CHAPTER 1. GAUSSIAN DIFFUSIONS

1. Gaussian diffusions. Probabilistic and analytic approaches

The simplest case of a parabolic second order equation is the Gaussian diffusion, whose Green function can be written explicitly as the exponential of a quadratic form. This chapter is devoted to this simplest kind of diffusion equation. In the first section we collect some well-known general facts about Gaussian diffusions pointing out the connection between probabilistic and analytic approaches to its investigation. In the next section the complete classification of its small time asymptotics is given, which is due essentially to Chaleyat-Maurel [CME]. We give a slightly different exposition stressing also the connection with the Young schemes. Sections 1.3-1.5 are devoted to the long time behaviour of Gaussian and complex stochastic Gaussian diffusions, and their (deterministic) perturbations.

A *Gaussian diffusion operator* is a second order differential operator of the form

$$L = \left(Ax, \frac{\partial}{\partial x}\right) + \frac{1}{2} tr \left(G \frac{\partial^2}{\partial x^2}\right), \tag{1.1}$$

where $x \in \mathcal{R}^m$, A and G are $m \times m$-matrices, the matrix G being symmetric and non-negative-definite. The corresponding parabolic equation $\partial u/\partial t = Lu$ can be written more explicitly as

$$\frac{\partial u}{\partial t} = A_{ij} x_i \frac{\partial u}{\partial x_j} + \frac{1}{2} G_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}. \tag{1.2}$$

The *Green function* of the corresponding Cauchy problem is by definition the solution $u_G(t, x, x_0)$ of (1.2) with initial condition

$$u_G(0, x, x_0) = \delta(x - x_0),$$

where δ is Dirac's δ-function.

We shall say that a square matrix is degenerate or singular, if its determinant vanishes.

The following fact is well-known.

Proposition 1.1 *If the matrix*

$$E = E(t) = \int_0^t e^{A\tau} G e^{A'\tau} \, d\tau \tag{1.3}$$

is non-singular (at least for small $t > 0$), then

$$u_G(t, x, x_0) = (2\pi)^{-m/2} (\det E(t))^{-1/2} \exp\{-\frac{1}{2} \left(E^{-1}(x_0 - e^{At}x), x_0 - e^{At}x\right)\}. \tag{1.4}$$

Non-singularity of E is a necessary and sufficient condition for the Green function of (1.2) to be smooth in x, t for $t > 0$.

We shall sketch several proofs of this simple but important result.

First proof. Let $f(t,p)$ be the Fourier transform of $u_G(t,x,x_0)$ with respect to the variable x. Then $f(t,p)$ satisfies the equation

$$\frac{\partial f}{\partial t} = -\left(A'p, \frac{\partial f}{\partial p}\right) - \left(\frac{1}{2}(Gp,p) + tr\, A\right) f$$

and the initial condition $f(0,p) = (2\pi)^{-m/2}\exp\{-ipx_0\}$. Solving this linear first order partial differential equation by means of the standard method of characteristics, yields

$$f = (2\pi)^{-m/2}\exp\{-i(x_0, e^{-A't}p) - \frac{1}{2}(E(t)e^{-A't}p, e^{-A't}p) - t\, tr\, A\}.$$

Taking the inverse Fourier transform of f and changing the variable of integration $p \mapsto q = e^{-A't}p$ one gets

$$u_G = (2\pi)^{-m}\int \exp\{i(e^{At}x - x_0, q) - \frac{1}{2}(E(t)q,q)\}\, dq,$$

which is equal to (1.4).

Second proof. This proof uses the theory of Gaussian stochastic processes. Namely, we associate with the operator (1.2) the stochastic process defined by the stochastic differential equation

$$dX = AX\, dt + \sqrt{G}\, dW, \tag{1.5}$$

where $\sqrt{G}$ is the symmetric non-negative-definite square root of G and W is the standard m-dimensional Brownian motion (or Wiener process). Its solution with initial data $X(0) = X_0$ is given by

$$X(t) = e^{At}X_0 + \int_0^t e^{A(t-\tau)}\sqrt{G}\, dW(\tau).$$

Direct calculations show that the correlation matrix of the process $X(t)$ is given by formula (1.4). Therefore, the probability density of the transition $x \to x_0$ in time t is given by (1.3) and by the general theory of diffusion processes (see, e.g. [Kal]), this transition probability is just the Green function for the Cauchy problem of equation (1.2).

Other proofs. Firstly, one can check by direct calculations that the function given by (1.4) satisfies equation (1.2) and the required initial condition. Secondly, one can deduce (1.4) using the WKB method, as shown at the end of Section 3.1. Lastly, one can also get (1.4) by the method of "Gaussian substitution", which will be described in Section 1.4, where the Green function for stochastic complex Gaussian diffusions will be constructed in this way.

We discuss now the connection between the non-singularity property of E and a general analytic criterion for the existence of a smooth Green function for second order parabolic equations. It is convenient to work in coordinates, where the matrix G is diagonal. It is clear that the change of the variables $x \to Cx$, for some non-singular matrix C, changes the coefficients of the operator L by the law

$$A \to CAC^{-1}, \quad G \to CGC'.$$

Therefore, one can always choose coordinates such that

$$L = (Ax, \frac{\partial}{\partial x}) + \frac{1}{2} \sum_{j=n+1}^{m} \frac{\partial^2}{\partial x_j^2}, \tag{1.6}$$

where $m - n = rank\, G$. It is convenient to introduce a special notation for the coordinates involving the second derivatives. From now we shall denote by $x = (x_1, ..., x_n)$ the first n coordinates and by $y = (y_1, ..., y_k)$, where $k = m - n$, the remaining ones. In other words, the coordinate space is considered to be decomposed into the direct sum

$$\mathcal{R}^m = \mathcal{R}^{n+k} = \mathcal{R}^n \oplus \mathcal{R}^k = X \oplus Y, \tag{1.7}$$

and L can be written in the form

$$L = L_0 + \frac{1}{2}\Delta_y, \tag{1.6'}$$

where Δ_y is the Laplace operator in the variables y, and

$$L_0 = (A^{xx}x, \frac{\partial}{\partial x}) + (A^{xy}y, \frac{\partial}{\partial x}) + (A^{yx}x, \frac{\partial}{\partial y}) + (A^{yy}y, \frac{\partial}{\partial y})$$

with

$$A = \begin{pmatrix} A^{xx} & A^{xy} \\ A^{yx} & A^{yy} \end{pmatrix} \tag{1.8}$$

according to the decomposition (1.7). The operator (1.6') has the so called Hörmander form. Application of the general theory of such operators (see, e.g. [IK]) to the case of the operator (1.6') gives the following result.

Proposition 1.2. *Let Id be the ideal generated by $\frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_k}$ in the Lie algebra of linear vector fields in $\mathcal{R}^{n+k}$ generated by L_0 and $\frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_k}$. The equation*

$$\frac{\partial u}{\partial t} = (L_0 + \frac{1}{2}\Delta_y)u$$

in $\mathcal{R}^{n+k}$ has a smooth Green function for $t > 0$ if and only if the dimension of the ideal Id is maximal, i.e. it is equal to $n + k$.

Due to Proposition 1.1, the condition of Proposition 1.2 is equivalent to the non-singularity of the matrix E. We shall prove this fact directly in the next section by giving as well the classification of the main terms of the small time asymptotics of the matrix E^{-1} for the processes satisfying the conditions of Proposition 1.2. We conclude this section by a simple description of the ideal Id.

Lemma 1.1 *The ideal Id is generated as a linear space by the vector fields*

$$\frac{\partial}{\partial y_j}, [\frac{\partial}{\partial y_j}, L_0], [[\frac{\partial}{\partial y_j}, L_0], L_0], ..., \quad j = 1, ..., k,$$

or, more explicitly, by $\frac{\partial}{\partial y_j}$, $j = 1, ...k$, and the vector fields, whose coordinates in the basis $\{\frac{\partial}{\partial x_i}\}$, $i = 1, ..., n$, are given by the columns of the matrices $(A^{xx})^l A^{xy}$, $l = 0, 1, ...$

Proof.

$$[\frac{\partial}{\partial y_j}, L_0] = (A^{yy})_{ij}\frac{\partial}{\partial y_i} + (A^{xy})_{ij}\frac{\partial}{\partial x_i}.$$

Therefore, taking the first order commutators we obtain the vector fields $v_j^1 = (A^{xy})_{ij}\frac{\partial}{\partial x_i}$, whose coordinates in the basis $\{\frac{\partial}{\partial x_i}\}$ are given by the columns of the matrix A^{xy}. The commutators $[v_i^1, \frac{\partial}{\partial y_j}]$ do not produce new independent vectors. Therefore, new vector fields can be obtained only by the second order commutators

$$[v_j^1, L_0] = (A^{xy})_{ij}(A^{xx})_{li}\frac{\partial}{\partial x_l} + (A^{xy})_{ij}(A^{yx})_{li}\frac{\partial}{\partial y_l},$$

which produce a new set of vector fields $v_j^2 = (A^{xx}A^{xy})_{lj}\frac{\partial}{\partial x_l}$ whose coordinates in the basis $\{\frac{\partial}{\partial x_i}\}$ are the columns of the matrix $A^{xx}A^{xy}$. The proof is completed by induction.

