that the the function $\varepsilon(p)$ has a Lipschitz-continuous derivative outside the origin in the sense that $\varepsilon'(p)$ exist as selfadjoint operators on D for all p, and for an arbitary $\xi > 0$ and an arbitary $v \in D$

$$\|[\varepsilon(\xi + p) - \varepsilon(\xi) - p\varepsilon'(\xi)]v\| = O(|p|^2) \tag{68}$$

uniformly for p from an arbitrary compact interval.

Next, let us fix a unitary operator σ in H. The operators $\hat{\varepsilon}$ and σ describe respectively the free continuous evolution and the jumps of a quantum system.

For an arbitrary selfadjoint operator E in H, which is defined on D and commutes with all $\varepsilon(p)$, and an arbitrary positive number ξ, we define the operators $\omega^{\mp}_{E,\xi} = \omega^{\mp}_{E,\xi}(h)$ in L^2_H by the formula

$$\omega^{\mp}_{E,\xi}(h) = \hat{\varepsilon}_{\pm(E+\xi/h)z}(h) - \varepsilon(\xi)/h, \tag{69}$$

where the notations (3) and (64) were used. Thus, equivalently

$$\omega^{\mp}_{E,\xi}(h) = \frac{1}{h}e^{\pm i(E+\xi/h)z}(\varepsilon(-hi\partial_z) - \varepsilon(\xi))e^{\mp i(E+\xi/h)z}. \tag{70}$$

Due to (67) it follows that $\omega^{\mp}_{E,\xi}(h)$ are ΨDO with the symbol

$$\frac{1}{h}\left(\varepsilon(h(p \mp E) \mp \xi) - \varepsilon(\xi)\right). \tag{71}$$

From (70) it follows that the operators $\omega^{\mp}_{E,\xi}(h)$ are selfadjoint and generate the unitary evolutions by the formula

$$(\exp\{-it\omega^{\mp}_{E,\xi}(h)\}\varphi)(z) = e^{\pm i(E+\xi/h)z}\int_{-\infty}^{\infty} e^{ikz}\exp\{-it(\varepsilon(hk)-\varepsilon(\xi))/h\}f(k)\,dk \tag{72}$$

for φ given respectively as

$$\varphi(z) = e^{\pm i(E+\xi/h)z}\int_{-\infty}^{+\infty} e^{ikz}f(k)\,dk, \quad f \in L^2_H. \tag{73}$$

Equivalently, due to (71) one can write

$$(\exp\{-it\omega^{\mp}_{E,\xi}(h)\}\varphi)(z) = \int_{-\infty}^{\infty} e^{ikz}\exp\{-it[\varepsilon(h(p \mp E) \mp \xi) - \varepsilon(\xi)]/h\}f(k)\,dk \tag{74}$$

for φ given by (63).

The following statement collects the simplest properties of evolutions (72) in the shifted Hardy classes.

Proposition 10 *(i) The space $\mathcal{H}^-_H(E+\xi/h)$ (respectively $\mathcal{H}^+_H(E+\xi/h)$) is invariant under the evolution generated by $\omega^-_{E,\xi}(h)$ (respectively $\omega^+_{E,\xi}(h)$).*

(ii) The operator $R\sigma_{(E+\xi/h)z}$ defines the isometric isomorphism

$$R\sigma_{(E+\xi/h)z} \colon \mathcal{H}^-_H(E+\xi/h) \mapsto \mathcal{H}^+_H(E+\xi/h).$$

(iii) the operator $R\sigma_{(E+\xi/h)z}$ conjugates the evolutions generated by $\omega^{\pm}_{E,\xi}(h)$, namely

$$R\sigma_{(E+\xi/h)z}\exp\{-it\omega^-_{E,\xi}(h)\} = \exp\{-it\omega^+_{E,\xi}(h)\}R\sigma_{(E+\xi/h)z}.$$

