Chapter 3. SEMICLASSICAL APPROXIMATION FOR REGULAR DIFFUSION

1. Main ideas of the WKB-method with imaginary phase

In this chapter we construct exponential WKB-type asymptotics for solutions of equations of type

$$h\frac{\partial u}{\partial t} = \frac{h^2}{2} tr \left(G(x) \frac{\partial^2 u}{\partial x^2} \right) + h \left(A(x), \frac{\partial u}{\partial x} \right) - V(x), \tag{1.1}$$

where $t \leq 0, x \in \mathcal{R}^m$, V, A and G are smooth real, vector-valued, and matrix-valued functions on $\mathcal{R}^m$ respectively, $G(x)$ is symmetric non-negative, and h is a positive parameter. Equivalently, one can write equation (1.1) in the "pseudo-differential form"

$$h\frac{\partial u}{\partial t} = H\left(x, -h\frac{\partial}{\partial x}\right) u = Lu \tag{1.2}$$

with the Hamiltonian function

$$H(x,p) = \frac{1}{2}(G(x)p,p) - (A(x),p) - V(x). \tag{1.3}$$

Our main aim will be the construction of the *Green function* of equation (1.1), i.e. of the solution $u_G(t,x,x_0)$ with Dirac initial data

$$u_G(t,x,x_0) = \delta(x - x_0). \tag{1.4}$$

The solution of the Cauchy problem for equation (1.1) with general initial data $u_0(x)$ can be then given by the standard integral formula

$$u(t,x) = \int u_G(t,x,x_0) u_0(x_0)\, dx_0. \tag{1.5}$$

In this introductory section we describe the main general steps of the construction of the formal asymptotic solution for the problem given by (1.1) and (1.4), presenting in a compact but systematic way rather well-known ideas (see, e.g. [MF1], [M1],[M2], [KM2]), which were previously used only for non-degenerate diffusions, i.e. when the matrix G in (1.1) was non-degenerate (and usually only for some special cases, see [MC1],[DKM1], [KM2]). Here we shall show that these ideas can also be applied for the case of regular (in particular degenerate) Hamiltonians of type (1.3) introduced and discussed in the previous chapter from the point of view of the calculus of variations. In fact, the results of the previous chapter form the basis that allows us to carry out successfully (effectively and rigorously) the general steps described in this section. Moreover, it seems that regular Hamiltonians form the most general class, for which it can be done in this way. As we shall see in Section 3.6, for non-regular Hamiltonians, the procedure must be modified essentially, even at the level of formal expansions, if one is interested in small time asymptotics, but for small h asymptotics (with fixed t), this procedure seems to lead to correct results even for non-regular degenerate diffusions. We shall construct two types of asymptotics for (1.1), (1.4), namely, small time asymptotics, when $t \to 0$ and h is fixed, say $h = 1$, and (global) small diffusion asymptotics, when t is any finite number and $h \to 0$.

Step 1. One looks for the asymptotic solution of (1.1), (1.4) for small h in the form

$$u_G^{as}(t, x, x_0, h) = C(h)\phi(t, x, x_0) \exp\{-S(t, x, x_0)/h\}, \tag{1.6}$$

where S is some non-negative function called the action or entropy, and $C(h)$ is a normalising coefficient.

In the standard WKB method traditionally used in quantum mechanics to solve the Schrödinger equation, one looks for the solutions in "oscillatory form"

$$C(h)\phi(t, x, x_0) \exp\{-\frac{i}{h} S(t, x, x_0)\} \tag{1.7}$$

with real functions ϕ and S called the amplitude and the phase respectively. For this reason one refers sometimes to the ansatz (1.6) as to the WKB method with imaginary phase, or as to exponential asymptotics, because it is exponentially small outside the zero-set of S. The difference between the asymptotics of the types (1.6) and (1.7) is quite essential. On the one hand, when justifying the standard WKB asymptotics of type (1.7) one should prove that the exact solution has the form