2. Classification of Gaussian diffusions by Young schemes

Let $H_0 = A^{xy}Y$ and let H_m, $m = 1, 2, ...$, be defined recurrently by the equation

$$H_m = A^{xx}H_{m-1} + H_{m-1}.$$

In coordinate description, H_m is the subspace of $X = \mathcal{R}^n$ generated by the columns of the matrices A^{xy},$A^{xx}A^{xy}$,..., $(A^{xx})^m A^{xy}$. Let M be the minimal natural number such that $H_M = H_{M+1}$. Clearly M is well defined and $0 \le M \le n - dim\, H_0$. Moreover, $dim\, Id = n + k$, iff $H_M = X$, or equivalently iff $dim\, H_m = n$.

Lemma 2.1. *If the correlation matrix E given by (1.3) corresponding to the operator (1.6') is non-singular, then $dim\, Id = n + k$.*

Proof. Suppose $dimId < n + k$ and consequently $dim\, H_M < n$. Then one can choose a basis in X whose first $n - dim\, H_M$ vectors belong to the orthogonal complement $H_M^{\perp}$ of H_M. In this basis, the first $n - dim\, H_M$ rows of the matrix A given by (1.8) vanish and therefore the matrix $e^{AT}Ge^{A'T}$ has the same property for all t, and therefore so does the matrix E. Thus E is singular.

These arguments show in fact that if $dim\,Id < n+k$, one can reduce the Gaussian process defined by the operator L to a process living in a Eucleadian space of lower dimension. From now on we suppose that $dim\,Id = n+k$ and thus $H_M = X$. The natural number $M+1$ will be called further the degree of singularity of the Gaussian diffusion. A finite non-increasing sequence of natural numbers is called a *Young scheme*. Young schemes play an important role in the representation theory of classical groups. The (clearly non-increasing) sequence $\mathcal{M}$ of $M+2$ numbers $m_{M+1} = \dim Y = k$, $m_M = \dim H_0$, $m_{M-1} = \dim H_1 - \dim H_0$,...,$m_0 = \dim H_M - \dim H_{M-1}$ will be called the Young scheme of the operator (1.6'). As we shall show these schemes completely define the main term of the small time asymptotics of the inverse matrix E^{-1} and therefore of the transition probability or the Green function of the corresponding Gaussian diffusion. To this end , let us decompose $X = H_M$ in the orthogonal sum:

$$X = X_0 \oplus ... \oplus X_{M-1} \oplus X_M, \tag{2.1}$$

where X_J, $J = 0,...,M$ are defined by the equation $X_{M-J} \oplus H_{J-1} = H_J$, i.e. each X_{M-J} is the orthogonal complement of H_{J-1} in H_J. To simplify the notation we shall sometimes denote Y by X_{M+1}. The coordinates are therefore decomposed in series $(x,y) = (x^0,...,x^{M+1})$ with the dimension of each series $x^J = (x_1^J,...,x_{m_J}^J)$, $J = 0,...,M+1$, being defined by the entry m_J of the Young scheme $\mathcal{M}$. Evidently in these coordinates the blocks $A_{J,J+I}$ of the matrix A vanish whenever $I > 1$ for all $J = 0,...,M$. Let A_J, $J = 0,...,M$, denote the blocks $A_{J,J+1}$ of A, which are $(m_J \times m_{J+1})$-matrices (with m_J rows and m_{J+1} columns) of rank m_J, and let

$$\alpha_J = A_J A_{J+1} ... A_M, \quad J = 0,...,M.$$

Let us find the main term $E^0_{IJ}(t)$ of the expansion in small t of the blocks $E_{IJ}(t)$ of the correlation matrix (1.3).

Lemma 2.2. *In the chosen coordinates, the blocks $E_{IJ}(t)$ of the matrix (1.3) are given by*

$$E_{IJ}(t) = E^0_{IJ}(t)(1+O(t)) = \frac{t^{2M+3-(I+J)}\alpha_I\alpha'_J(1+O(t))}{(2M+3-(I+J))(M+1-I)!(M+1-J)!}. \tag{2.2}$$

Proof. Let us calculate first the main term of the expansion of the blocks of the matrix $\Omega(t) = e^{At}Ge^{A't}$ in the integral in (1.3), taking into account that according to our assumptions the block $G_{M+1,M+1}$ of G is the unit matrix and all other blocks of G vanish. Writing

$$\Omega(t) = \sum_{p=0}^{\infty} t^p \Omega(p),$$

one has

$$\Omega(p) = \sum_{q=0}^{p} \frac{1}{q!(p-q)!} A^q G (A^{p-q})'. \tag{2.3}$$

It is easy to see that for $p < (2M+2) - (I+J)$ the blocks $\Omega(p)_{IJ}$ vanish and for $p = 2M+2-(I+J)$ only one term in sum (2.3) survives, namely that with $q = M+1-I$, $p-q = M+1-J$. Consequently, for this value of p

$$\Omega(p)_{IJ} = \frac{1}{(M-I+1)!}\frac{1}{(M-J+1)!}\alpha_I\alpha'_J,$$

which implies (2.2).

It turns out that the matrix $E^0(t) = \{E^0_{IJ}(t)\}$ is invertible and its inverse in appropriate coordinates depends only on the Young scheme $\mathcal{M}$. The following result is crucial.

Lemma 2.3. *There exist orthonormal coordinates in Y and (not necessarily orthonormal) coordinates in X_J, $J = 0, ..., M$, such that in these coordinates, each matrix A_J has the form*

$$A_J = (0, 1_{m_J}), \quad J = 0, ..., M,$$

where 1_{m_J} denote the $(m_J \times m_J)$ unit matrix.

Remark. We need the coordinates in Y to be orthonormal in order not to destroy the simple second order part of the operator (1.6').

Proof. Consider the chain

$$X_{M+1} \to X_M = H_0 \to H_1 = X_{M-1} \oplus H_0$$

$$\to H_2 = X_{M-2} \oplus H_1 \to ... \to H_M = X_0 \oplus H_{M-1},$$

where the first arrow stands for the linear map A^{xy} and all other stand for A^{xx}. Taking the projection on the first term in each $X_{M-J} \oplus H_{J-1}$ yields the chain

$$X_{M+1} \to X_M \to X_{M-1} \to X_{M-2} \to ... \to X_0, \tag{2.4}$$

where each arrow stands for the composition $\tilde{A} = Pr \circ A$ of the map A and the corresponding projection. Since

$$X_{M+1} \supset Ker\tilde{A}^{M+1} \supset ... \supset Ker\tilde{A}^2 \supset Ker\tilde{A},$$

one can expand $Y = X_{M+1}$ as the orthogonal sum

$$X_{M+1} = Ker\tilde{A}^1 \oplus (Ker\tilde{A}^2 \ominus Ker\tilde{A}^1) \oplus ...$$

$$\oplus (Ker\tilde{A}^{M+1} \ominus Ker\tilde{A}^M) \oplus (X_{M+1} \ominus Ker\tilde{A}^{M+1}),$$

where $Ker\tilde{A}^j \ominus Ker\tilde{A}^{j-1}$ means the orthogonal complement of $Ker\tilde{A}^{j-1}$ in $Ker\tilde{A}^j$. Now choose an orthonormal basis in X_{M+1} which respects this decomposition, i.e. the first $(m_{M+1} - m_M)$ elements of this basis belong to the first term of this decomposition, the next $(m_M - m_{M-1})$ elements belong to the second term and so on. The images of the basis vectors under the action of $\tilde{A}^j$ in the chain (2.4) define the basis in X_{M-j+1} (not necessarily orthonormal).

The coordinates in $X \oplus Y$ defined in such a way satisfy the requirements of the Lemma.