(ih) if φ is given by

$$\varphi(z) = e^{i(E+\xi/h)z} \int_{-\infty}^{+\infty} e^{ikz} f(k)\, dk, \quad f \in L_H^2, \tag{75}$$

then

$$(\exp\{-it\omega^+_{E,\xi}(h)\} R\sigma_{(E+\xi/h)z}\varphi)(z)$$
$$= e^{-i(E+\xi/h)z}\sigma \int_{-\infty}^{\infty} e^{-ikz} \exp\{-it(\varepsilon(hk) - \varepsilon(\xi))/h\} f(k)\, dk. \tag{76}$$

Proof. (i) and (ii) follow from (72), (73) and definitions (2), (61). (iii) follows from (iv). To prove (76) we note that for φ of form (75)

$$(R\sigma_{(E+\xi/h)z}\varphi)(z) = e^{-i(E+\xi/h)z}\sigma \int_{-\infty}^{+\infty} e^{ikz} f(-k)\, dk.$$

Hence, the l.h.s. of (76) equals

$$e^{-i(E+\xi/h)z}\sigma \int_{-\infty}^{+\infty} e^{ikz} \exp\{-it(\varepsilon(hk) - \varepsilon(\xi))/h\} f(-k)\, dk,$$

which coincides with the r.h.s. of (76) because the function $\varepsilon(p)$ is even.

Now we are going to prove that the evolutions (72) converge strongly to the unitary evolutions (14) giving the solutions to equations (12).

Theorem 5 *(i) For any $\xi > 0$, and $T > 0$ the evolutions (72) converge strongly to the evolutions $\exp\{-it\varepsilon'(\xi)(E \pm i\partial_z)\}$ as $h \to 0$ uniformly for $t \in [0, T]$.*

(ii) if ε is of linear growth, i.e. if $\varepsilon'(p) = c + O(p^{-2})$ for $p \to \infty$ with some constant operator c, then the convergence is uniform on any shifted Hardy class $\mathcal{H}^{\mp}(E + p)$ for any fixed p. Moreover, one can put the operator c everywhere instead of $\varepsilon'(\xi)$.

Proof. We shall prove only (i). The case (ii) is obtained by similar manipulations. Thus, we need to prove that for all $\varphi \in L_H^2$

$$\Big(\exp\{-it\omega^{\mp}_{E,\xi}(h)\} - \exp\{-it\varepsilon'(\xi)(E \pm i\partial_z)\}\Big)\, \varphi$$

tends to zero as $h \to 0$. Notice first that it is enough to prove the statement for $E = 0$, because the process of dressing all operators in e^{iEz} does not change the required convergence. Thus, let $E = 0$. Next, due to (74) and the Parceval identity the required convergence is equivalent to the statement that for any $f \in L_H^2$

$$(\exp\{-it[\varepsilon(hk \mp \xi) - \varepsilon(\xi)]/h\} - \exp\{\pm itk\varepsilon'(\xi))\, f(k)$$

tends to zero in L_H^2 as $h \to 0$. This is the same as to prove that

$$(\exp\{-it[\varepsilon(hk \mp \xi) - \varepsilon(\xi)]/h \mp itk\varepsilon'(\xi)\} - 1)\, f(k)$$

tends to zero. Hence, we need to prove that

$$\lim_{h\to 0}\int_{-\infty}^{\infty} \| (\exp\{-it[\varepsilon(hk \mp \xi) - \varepsilon(\xi) \pm k\varepsilon'(\xi)]/h\} - 1)\, f(k)\|^2 \, dk = 0 \qquad (77)$$

for all $f \in L^2_H$.

Since any such f can be approximated in L^2_H by a step function, it is sufficient to prove (77) for f of the form

$$f(k) = \sum_{m=1}^{n} f_m \chi_{M_m}(k)$$

with some integer n, vectors $f_m \in H$ and bounded intervals M_m. Since D is dense in H, and n is finite it is therefore enough to prove (77) only for f of the form

$$f(k) = f\chi_M(k), \quad f \in D$$

with a bounded interval M. For such $f(k)$ the statement (77) follows directly from assumption (68).