Corollary. *In the coordinates of Lemma 2.3, if $J \geq I$, then*

$$\alpha_I = (0, 1_{m_I}), \quad \alpha_I \alpha'_J = (0, 1_{m_I}), \tag{2.5}$$

where the matrix 0 *in the first and second formulas are vanishing matrices with m_I rows and $m_M - m_I$ (resp. $m_J - m_I$) columns.*

Lemma 2.4. *If $E^0_{IJ}(t)$ is defined by (2.2) and (2.5), then by changing the order of the basis vectors one can transform the matrix $E^0(t)$ to the block-diagonal matrix with m_0 square blocks Λ_{M+2}, $(m_0 - m_1)$ square blocks Λ_{M+1},..., $(m_M - m_{M-1})$ blocks Λ_2, and $m_{M+1} - m_M$ one-dimensional blocks $\Lambda_1 = t$, where $\Lambda_p(t)$ denotes the $(p \times p)$-matrix with entries*

$$\Lambda_p(t)_{ij} = \frac{t^{2p+1-(i+j)}}{(2p+1-(i+j))(p-i)!(p-j)!}, \quad i, j = 1, ..., p. \tag{2.6}$$

Proof. It is straightforward. Let us only point out that the block representation of E^0 in the blocks E^0_{IJ} corresponds to the partition of the coordinates in the parts corresponding to the rows of the Young scheme $\mathcal{M}$, and the representation of E^0 in the block-diagonal form with blocks (2.6) stands for the partition of coordinates in parts corresponding to the columns of the Young scheme $\mathcal{M}$.

Example. If the Young scheme $\mathcal{M} = (3, 2, 1)$, i.e. if $M = 1$, $dimY = dimX = 3$ and $X = X_0 \oplus X_1$ with $\dim X_0 = 1$, $\dim X_1 = 2$, the matrices A and $E^0(t)$ in the coordinates of Lemma 2.3 have the forms respectively

$$\begin{pmatrix} \star & 0 & 1 & 0 & 0 & 0 \\ \star & \star & \star & 0 & 1 & 0 \\ \star & \star & \star & 0 & 0 & 1 \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} \frac{t^5}{5\cdot 2\cdot 2} & 0 & \frac{t^4}{4\cdot 2} & 0 & 0 & \frac{t^3}{3\cdot 2} \\ 0 & \frac{t^3}{3} & 0 & 0 & \frac{t^2}{2} & 0 \\ \frac{t^4}{4\cdot 2} & 0 & \frac{t^3}{3} & 0 & 0 & \frac{t^2}{2} \\ 0 & 0 & 0 & t & 0 & 0 \\ 0 & \frac{t^2}{2} & 0 & 0 & t & 0 \\ \frac{t^3}{3\cdot 2} & 0 & \frac{t^2}{2} & 0 & 0 & t \end{pmatrix},$$

where the entries denoted by $\star$ are irrelevant. Clearly by change of order of the basis, E^0 can be transformed to

$$\begin{pmatrix} \frac{t^5}{5\cdot 2\cdot 2} & \frac{t^4}{4\cdot 2} & \frac{t^3}{3\cdot 2} & 0 & 0 & 0 \\ \frac{t^4}{4\cdot 2} & \frac{t^3}{3} & \frac{t^2}{2} & 0 & 0 & 0 \\ \frac{t^3}{3\cdot 2} & \frac{t^2}{2} & t & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{t^3}{3} & \frac{t^2}{2} & 0 \\ 0 & 0 & 0 & \frac{t^2}{2} & t & 0 \\ 0 & 0 & 0 & 0 & 0 & t \end{pmatrix} = \begin{pmatrix} \Lambda_3(t) & 0 & 0 \\ 0 & \Lambda_2(t) & 0 \\ 0 & 0 & \Lambda_1(t) \end{pmatrix}.$$

Lemma 2.5.

$$(i) \qquad \det \Lambda_p(t) = t^{p^2} \frac{2! \cdot ... \cdot (p-1)!}{p!(p+1)!...(2p-1)!},$$

(ii) *the matrix* $E^0(t)$ *is non-singular and in the coordinates of Lemma 2.3 its determinant is equal to*

$$\det E^0(t) = (\det \Lambda_1(t))^{m_{M+1}-m_M} (\det \Lambda_2(t))^{m_M - m_{M-1}} ... (\det \Lambda_{M+2}(t))^{m_0 - m_1}, \tag{2.7}$$

(iii) the maximal negative power of t in the small time asymptopics of the entries of $(E^0(t))^{-1}$ is $-(2M+3)$ and there are exactly m_0 entries that have this maximal power.

Proof. (ii) and (iii) follow directly from Lemma 2.4. To prove (i) notice that

$$\det \Lambda_p(t) = [2!...(p-1)!]^{-2} \det \lambda_p(t),$$

where $\lambda_p(t)$ is the matrix with entries

$$\lambda_p(t)_{ij} = \frac{t^{2p+1-(i+j)}}{2p+1-(i+j)}. \tag{2.8}$$

In order to see clearly the structure of these matrices, let us write down explicitly the first three representatives:

$$\lambda_1(t) = t, \quad \lambda_2(t) = \begin{pmatrix} \frac{t^3}{3} & \frac{t^2}{2} \\ \frac{t^2}{2} & t \end{pmatrix}, \quad \lambda_3(t) = \begin{pmatrix} \frac{t^5}{5} & \frac{t^4}{4} & \frac{t^3}{3} \\ \frac{t^4}{4} & \frac{t^3}{3} & \frac{t^2}{2} \\ \frac{t^3}{3} & \frac{t^2}{2} & t \end{pmatrix}.$$

The determinant of these matrices is well known (see, e.g. [Ak]) and can be easily found to be:

$$\det \lambda_p(t) = \frac{[2! \cdot 3! \cdot ... \cdot (p-1)!]^3}{p!(p+1)!...(2p-1)!} t^{p^2}, \tag{2.9}$$

which implies (i).

The entries of the matrices $\Lambda_p(t)^{-1}$ and therefore of $(E^0(t))^{-1}$ can be calculated by rather long recurrent formulas. It turns out that these entries are always integer multipliers of negative powers of t, for example, $(\Lambda_1(t))^{-1} = \frac{1}{t}$,

$$(\Lambda_2(t))^{-1} = \begin{pmatrix} \frac{12}{t^3} & -\frac{6}{t^2} \\ -\frac{6}{t^2} & \frac{4}{t} \end{pmatrix}, \quad (\Lambda_3(t))^{-1} = \begin{pmatrix} \frac{6!}{t^5} & -\frac{6!}{2t^4} & \frac{60}{t^3} \\ -\frac{6!}{2t^4} & \frac{192}{t^3} & -\frac{36}{t^2} \\ \frac{60}{t^3} & -\frac{36}{t^2} & \frac{9}{t} \end{pmatrix}.$$

Therefore, we have proved the following result.

Theorem 2.1. *For an arbitrary Gaussian diffusion in the Eucleadian space* $\mathcal{R}^m$ *whose correlation matrix is non-singular (or equivalently, whose transition probability has smooth density), there exists a scheme* $\mathcal{M} = (m_{M+1}, m_M, ..., m_0)$ *such that* $\sum_{J=0}^{M+1} m_J = m$, *and a coordinate system* z *in* $\mathcal{R}^m$ *such that in these coordinates the inverse* $E(t)^{-1}$ *of the correlation matrix (1.3) has the entries*

$$E^{-1}_{ij}(t) = (E^0)^{-1}_{ij}(t)(1 + (t)),$$

where $E^0(t)$ is an invertible matrix that depends only on Y_L, and which is described in Lemmas 2.4, 2.5. The Green function for small t has the form

$$u_G(t,z;z_0) = (2\pi)^{-n/2}(1+O(t))\exp\{-\frac{1}{2}(E^{-1}(t)(z_0 - e^{At}z), z_0 - e^{At}z)\}$$

$$\times[(\det\Lambda_1(t))^{k-m_M}(\det\Lambda_2(t))^{m_M-m_{M-1}}...(\det\Lambda_{M+2}(t))^{m_0-m_1}]^{-1/2}. \qquad (2.10)$$

In particular, the coefficient of the exponential in (2.8) has the form of a constant multiple of $t^{-\alpha}$ with

$$\alpha = \frac{1}{2}[(m_{M+1} - m_M) + 2^2(m_M - m_{M-1} + ... + (M+2)^2 m_0]. \qquad (2.11)$$

Conversely we have

Theorem 2.2. *For any Young scheme $\mathcal{M}$ satisfying the conditions of Theorem 2.1, there exists a Gaussian diffusion, for which the small time asymptotics of its Green function is (2.8). Moreover, there exists a Gaussian diffusion for which the matrix $E^0(t)$ of the principle term of the asymptotic expansion of E is the exact correlation matrix.*

For example, if in the example with $\mathcal{M} = (3,2,1)$ considered above, one places zero instead of all the entries denoted by "stars" in the expression for A, one gets a diffusion for which $E^0(t)$ is the exact correlation matrix.

Notice that the case of a Young scheme consisting of only one element (i.e. the case $M+1=0$) corresponds to the case of non-singular diffusion.