Thus the Dirac type evolution (14) with the unbounded generator ∂_z is obtained as a limit as $h \to 0$ of a rather general Schrödinger evolution with bounded below Hamiltonians. We deduced this limit only for the case of a single-kick equation. The generalisations to a multi-dimensional case are straightforward.

As an example of $\varepsilon(p)$ satisfying conditions (ii) of the theorem, one can take the standard relativistic Hamiltonian $\varepsilon(p) = |p|$, or more generally $\varepsilon(p) = \sqrt{p^2 + m^2}$ (see [10] for a detailed discussion of this example and a proof of the theorem 5 for it). In such cases, the limit $h \to 0$ is equivalent to the limit $\xi \to \infty$, i.e. it is the limit of asymptotically infinite momentum (ultra-relativistic limit). As an example of $\varepsilon(p)$ satisfying the assumptions (i), but not (ii), one can take the symbol of the standard non-relativistic Schrödinger operator $p^2/2m$. In this case our limit $h \to 0$ is equivalent to the limit $\xi \to \infty$, $m \to \infty$ with ξ/m tending to a constant. This limit describes the infinitely heavy particles with a constant speed.

8 Appendix: quantum filtering equations.

For completeness and convenient reference, we give here an essentially simplified and modified version of the deduction [9] of a quantum filtering equation for continuous quantum observations.

Suppose a quantum particle (or any other system) X is described by wave functions $\varphi(x)$ from $L^2(\mathbb{R}^d)$, and a quantum meter (or pointer) is described by wave functions $f(y)$ from another copy of $L^2(\mathbb{R}^d)$. Then the complex quantum system particle + meter can be naturally described by functions of two variable

$\psi(x,y)$ from $L^2(\mathbb{R}^{2d})$. Suppose that at the moment of measurement the states of the particle and the pointer are not coherent, i.e. the state of the complex system has the form $\psi(x,y) = \varphi(x)f(y)$, where usual normalisation of the wave functions is assumed, i.e. $|\varphi(x)|^2$ and $|f(y)|^2$ are supposed to define probability densities. Moreover, we suppose for simplicity that the state of the pointer $f(y)$ is an everywhere real positive function: $f(y) > 0$ for all y. The unsharp (and non-direct) mesurements of the position of the particle are made by observing the position y of the meter. The effect of such a measurement on the whole system is usually describes by a unitary operator U in $L^2(\mathbb{R}^{2d})$. We shall reduce the discussion to the case of the von Neumann [26] model of unsharp measurement with U being the shift in the varible y: $U = exp\{-ax\partial_y\}$, where a is a positive number (the coupling constant for the interaction of the particle and the meter). This unitary operator corresponds formally to the singular interaction Hamiltonian $-iax\delta(t)\partial_y$ considered in [26], since formally

$$exp\{-ax\partial y\} = \exp\{-i\int_{-\infty}^{\infty}(-iax\delta(t)\partial_y)\,dt\}.$$

Thus, if y was measured, the system is supposed to experience the unitary jump

$$U : \varphi(x)f(y) \mapsto \varphi(x)f(y-ax).$$

and the distribution of x becomes to be given by the density

$$c(y)|\varphi(x)|^2f^2(y-ax), \quad c(y) = \left(\int |\varphi(x)|^2f^2(y-ax)\,dx\right)^{-1}.$$

Thus the (conditional) state of the particle itself must be $\sqrt{c(y)}\varphi(x)f(y-ax)$ up to a multiplier of the unit amplitude which we omit for simplicity (the inclusion of a nontrivial multiplier would give a certain gauge transformation to the states of the particle under consideration). Putting the wave function of a particle to be

$$\varphi_y(x) = \varphi(x)f(y-ax)/f(y),$$

we normalise this wave function not pointwise (for each y), but on the initial probability of the pointer:

$$\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|\varphi_y(x)|^2\,dx\right)f^2(y)\,dy = 1,$$

which is more natural for the probabilistic interpretation of the process of measurement. With this normalisation, the transformation of the state of the particle by the measurement is described by the multiplication operator

$$G(y): \quad \varphi(x) \mapsto \varphi(x)f(y-ax)/f(y), \tag{78}$$

depending on the measured value of y.

Supposing that (i) between the measurements the evolution of a particle is described by a Hamiltonian E, (ii) the measurement has been performed at times $z_1 < ... < z_n$, (iii) after each measurement the pointer has been returned to the same initial state $f(y)$, we conclude that the evolution of the wave function of the particle is described by formula (24) with $\sigma_k = G(y(z_k))$. In other words, this evolution satisfies the equation of form (22).

Under the assumptions that the times $z_1, ..., z_n$ of measurements are randomly distributed according to the standard Poisson process n_t and that the results of measurements are independent, the aposterior evolution of the state φ (given by the totallity of all equations (22) with various values of n and the measurement outcomes $y(.)$) can be described by the stochastic equation driven by a Poisson noise

$$d\varphi + iE\varphi\, dt = (G(y(n_t)) - 1)\varphi\, dn_t, \tag{79}$$

with coefficients depending on random jumps y. Introducing the random point measure

$$\mu(dt\, dy) = \sum_s \delta_{s,y(s)}(t, y)$$

on $\mathbb{R}^+ \times R^d$, where the sum is over all random jump times s and $y(s)$ denotes the size of the corresponding random jumps (distributed with the probability density $f^2(y)$), one can define an operator-valued compound Poisson process

$$P(t) = \int_{[0,t)\times\mathbb{R}^d} (G(y(s)) - 1)\mu(dt\, dy) = \sum_{s<t} (G(y(s)) - 1), \tag{80}$$

Equation (79) can be then rewritten in the equivalent form

$$d\varphi + iE\varphi\, dt = dP(t)\varphi. \tag{81}$$

Equations (79) (or (81)) are the quantum filtering equations describing the aposterior dynamics of the wave function of a quantum particle under a continuous observation of counting type.

Now, we shall consider various limits of equation (81) as the intensity of the underlying Poisson process n_t tends to infinity. These limits were used in [9] to deduce the quantum filtering equations on the aposterior quantum state corresponding to the measurement of diffusion type. Here, unlike [9], we shall not use the heavy machinery of quantum stochastic calculus, but rather the standard probabilistic tools.

Let us recall (see e.g. [28]) that to an arbitrary finite Borel measure ν on $\mathbb{R}^d \setminus \{0\}$ and an arbitrary smooth bounded mapping g from $\mathbb{R}^d$ to itself, there corresponds a $\mathbb{R}^d$-valued compound Poisson process $Y = \{Y(t)\}$ (defined on a appropriate probability space):

$$Y(t) = g(y(z_1)) + ... + g(y(z_n)), \tag{82}$$

where $z_1 < ... < z_n \leq t$ are the jump times before t of a Poisson process of the intensity $\lambda_\nu = \nu(\mathbb{R}^d \setminus \{0\})$, and $y(z_j)$ are independent identically distrubuted random variables with the probability law $\nu(dy)/\lambda_\nu$. Without loss of generality, one can consider g to be an identity (which is usually done in the text books), but we included g, because it is present explicitly in the examples we are discussing. The characteristic function of the r.v. $Y(t)$ is given by the formula $\exp\{t\psi(p)\}$, where $\psi(p)$ is called the characteristic exponent of the compound Poisson process Y and has the form

$$\psi(p) = \int (e^{ipy} - 1)\nu_g(dy) = \int (e^{ipg(y)} - 1)\nu(dy),$$

where ν_g is the transformation of the measure ν under the mapping g. Let a pair of positive numbers a and b is given. Let us define a compound Poisson process Y_b^a, which is obtained from Y by (i) multiplying all $g(y(z_j))$ in (82) by a and (ii) by changing the intensity of jumps z_j from λ_ν to $b^{-1}\lambda_\nu$. Y_b^a is again a compound Poisson process with the characteristic exponent

$$\psi_b^a(p) = b^{-1} \int (e^{iapy} - 1)\nu_g(dy) = b^{-1} \int (e^{iapg(y)} - 1)\nu(dy).$$