3. Long time behaviour of the Green function for Gaussian diffusions

This section lies somewhat apart from the main line of the exposition. Here the large time asymptotics is discussed for some classes of Gaussian diffusions including the most commonly used Ornstein-Uhlenbeck and oscillator processes. One aim of this section is to demonstrate that the small time asymptotics classification given in the previous section has little to do with the large time behaviour. Even the property of non-singularity of the matrix G of second derivatives in the expression for the corresponding operator L has little relevance. The crucial role in the long time behaviour description belongs to the eigenvectors of the matrix A together with a "general position" property of these eigenvectors with respect to the matrix G. We consider two particular cases of the operator (1.1) with the matrix A being antisymmetric and with A having only real eigenvalues.

First let A be antisymmetric so that the evolution e^{At} is orthogonal and there exists a unitary matrix U such that $U^{-1}AU$ is diagonal. Let the rank of A be $2n \leq m$. Then one can write down the spectrum of A as

$$i\lambda_1, ..., i\lambda_k, -i\lambda_1, ..., -i\lambda_k, 0, ..., 0$$

with $\lambda_1 > \lambda_2 > ... > \lambda_k > 0$, and to order the unit eigenvectors as follows

$$v_1^1, ..., v_{j_1}^1, v_1^2, ..., v_{j_2}^2, ..., v_1^k, ..., v_{j_k}^k,$$

$$v_1^{k+1},...,v_{j_1}^{k+1},v_1^{k+2},...,v_{j_2}^{k+2},...,v_1^{2k},...,v_{j_k}^{2k},v_1^{2k+1},...,v_{m-2n}^{2k+1}, \tag{3.1}$$

where $j_1+...+j_k=n$, the vectors $v_1^l,...,v_{j_l}^l$, $l=1,...,k$, and their complex conjugates $v_1^{k+l}=\bar{v}_1^l,...,v_{j_l}^{k+l}=\bar{v}_{j_l}^l$ correspond to the eigenvalues $i\lambda_l$ and $-i\lambda_l$ respectively, and the vectors $v_1^{2k+1},...,v_{m-2n}^{2k+1}$ belong to the kernel of A. With these notation, the columns of U are the components of the vectors (3.1) and an arbitrary operator $B:\mathcal{R}^m\mapsto\mathcal{R}^m$ is represented in the basis (3.1) by a matrix $\beta=U^\star BU$ given by rectangular blocks β_{IJ}, $I,J=1,...,2k+1$. The correlation matrix (1.3) becomes $U\tilde{E}(t)U^\star$ with

$$\tilde{E}(t)=\int_0^t D(s)U^\star GUD^\star(s)\,ds=\int_0^t D(s)\Gamma D^\star(s)\,ds,$$

where $D(s)$ is diagonal with diagonal elements $e^{\pm i\lambda_j}$ and 1, and the matrix $\Gamma=U^\star GU$ consists of the blocks

$$(\Gamma_{IJ})_{lp}=(\bar{v}_l^I,Gv_p^J),\quad l=1,...,I,\,p=1,...J.$$

For $I=J$ these blocks are clearly nonnegative-definite selfadjoint $(j_I\times j_I)$-matrices.

Proposition 3.1. *If for all $I=1,...,2k+1$ the square blocks Γ_{II} are non-singular, then*

$$\det E(t)=t^m\prod_{I=1}^{2k+1}\det\Gamma_{II}(1+O(t)),$$

as $t\to\infty$, moreover

$$(\tilde{E}^{-1})_{IJ}=\begin{cases}(t\Gamma_{II})^{-1}(1+O(\frac{1}{t})),\ J=I,\\ O(t^{-2}),\quad J\neq I.\end{cases}$$

Proof. There are algebraic manipulations, which we omit.

Notice that the non-singularity assumption in Proposition 3.1 is quite different from the non-singularuty assumption of the matrix G that defines the second order part of the diffusion operator. In order to meet the hypothesis of Proposition 3.1 it is enough that the rank of G be equal to the maximal multiplicity of the eigenvalues of A. For example, if the eigenvalues of A are different, then the hypothesis of Proposition 3.1 means just that $(\bar{v}_j,Gv_j)\neq 0$ for all eigenvalues v_j of A, and it can be satisfied by the one-dimensional projection. From Proposition 3.1 it follows that the large time asymptotics of the Green function (1.4) in this situation is similar to the standard diffusion with the unit matrix G and vanishing drift.

Corollary. *Let the hypothesis of Proposition 3.1 hold and let all (necessarily positive) eigenvalues of all blocks $(\Gamma^{II})^{-1}$, $I=1,...,2k+1$, lie inside the interval*

$[\beta_1, \beta_2]$. *Then for arbitrary* $\epsilon > 0$ *and sufficiently large* t, *the Green function (1.4) satisfies the two-sided estimates*

$$(2\pi t)^{-m/2} \left(\prod_{I=1}^{2k+1} \det \Gamma^{II} \right)^{-1/2} (1-\epsilon) \exp\{-\frac{1}{2}\beta_2 \|x_0 - e^{At}x\|^2\} \le u_G(t, x; x_0)$$

$$\le (2\pi t)^{-m/2} \left(\prod_{I=1}^{2k+1} \det \Gamma^{II} \right)^{-1/2} (1+\epsilon) \exp\{-\frac{1}{2}\beta_1 \|x_0 - e^{At}x\|^2\}.$$

An example of this situation is given by the stochastic process defined by the motion of the classical oscillator perturbed by a force given by white noise.

Now let us assume that A has only real eigenvalues λ_j. For simplicity assume A is diagonalisable with k positive eigenvalues $\mu_1 \ge ... \ge \mu_k > 0$, l vanishing eigenvalues, and $m-k-l$ negative eigenvalues $0 > -\nu_{k+l+1} \ge ... \ge -\nu_m$. The matrix (1.3) has the entries $E_{ij} = (e^{(\lambda_i+\lambda_j)t} - 1)(\lambda_i + \lambda_j)^{-1} G_{ij}$ if $\lambda_i + \lambda_j \ne 0$, and $E_{ij} = tG_{ij}$, if $\lambda_i + \lambda_j = 0$. One easily derives the following:

Proposition 3.2. *Let the quadratic matrices* B_1, B_2, B_3 *be non-degenerate, where* $(B_1)_{ij} = (\mu_i + \mu_j)^{-1} G_{ij}$ *with* $i, j = 1, ..., k$, $(B_2)_{ij} = tG_{ij}$ *with* $i, j = k+1, ..., k+l$, *and* $(B_3)_{ij} = (\nu_i + \nu_j)^{-1} G_{ij}$ *with* $i, j = k+l+1, ..., n$. *Then, as* $t \to \infty$,

$$\det E(t) = \exp\{2 \sum_{j=1}^{k} \mu_j t\} t^l \det B_1 \det B_2 \det B_3 (1 + O(t)),$$

moreover, $(E(t)^{-1})_{ij}$ *is exponentially small whenever* i *or* j *does not exceed* k, *and* $(E(t)^{-1})_{ij}$ *have a finite limit whenever both* $i, j > k+l$.

This result implies the corresponding asymptotics for the Green function (1.4). An example of this situation is given by the diffusion operator of the Orstein-Uhlenbeck process. The cases where A has nontrivial Jordan blocks can be considered similarly. Let us point out finally that two-dimensional diffusions described by

$$A = \begin{pmatrix} 0 & 1 \\ -\beta & 0 \end{pmatrix}, G = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ 0 & -\beta \end{pmatrix}, G = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

(the oscillator process and Orstein-Uhlenbeck process respectively) belong to the two different classes described in Propositions 3.1 and 3.2 respectively but from the point of view of the small time asymptotics classification of the previous section they belong to the same class given by the Young scheme (1,1).

The results of this section can be used to estimate the escape rate of transient Gaussian processes defined by equation (1.5) and also of perturbations of them; see [AHK1],[AK],[K2], and section 1.5.

4. Complex stochastic Gaussian diffusion

It is well-known that the Green function of the Cauchy problem for partial differential equations depending quadratically on position and derivatives, i.e. on x and $\partial/\partial x$, has Gaussian form, see e.g. Proposition 1.1 for the case of diffusion. It was realised recently that stochastic generalisations of such equations are of importance for many applications. We present here a simple method for effective calculation of the corresponding Green functions. However, in order not to get lost in complexities we shall not consider the most general case but reduce the exposition to a class of such equations which contains the most important examples for the theory of stochastic filtering, quantum stochastic analysis and continuous quantum measurements. Namely, let us consider the equation

$$d\psi = \frac{1}{2}(G\Delta\psi - \beta x^2\psi)\, dt + \alpha x\psi\, dB, \quad \psi = \psi(t, x, [B]), \tag{4.1}$$

where $x \in \mathcal{R}^m$, $dB = (dB^1, ..., dB^m)$ is the stochastic differential of the standard Brownian motion in $\mathcal{R}^m$, and G, β, α are complex constants such that $|G| > 0$, $ReG \geq 0$, $Re\beta \geq |\alpha|^2$. The last two conditions ensure the conservativity of the system, namely that the expectation of $\|\psi\|^2$ is not increasing in time, which one checks by the formal application of the Ito formula. To justify these calculations one needs actually the well-posedness of the Cauchy problem for equation (4.1) that follows for instance from the explicit expression for the Green function given below. We suppose also for simplicity that ImG and $Im\beta$ are nonnegative.