Suppose now that the measure ν_g has at least three finite moments $\nu_g^j = \int |y|^j \nu_g(dy)$, $j = 1, 2, 3$. Then clearly

$$\lim_{a\to 0} \psi_a^a(p) = i\left(p, \int y\nu_g(dy)\right) = i\left(p, \int g(y)\nu(dy)\right), \tag{83}$$

and, if the r.h.s. of (83) vanishes, then

$$\lim_{a\to 0} \psi_{a^2}^a(p) = -\frac{1}{2} \int (p, y)^2 \nu_g(dy) = \frac{1}{2} p_j p_k \int y_j y_k \nu_g(dy). \tag{84}$$

Now the function on the r.h.s. of (83) (respectively (84)) is the characteristic exponent of the deterministic process $t \int x\nu_g(dx)$ (respectively the Wiener process with the correlation matrix $\{\int y_j y_k \nu_g(dy)\}$). Therefore, the equations (83) and (84) respectively mean that the process Y_a^a tends to a deterministic process and the process $Y_{a^2}^a$ tends to the Wiener process W with the correlation matrix $\{\int y_j y_k \nu_g(dy)\}$.

To apply these facts to equation (81), let us suppose that the function $f(y)$ is differentiable and write the Taylor expansion

$$G(y) - 1 = -a\frac{(f'(y), x)}{f(y)} + \frac{1}{2}a^2 \frac{(f''(y)x, x)}{f(y)} + O(a^3)$$

for the integrand in (80). Using (83) and (84) with $\nu(dy) = f^2(y)dy$ and assuming that f is decreasing sufficiently fast at infinity (for example, one can take the Gaussian function e^{-y^2}) we come to the following conclusions [9], [10]:

(i) if the intensity of the Poisson process in (79) is considered to be γ/a with a certain constant γ, then, as $a \to 0$, equation (79), (81) tends to the deterministic equation

$$d\varphi + iE\varphi\, dt = -\gamma(x, p_0)\varphi\, dt, \tag{85}$$

where $p_0 = \int f'(y)f(y)\, dy$ is the mean momentum of the meter at the initial state f,

(ii) if the meter is centralised, in the sense that the mean momentum p_0 vanishes, and if the intensity of the Poisson process in (79) is considered to be γ/a^2 with a certain constant γ, then, as $a \to 0$, equation (79), (81) tends to the stochastic equation of diffusion type

$$d\varphi + iE\varphi\, dt + \frac{1}{2}\gamma^2(Cx, x)\varphi\, dt = \gamma(x, dW)\varphi, \tag{86}$$

where W is the d-dimensional Wiener process with the correlation matrix C with the entries

$$c_{jk} = \int f'_j(y) f'_k(y)\, dy.$$

We did not prove here the convergence of the solutions of equation (79) to the solutions of equations (85), (86). This is not difficult to do for bounded operators E. In case of unbounded E, this question seems to be nontrivial and is closely connected with the question of well-posedness of general equations of type (86), which seemed to be considered till now only for some particular examples, see e.g. [22] and references therein.

Alternative deductions of equation (86) can be found e.g. in [17], [4], [22] its generalisations to the case of general classical noises are given in [6] and to the case of general quantum noises in [12], for various mathematical properties of this equation and its physical interpretations, the reader is referred to e.g. [14], [23],[29] and references therein.

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