Let us discuss the main examples. If G, α, β are real, (4.1) is the so called stochastic heat equation, which is also the simplest example of Zakai's equation [Za] of stochastic filtering theory. Its exact Green function was apparently first given in [TZ1] for $\alpha = G = \beta = 1$ and in [TZ2] for $\alpha = G = 1$, $\beta > 1$. In the latter case it was called in [TZ2] the stochastic Mehler formula, because in the deterministic case $\alpha = 0$, G, β positive this formula describes the evolution of the quantum oscillator in imaginary time and is sometimes called Mehler's formula. If α, G are purely imaginary and $Re\beta = |\alpha|^2$, (4.1) represents a unitary stochastic Schrödinger equation, which appeared recently in stochastic models of unitary evolution, see [HP], [TZ1], [K2]. It can be obtained by the formal quantisation of a classical system describing a Newtonian free particle or an oscillator perturbed by a white noise force (see next Section). The explicit Green function for that equation was given in [TZ1]. If G is imaginary, α real (more generally, α complex) and $Re\beta = |\alpha|^2$, equation (4.1) represents the simplest example of the Belavkin quantum filtering equation describing the evolution of the state of a free quantum particle or a quantum oscillator (when $Im\beta = 0$ or $Im\beta > 0$ respectively) under continuous (indirect but non-demolition) observation of its position, see [Be1],[Be2] and Appendix A. When $\alpha = \beta = 1$, $G = i$, the Green function was constructed in [BK], [K4] and a generalisation was given in [K1]. It is worth mentioning that in this case, it is physically more relevant (see Appendix A) to consider the Brownian motion B (which is interpreted as the output process) to have nonzero mean, and to be connected with the standard (zero-mean) Wiener process W (called in the filtering theory the innovation process) by the equation $dB = dW + 2\langle x\rangle_\psi\,dt$, where $\langle x\rangle_\psi = \int x|\psi|^2(x)\,dx\|\psi\|^{-1}$ is the mean position of the normalised state ψ. Since in quantum mechanics one is interested in normalised solutions, it is convenient to rewrite equation (4.1) (using Ito's formula) in terms of the normalised state $\phi = \psi\|\psi\|^{-1}$ and the "free" Wiener process W to obtain, in the simplest case $\alpha = \beta = 1$, $G = i$, the following norm preserving but nonlinear equation

$$d\phi = \frac{1}{2}\left(i\Delta\phi - (x - \langle x\rangle_\phi)^2\phi\right)\,dt + (x - \langle x\rangle_\phi)\phi\,dW. \tag{4.2}$$

This equation and its generalisations are extensively discussed in current physical literature on open systems, see e.g. [BHH], [QO], and also Appendix A. It is worth while noting that the equation

$$d\phi = \frac{1}{2}(\beta\Delta\phi - Gx^2\phi)\,dt + \alpha i\frac{\partial}{\partial p}\phi\,dB, \quad \phi = \phi(t, p, [B]), \tag{4.1$'$}$$

which can be obtained from (4.1) by Fourier transformation, decribes (under the appropriate choice of the parameters) the evolution of a quantum particle or an oscillator under the continuous observation of its momentum (see [Be1]).

In view of so many examples it seems reasonable to give a unified deduction of a formula describing all these situations, which is done below in Theorem 4.1.

To obtain the Green function for equation (4.1) we calculate first the dynamics of the Gaussian functions

$$\psi(x) = \exp\{-\frac{\omega}{2}x^2 + zx - \gamma\}, \tag{4.3}$$

where ω, γ and the coordinates of the vector z are complex constants and $Re\omega > 0$. It turns out that the Gaussian form of a function is preserved by the dynamics defined by (4.1).

Proposition 4.1. *For an initial function of Gaussian form (4.3) with arbitrary initial ω_0, z_0, β_0, solution to the Cauchy problem for (4.1) exists and has Gaussian form (4.3) for all $t > 0$. Moreover, the dynamics of the coefficients ω, z, γ is given by the differential equations*

$$\begin{cases} \dot\omega = -G\omega^2 + (\beta + \alpha^2) \\ dz = -\omega Gz\,dt + \alpha\,dB \\ \dot\gamma = \frac{1}{2}G(\omega - z^2) \end{cases} \tag{4.4}$$

Proof. Let us write down the dynamics of z with undetermined coefficients $dz = z_t\,dt + z_B\,dB$ and let us assume the dynamics of ω to be non-stochastic: $d\omega = \dot\omega\,dt$, $d\gamma = \dot\gamma\,dt$. (This is justified because inserting the differentials of ω or γ in (4.1) with non-vanishing stochastic terms, yields a contradiction.) Inserting ψ of form (4.3) in (4.1) and using Ito's formula, yields

$$-\frac{1}{2}\dot\omega x^2\,dt + x\,dz - \dot\gamma\,dt + \frac{1}{2}x^2 z_B^2\,dt$$

$$= \frac{1}{2}[G((-\omega x + z)^2 - \omega) - \beta x^2]\, dt + \alpha x\, dB.$$

Comparing coefficients of dB, x^2dt, xdt, dt yields (4.4).

Remark. For the purposes of quantum mechanics it is often convenient to express the Gaussian function (4.2) in the equivalent form

$$g^{\omega}_{q,p}(x) = c\exp\{-\frac{\omega}{2}(x-q)^2 + ipx\}, \tag{4.5}$$

where real q and p are respectively the mean position and the mean momentum of the Gaussian function. One deduces from (4.4) that the dynamics of these means under the evolution (4.1) is given by the equations

$$\begin{cases} dq = \frac{1}{Re\omega}[Im(G\omega)p - Re(\beta+\alpha^2)q]\, dt + \frac{Re\alpha}{Re\omega}\, dB \\ dp = -\frac{1}{Re\omega}[Im(\bar{\omega}(\beta+\alpha^2))q + |\omega|^2 pReG]\, dt + \frac{Im(\bar{\omega}\alpha)}{Re\omega}\, dB. \end{cases}$$

The solution of equation (4.4) can be found explicitly. Namely, let

$$\sigma = \sqrt{\frac{\beta+\alpha^2}{G}} \equiv \sqrt{\left|\frac{\beta+\alpha^2}{G}\right|}\exp\{\frac{i}{2}\arg\frac{\beta+\alpha^2}{G}\}. \tag{4.6}$$

Since $Im\, G \geq 0$ and $Re\,\beta \geq |\alpha|^2$ one sees that $-\frac{\pi}{2} \leq \arg\sigma \leq \frac{\pi}{2}$. The solution to the first equation in (4.4) is

$$\omega(t) = \begin{cases} \sigma\frac{\omega_0\coth(\sigma G t)+\sigma}{\omega_0+\sigma\coth(\sigma G t)}, & \sigma \neq 0 \\ \omega_0(1+tG\omega_0)^{-1}, & \sigma = 0. \end{cases} \tag{4.7}$$

In the case $\omega_0 \neq \sigma$ the first formula in (4.7) can be also written in the form

$$\omega = \sigma\coth(\sigma G t + \Omega(\omega_0)), \quad \Omega(\omega_0) = \frac{1}{2}\log\frac{\omega_0+\sigma}{\omega_0-\sigma}. \tag{4.8}$$

Thia implies the following result.

Proposition 4.2. *For an arbitrary solution of form (4.3) of equation (4.1)*

$$\lim_{t\to\infty}\omega(t) = \sigma.$$

From the second equation of (4.4) one gets

$$z(t) = \exp\{-G\int_0^t \omega(s)\, ds\}(z_0 + \int_0^t \exp\{G\int_0^\tau \omega(s)\, ds\}\, dB(\tau)). \tag{4.9}$$

Furthermore, since

$$\int_0^t \coth(\sigma G\tau + \Omega(\omega_0))\, d\tau = \frac{1}{G\sigma}\left[\log\sinh(\sigma G t + \Omega(\omega_0)) - \log\frac{\sigma}{\sqrt{\omega_0^2-\sigma^2}}\right],$$

inserting (4.7) in (4.9) yields

$$z(t) = (\sinh(\sigma G t + \Omega(\omega_0)))^{-1} \left[\frac{\sigma}{\sqrt{\omega_0^2 - \sigma^2}} z_0 + \alpha \int_0^t \sinh(\sigma G \tau + \Omega(\omega_0))\, dB(\tau) \right] \tag{4.10}$$

for $\sigma \neq 0$, and similarly for $\sigma = 0$

$$z(t) = (1 + tG\omega_0)^{-1}(z_0 + \alpha \int_0^t (1 + \tau G\omega_0)\, dB(\tau)). \tag{4.11}$$

From the last equation of (4.4) one gets

$$\gamma(t) = \gamma_0 + \frac{G}{2} \int_0^t (\omega(\tau) - z^2(\tau))\, d\tau, \tag{4.12}$$

and thus the following result is proved.

Proposition 4.3. *The coefficients of the Gaussian solution (4.3) of equation (4.1) are given by (4.7), (4.8), (4.10)-(4.12).*

We can prove now the main result of the Section.

Theorem 4.1. *The Green function $u_G(t, x; x_0)$ of equation (4.1) exists and has the Gaussian form*

$$u_G(t, x; x_0) = C_G^m \exp\{-\frac{\omega_G}{2}(x^2 + x_0^2) + \beta_G x x_0 - a_G x - b_G x_0 - \gamma_G\}, \tag{4.13}$$

where the coefficients C_G, ω_G, β_G are deterministic (they do not depend on the Brownian trajectory $B(t)$) and are given by

$$\omega_G = \sigma \coth(\sigma G t), \quad \beta_G = \sigma(\sinh(\sigma G t))^{-1}, \quad C_G = \left(\frac{2\pi}{\sigma} \sinh(\sigma G t) \right)^{-1/2}, \tag{4.14}$$

and

$$\omega_G = \beta_G = \frac{1}{tG}, \quad C_G = (2\pi t G)^{-1/2} \tag{4.15}$$

for the cases $\sigma \neq 0$ and $\sigma = 0$ respectively; the other coefficients are given by

$$a_G = \alpha(\sinh(\sigma G t))^{-1} \int_0^t \sinh(\sigma G \tau)\, dB(\tau), \tag{4.16}$$

$$b_G = \sigma G \int_0^t \frac{a(\tau)}{\sinh(\sigma G \tau)}\, d\tau, \quad \gamma_G = \frac{G}{2} \int_0^t a^2(\tau)\, d\tau \tag{4.17}$$

for $\sigma \neq 0$ and

$$a_G = \frac{\alpha}{t} \int_0^t \tau\, dB(\tau), \quad b_G = \int_0^t \frac{a(\tau)}{\tau}\, d\tau, \quad \gamma_G = \frac{1}{2} \int_0^t a^2(\tau)\, d\tau \tag{4.18}$$

for $\sigma = 0$.

Remark. It follows in particular that the Green function (4.13) is continuous everywhere except for the case when σG is purely imaginary, in which case (4.14) has periodical singularities.

Proof. Since the Dirac δ-function is the weak limit of Gaussian functions

$$\psi_0^\epsilon = (2\pi\epsilon)^{-1/2} \exp\{-(x-\xi)/2\epsilon\},$$

as $\epsilon \to 0$, we can calculate $u_G(t,x,x_0)$ as a limit of solutions ψ^ϵ of form (4.3) with initial data

$$\omega_0^\epsilon = \frac{1}{\epsilon}, \quad z_0^\epsilon = \frac{\xi}{\epsilon}, \quad \gamma_0^\epsilon = \frac{\xi^2}{2\epsilon} + \frac{1}{2}\log 2\pi\epsilon. \tag{4.19}$$

Since clearly

$$\Omega(\omega_0^\epsilon) = \epsilon\sigma + O(\epsilon^2), \quad (\omega_0^\epsilon - \sigma^2)^{-1/2} z_0^\epsilon \to x_0,$$

as $\epsilon \to 0$, substituting (4.19) in (4.7), (4.10), (4.11) yields

$$\lim_{\epsilon\to 0} \omega^\epsilon = \sigma \coth(\sigma G t),$$

$$\lim z^\epsilon = (\sinh(\sigma G t))^{-1} \left(\sigma x_0 + \alpha \int_0^t \sinh(\sigma G t)\, dB(\tau) \right)$$

for $\sigma \neq 0$ and

$$\lim_{\epsilon\to 0} \omega^\epsilon = \frac{1}{tG}, \quad \lim_{\epsilon\to 0} z^\epsilon = \frac{1}{tG} \left(x_0 + \alpha \int_0^t \tau G\, dB(\tau) \right)$$

for $\sigma = 0$, which implies (4.16),(4.18) and the first two formulas in (4.14), (4.15). Let us calculate γ. If $\sigma \neq 0$,

$$\int_0^t \omega^\epsilon(\tau)\, d\tau = \sigma \int_0^t \coth(\sigma G\tau + \Omega(\omega_0^\epsilon))\, d\tau$$

$$= \frac{\log\sinh(\sigma G t + \Omega(\omega_0^\epsilon) - \log\frac{\sigma}{\sqrt{\omega_0^\epsilon - \sigma^2}}}{G} = \frac{\log\sinh(\sigma G t) - \log(\epsilon\sigma) + o(1)}{G},$$

and

$$\int_0^t z^2(\tau)\, d\tau = -\frac{\sigma}{G}\xi^2 \coth(\sigma G t) + \frac{\xi^2}{G\epsilon} + o(1)$$

$$+ \int_0^t \frac{2x_0\sigma\alpha \int_0^\tau \sinh(\sigma G s)\, dB(s) + \left(\alpha \int_0^\tau \sinh(\sigma G s)\, dB(s)\right)^2}{\sinh^2(\sigma G t)}\, d\tau.$$

Substituting these formulas in (4.12) and taking the limit as $\epsilon \to 0$ yields the remaining formulas. The simpler case $\sigma = 0$ is dealt with similarly.

It is easy now to write down the Green function $\tilde{u}_G(t,p,p_0)$ of equation (4.1'). Since (4.1') is obtained from (4.1) by the Fourier transformation,

$$\tilde{u}_G(t,p,p_0) = \frac{1}{(2\pi)^m} \int\int u_G(t,x,\xi)\exp\{i\xi p_0 - ixp\}\, d\xi dx. \qquad (4.20)$$

To evaluate this integral it is convenient to change the variables (x,ξ) to $y = x+\xi$, $\eta = x - \xi$. Then (4.20) takes the form

$$\tilde{u}_G = \left(\frac{C_G}{(4\pi)}\right)^m e^{-\gamma_G} \int \exp\{-\frac{\omega_G - \beta_G}{4}y^2 - \frac{1}{2}(a_G + b_G - ip_0 + ip)y\}\, dy$$

$$\times \int \exp\{-\frac{\omega_G + \beta_G}{4}\eta^2 - \frac{1}{2}(a_G - b_G + ip_0 + ip)\eta\}\, d\eta.$$

It is easy to evaluate these Gaussian integrals using the fact that $\omega_G^2 - \beta_G^2 = \sigma^2$. This yields the following result.

Proposition 4.4. *The Green function of equation (4.1') has the form*

$$\tilde{u}_G(t,p;p_0) = \left(\frac{C_G}{\sigma}\right)^m \exp\{-\frac{\omega_G}{2\sigma^2}(p^2+p_0^2) + \frac{\beta_G}{\sigma^2}pp_0 - \tilde{a}_G p - \tilde{b}_G p_0 - \tilde{\gamma}_G\}, \quad (4.21)$$

where

$$\tilde{a}_G = -\frac{i}{\sigma^2}(\omega_G a_G + \beta_G b_G), \quad \tilde{b}_G = \frac{i}{\sigma^2}(\omega_G b_G + \beta_G a_G),$$

$$\tilde{\gamma}_G = \gamma_G - \frac{\omega_G(a_G^2 + b_G^2)}{2\sigma^2} - \frac{\beta_G a_G b_G}{\sigma^2}.$$

The explicit formulas for the Green functions of equations (4.1), (4.1') can be used in estimating the norms of the corresponding integral operators (giving the solution to the Cauchy problem of these equations) in different L_p spaces, as well as in some spaces of analytic functions, see [K7].

The Gaussian solutions constructed here can serve as convenient examples to test various asymptotic methods. Moreover, they often present the principle term of an asymptotic expansion with respect to a small time or a small diffusion coefficient for more complicated models. This will be explained in detail in the following chapters. Furthermore, since the Gaussian solutions are globally defined for all times, they can be used to study the behaviour of the solutions as time goes to infinity and to provide a basis for scattering theory. In the next section we give some results in this direction. To conclude this section let us mention another interesting property of equation 4.1 and the corresponding nonlinear equation 4.2. It was shown that the Gaussian form is preserved by the evolution defined by these equations. However, they certainly have non-Gaussian solutions as well. An interesting fact concerning these solutions is the effect of Gaussianisation, which means that every solution is asymptotically Gaussian as $t \to \infty$. Moreover, unlike the case of the unitary Schrödinger equation of a free quantum particle, where all solutions are asymptotically free waves e^{ipx} (that is, Gaussian packets with the infinite dispersion), the solutions of (4.1), (4.2) tend to a Gaussian function (4.3) with a fixed finite non-vanishing ω. This fact has an important physical interpretation (it is called the watchdog effect for continuous measurement, or the continuous collapse of the quantum state). For Gaussian solutions it is obvious (see Proposition 4.2) and was observed first in [Di2], [Be2]. For general initial data it was proved by the author in [K4] (with improvements in [K7]). We shall now state (without proof) the precise result.

Theorem 4.2 [K4],[K7]. *Let ϕ be the solution of the Cauchy problem for equation (4.2) with an arbitrary initial function $\phi_0 \in L_2$, $\|\phi_0\| = 1$. Then for a.a. trajectories of the innovating Wiener process W,*

$$\|\phi - \pi^{1/4} g^{1-i}_{q(t),p(t)}\| = O(e^{-\gamma t})$$

as $t \to \infty$, for arbitrary $\gamma \in (0,1)$, where

$$\begin{cases} q(t) = q_W + p_W t + W(t) + \int_0^t W(s)\, ds + O(e^{-\gamma t}) \\ p(t) = p_W + W + O(e^{-\gamma t}) \end{cases}$$

for some random constants q_W, p_W.

It follows in particular that the mean position of the solution behaves like the integral of the Brownian motion, which provides one of the motivations for the study of this process in the next section.

We note also that finite dimensional analogues of the localisation under continuous measurement and its applications are discussed in [K8], [Ju], [K14], where the notion of the *coefficient of the quality of measurement* was introduced to estimate this localisation quantitatively.

5. The escape rate for Gaussian diffusions and scattering theory for its perturbations

In this Section we show how the results of the two previous Sections can be used to estimate the escape rate for Gaussian diffusions and to develop the scattering theory for small perturbations. The results of this section will not be used in other parts of the book. We shall show first how one estimates the escape rate in the simplest nontrivial example, namely for the integral of the Brownian motion. Then more general models will be discussed including the stochastic Schrödinger equation.

Let $W(t)$ be the standard Wiener process in $\mathcal{R}^n$. Consider the system

$$\dot{x} = p, \quad dp = -\epsilon V'(x)\, dt + dW \tag{5.1}$$

with some positive smooth function $V(x)$ and some $\epsilon > 0$. This system describes a Newtonian particle in a potential field V disturbed by a white noise force. The global existence theorem for this system was proved in [AHZ]. Firstly, if V vanishes, the solution $x(t)$ of (3.1) is simply the integral of the Brownian motion $Y(t) = \int_0^t W(s)\, ds$.

Theorem 5.1 [K6],[K2]. *If $n > 1$, then, almost surely, $Y(t) \to \infty$, as $t \to \infty$, moreover,*

$$\liminf_{t\to\infty}(|Y(t)|/g(t)) = \infty \tag{5.2}$$

for any positive increasing function $g(t)$ on $\mathcal{R}_+$ such that $\int_0^\infty (g(t)t^{-3/2})^n\,dt$ is a convergent integral.

Remark. For example, for any $\delta > 0$ the function

$$g(t) = t^{\frac{3}{2}-\frac{1}{n}-\delta}$$

satisfies the conditions of the Theorem.

Proof. For an event B in the Wiener space we shall denote by $P(B)$ the probability of B with respect to the standard Wiener measure. The theorem is a consequence of the following assertion. Let A be a fixed positive constant and let $B^t_{A,g}$ be the event in Wiener space which consists of all trajectories W such that the set $\{Y(s) : s \in [t, t+1]\}$ has nonempty intersection with the cube $[-Ag(t), Ag(t)]^n$. Then

$$P(B^t_{A,g}) = (O(g(t)t^{-3/2}) + O(t^{-1}))^n. \tag{5.3}$$

In fact, (5.3) implies that $\sum_{m=1}^\infty P(B^m_{A,g}) < \infty$, if the conditions of the Theorem hold. Then by the first Borell-Cantelli lemma, only a finite number of the events $B^m_{A,\beta}$ can hold. Hencs there exists a constant T such that $Y(t) \notin [-Ag([t]), Ag([t])]^n$ for $t > T$, where $[t]$ denotes the integer part of t. This implies the result.

Let us now prove (5.3). Clearly, it is enough to consider the case $n = 1$. The density $p_t(x, y)$ of the joint distribution of $W(t)$ and $Y(t)$ is well known to be

$$p_t(x, y) = \frac{\sqrt{3}}{\pi t^2}\exp\left\{-\frac{2}{t}x^2 + \frac{6}{t^2}xy - \frac{6}{t^3}y^2\right\}.$$

In particular,

$$p_t(x, y) \le \frac{\sqrt{3}}{\pi t^2}\exp\left\{-\frac{x^2}{2t}\right\}. \tag{5.4}$$

It is clear that

$$P(B^t_{A,g}) = P(Y(t) \in [-Ag(t), Ag(t)])$$

$$+2\int_{Ag(t)}^\infty dy \int_{-\infty}^{+\infty} p_t(x, y) P\left(\min_{0\le\tau\le1}(y + \tau x + \int_0^\tau W(s)\,ds) < Ag(t)\right) dx \tag{5.5}$$

The first term of (5.5) is given by

$$\frac{1}{\sqrt{2\pi t^3}}\int_{-Ag(t)}^{Ag(t)} \exp\left\{-\frac{y^2}{2t^3}\right\} dy$$

and is of order $O(g(t)t^{-3/2})$. The second term can be estimated from above by the integral

$$2\int_{Ag(t)}^{\infty} dy \int_{-\infty}^{+\infty} p_t(x,y)P\left(\min_{0\le\tau\le1} \tau x + \min_{0\le\tau\le1} W(\tau) < Ag(t)-y\right) dx.$$

We decompose this integral in the sum $I_1+I_2+I_3$ of three integrals, whose domain of integration in the variable x are $\{x\ge 0\}$, $\{Ag(t)-y\le x\le 0\}$, and $\{x<Ag(t)-y\}$ respectively. We shall show that the integrals I_1 and I_2 are of order $O(t^{-3/2})$ and the integral I_3 is of order $O(t^{-1})$, which will complete the proof of (5.3).

It is clear that

$$I_1 = 2\int_{Ag(t)}^{\infty} dy \int_{0}^{\infty} p_t(x,y)P\left(\min_{0\le\tau\le1} W(\tau) < Ag(t)-y\right) dx.$$

Enlarging the domain of integration in x to the whole line, integrating over x, and using the well known distribution for the minimum of the Brownian motion we obtain

$$I_1 \le \frac{2}{\pi\sqrt{t^3}}\int_{Ag(t)}^{\infty} \exp\left\{-\frac{y^2}{2t^3}\right\} dy \int_{y-Ag(t)}^{\infty} \exp\left\{-\frac{z^2}{2}\right\} dz.$$

Changing the order of integration we can rewrite the last expression in the form

$$\frac{2}{\pi\sqrt{t^3}}\int_{0}^{\infty} \exp\left\{-\frac{z^2}{2}\right\} dz \int_{Ag(t)}^{Ag(t)+z} \exp\left\{-\frac{y^2}{2t^3}\right\} dy.$$

Consequently,

$$I_1 \le \frac{2}{\pi\sqrt{t^3}}\int_{0}^{\infty} z\exp\left\{-\frac{z^2}{2}\right\} dz = O(t^{-3/2}).$$

We continue with I_2. Making the change of variable $x\mapsto -x$ we obtain

$$I_2 = 2\int_{Ag(t)}^{\infty} dy \int_{0}^{y-Ag(t)} p_t(-x,y)P\left(\min_{0\le\tau\le1} < Ag(t)-y+x\right) dx.$$

Making the change of the variable $s=y-Ag(t)$ and using the distribution of the minimum of the Brownian motion we obtain that

$$I_2 = 2\int_{0}^{\infty} ds \int_{0}^{s} p_t(-x,s+Ag(t))\, dx\sqrt{\frac{2}{\pi}}\int_{s-x}^{\infty} \exp\left\{-\frac{z^2}{2}\right\} dz.$$

Estimating $p_t(x,y)$ by (5.4) and changing the order of integration we get

$$I_2 \le \frac{4}{\sqrt{2\pi}}\frac{\sqrt{3}}{\pi t^2}\int_{0}^{\infty} dz\exp\left\{-\frac{z^2}{2}\right\}\int_{0}^{\infty} dx\exp\left\{-\frac{x^2}{2t}\right\}\int_{x}^{x+z} ds.$$

The last integral is clearly of order $O(t^{-3/2})$. It remains to estimate the integral I_3. We have

$$I_3 = 2\int_{Ag(t)}^{\infty} dy \int_{y-Ag(t)}^{\infty} p_t(-x,y)\,dx = 2\int_0^{\infty} p_t(-x,y)\,dx \int_{Ag(t)}^{Ag(t)+x} dy$$
$$\leq \frac{2\sqrt{3}}{\pi t^2}\int_0^{\infty} x \exp\{-\frac{x^2}{2t}\}\,dx = O(t^{-1}).$$

The proof is complete.

It is evident that the method of the proof is rather general and can be applied to other processes whenever a reasonable estimate for the transition probability at large times is available. For example, one easily obtains the following generalisation of the previous result.

Theorem 5.2. *Let Y_k be the family of processes defined recurrently by the formulas $Y_k = \int_0^t Y_{k-1}(s)\,ds$, $k = 1, 2, ...$, with $Y_0 = W$ being the standard n-dimensional Brownian motion. If $n > 1$ and $f(t)$ is an increasing positive function for which the integral $\int_1^{\infty}(f(t)t^{-(k+1/2)})^n\,dt$ is convergent, then*

$$\liminf_{t\to\infty}(|Y_k(t)|/f(t)) = +\infty$$

with probability one.

The same method can be used to estimate the rate of escape for the processes discussed in Section 3. For example, for the Ornstein-Uhlenbeck process defined by the stochastic differential system

$$\begin{cases} dX = v\,dt, \\ dv = -\beta v\,dt + dW \end{cases} \tag{5.6}$$

with constant $\beta > 0$, the application of this method together with the estimate of Proposition 3.2 leads to the following result.

Theorem 5.3 [AK]. *Let $n \geq 3$ and let $f(t)$ be an increasing positive function such that $\int_1^{\infty}(f(t)/\sqrt{t})^n\,dt < \infty$. Let $X(t,[W]), v(t,[W])$ denote a solution of (5.6). Then almost surely*

$$\liminf_{t\to\infty}(|X(t,[W])|/f(t)) = \infty. \tag{5.7}$$

Similar results were obtained in [AK] for the processes described in Proposition 3.1. Infinite-dimensional generalisations of these results are also given in [AK]. Theorems 5.1-5.3 allow to develop the scattering theory for small perturbations of the corresponding Gaussian diffusions. For example, the following result is a simple corollary of Theorem 5.1 and standard arguments of deterministic scattering theory.

Theorem 5.4 [AHK1]. *Let $n > 2$ and let the vector valued function $F(x) = V'(x)$ is uniformly bounded, locally Lipschitz continuous and suppose furthermore that there exist constants $C > 0$ and $\alpha > 4n/(3n-2)$ such that*

$$|K(x)| \leq C|x|^{-\alpha} \quad \forall\, x \in \mathcal{R}^n, \tag{5.8}$$

$$|K(x) - K(y)| \le Cr^{-\alpha}|x - y| \;\; \forall \, x, y : |x|, |y| > r. \tag{5.9}$$

Then for any pair $(x_\infty, p_\infty) \in \mathcal{R}^{2n}$ and for almost all W there exists a unique pair (x_0, p_0) (depending on W) such that the solution $(\tilde{x}, \tilde{p})$ to the Cauchy problem for system (5.1) with initial data (x_0, p_0) has the following limit behaviour:

$$\lim_{t\to\infty} (\tilde{p}(t) - W(t) - p_\infty) = 0, \tag{5.10}$$

$$\lim_{t\to\infty} (\tilde{x}(t) - \int_0^t W(s)\, ds - x_\infty - tp_\infty) = 0. \tag{5.11}$$

Moreover, the mapping $\Omega_+([W]) : (x_\infty, p_\infty) \mapsto (x_0, p_0)$, which can naturally be called the random wave operator, is an injective measure preserving mapping $\mathcal{R}^{2n} \mapsto \mathcal{R}^{2n}$.

It is worth mentioning that the assumptions on the force F in the theorem are weaker than those usually adopted to prove the existence of wave operators for deterministic Newtonian systems. In particular, the long range Coulomb potential satisfies the assumption of Theorem 5.4. The reason for this lies in Theorem 5.1 which states that a particle driven by white noise force tends to infinity faster than linearly in time. The question whether Ω_+ is surjective or not can be considered as the question of asymptotic completeness of the wave operator Ω_+. The following weak result was obtained by means of Theorem 5.1 and certain estimates for the probability density for the processes defined by the system (5.1).

Theorem 5.5 [AHK2]. *Let $F(x) = V'(x)$ be bounded locally Lipschitz continuous function from $L_2(\mathcal{R}^n)$ and $n > 2$. Then there exists $\epsilon_0 > 0$ such that for arbitrary $\epsilon \in (0, \epsilon_0]$ and any (x_0, p_0) there exists with probability one a pair (x_∞, p_∞) such that (5.10),(5.11) hold for the solution of the Cauchy problem for (5.1) with initial data (x_0, p_0).*

Hence, if the conditions of Theorems 5.4 and 5.5 are satisfied, then for small $\epsilon > 0$ the random wave operator for the scattering defined by system (5.1) exists and is a measure preserving bijection (i.e. it is complete).

Similarly one can obtain the existence of the random wave operator for small perturbations of the Ornstein-Uhlenbeck process (5.6) (see details in [AK]).

The stochastic Newtonian system (5.1) formally describes the dynamics of particle in the (formal) potential field $V(x) - x\dot{W}$. The formal Schrödinger equation for the corresponding quantised system would have the form

$$ih\dot{\psi} = (-\frac{h^2}{2}\Delta + V(x))\psi - x\psi\dot{W}. \tag{5.12}$$

To write this equation in a rigorous way, one should use stochastic differentials and thus one obtains

$$ih\, d\psi = (-\frac{h^2}{2}\Delta + V(x))\psi\, dt - x\psi\, d_S W. \tag{5.13}$$

Using the transformation rule for going from the Stratonovich differential to the Ito one $\psi\, d_S W = \psi dW + \frac{1}{2} d\psi dW$ one gets the Ito form of stochastic Schrödinger equation

$$ih\, d\psi = (-\frac{h^2}{2}\Delta + V(x))\psi\, dt - \frac{i}{2h}x^2\psi\, dt - x\psi\, dW. \qquad (5.14)$$

This equation is one of the simplest (and also most important) examples of a Hudson-Parthasarathy quantum stochastic evolution (with unbounded coefficients) [HP] describing in general the coupling of a given quantum system with boson reservoir (the latter being, in particular, the simplest model of a measuring apparatus). Formally, one can easily verify (using Ito calculus) that the evolution defined by (5.14) is almost surely unitary. To make these calculations rigorous one should use the well-posedness theorem for the Cauchy problem of equation (5.14) obtained in [K1] for measurable bounded potentials V. The idea of the proof is to develop a perturbation theory, starting from equation (5.14) with vanishing potential, i.e. from the equation

$$ih\, d\phi = -\frac{h^2}{2}\Delta\phi\, dt - \frac{i}{2h}x^2\phi\, dt - x\phi\, dW. \qquad (5.15)$$

This equation has the form (4.1) with purely imaginary α, G and real β and was considered in detail in the previous Section. The properties of equation (5.15) obtained there can be used also for the development of the scattering theory for equation (5.14). Namely, using Theorem 5.1 and the Gaussian solutions (4.2) of equation (5.15) as the test solutions for the Cook method [Coo] one obtains (see details of the proof in [K2]) the existence of the wave operator for the scattering defined by the stochastic Schrödinger equation (5.14), namely, the following result.

Theorem 5.6 [K2]. *Let the potential* V *in (5.14) belong to the class* $L_r(\mathcal{R}^n)$ *for some* $r \in [2, n)$ *and let the dimension* n *be greater than* 2. *Then for each solution of (4.3) (defined by an initial function* $\psi_0 \in L_2(\mathcal{R}^n)$*) there exists with probability one a solution* ϕ *of (5.15) such that, in* $L_2(\mathcal{R}^n)$,

$$\lim_{t\to\infty}(\psi(t) - \phi(t)) = 0.$$

This result is a more or less straightforward generalisation of the corresponding deterministic result. Apparently a deeper theory is required for the consideration of the perturbations of the general equation (4.1), because already the "free" dynamics for this case is much more complicated, as Theorem 4.2 states.