$$C(h)\phi(t, x, x_0) \exp\{-\frac{i}{h} S(t, x, x_0)\} + O(h) \tag{1.8}$$

(additive remainder), which can be proved under rather general conditions by L^2 methods of functional analysis [MF1]. For asymptotics of form (1.6) this type of justification would make no sense, because the expression (1.6) is exponentially small outside the zero-set of S. Thus, to justify (1.6) one should instead prove that the exact solution has the form

$$C(h)\phi(t, x, x_0) \exp\{-S(t, x, x_0)/h\}(1 + O(h)), \tag{1.9}$$

which must be carried out by some special pointwise estimates. Because of the multiplicative remainder in (1.9) one calls asymptotics of this type multiplicative. On the other hand, essential difference between (1.6) and (1.7) lies in the fact that if one adds different asymptotic expressions of form (1.9), then, unlike the case of the asymptotics (1.8), in the sum only the term with the minimal entropy survives at each point (because other terms are exponentially small in compared with this one), and therefore for the asymptotics (1.9) the superposition principle transforms into the idempotent superposition principle $(S_1, S_2) \mapsto \min(S_1, S_2)$ at the level of actions. For a detailed discussion of this idempotent superposition principle and its applications see [KM1],[KM2].

It seems that among parabolic differential equations only second order equations can have asymptotics of the Green function of form (1.9). Considering more general pseudo-differential equations one gets other classes, which enjoy this property, for example, the so called tunnel equations introduced in [M1], [M2] (see Chapter 6).

Inserting (1.6) in (1.1) yields

$$h\left(\frac{\partial\phi}{\partial t}-\frac{1}{h}\phi\frac{\partial S}{\partial t}\right)=\frac{h^2}{2}tr\,G(x)\left(\frac{\partial^2\phi}{\partial x^2}-\frac{\phi}{h}\frac{\partial^2 S}{\partial x^2}\right)+h\left(A(x),\frac{\partial\phi}{\partial x}-\frac{1}{h}\phi\frac{\partial S}{\partial x}\right)$$

$$+\frac{1}{2}\left(G(x)\frac{\partial S}{\partial x},\frac{\partial S}{\partial x}\right)\phi-h\left(G(x)\frac{\partial S}{\partial x},\frac{\partial\phi}{\partial x}\right)-V(x). \tag{1.10}$$

Comparing coefficients of h^0 yields the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t}+H\left(x,\frac{\partial S}{\partial x}\right)=0 \tag{1.11}$$

corresponding to the Hamiltonian function (1.3), or more explicitly

$$\frac{\partial S}{\partial t}+\frac{1}{2}\left(G(x)\frac{\partial S}{\partial x},\frac{\partial S}{\partial x}\right)-\left(A(x),\frac{\partial S}{\partial x}\right)-V(x)=0. \tag{1.12}$$

Comparing coefficients of h one gets the so called transport equation

$$\frac{\partial\phi}{\partial t}+\left(\frac{\partial\phi}{\partial x},\frac{\partial H}{\partial p}(x,\frac{\partial S}{\partial x})\right)+\frac{1}{2}\,tr\left(\frac{\partial^2 S}{\partial x^2}\frac{\partial^2 H}{\partial p^2}(x,\frac{\partial S}{\partial x})\right)\phi(x)=0, \tag{1.13}$$

or more explicitly

$$\frac{\partial\phi}{\partial t}-\left(A(x),\frac{\partial\phi}{\partial x}\right)+\left(G(x)\frac{\partial S}{\partial x},\frac{\partial\phi}{\partial x}\right)+\frac{1}{2}tr\left(G(x)\frac{\partial^2 S}{\partial x^2}\right)\phi=0. \tag{1.13$'$}$$

Therefore, if S and ϕ satisfy (1.12), (1.13), then the function u of form (1.6) satisfies equation (1.1) up to a term of order h^2, i.e.

$$h\frac{\partial u_G^{as}}{\partial t}-H\left(x,-h\frac{\partial}{\partial x}\right)u_G^{as}=-\frac{h^2}{2}C(h)\,tr\left(G(x)\frac{\partial^2\phi}{\partial x^2}\right)\exp\{-\frac{S(t,x)}{h}\}. \tag{1.14}$$

As is well known and as was explained in the previous chapter, the solutions of the Hamilton-Jacobi equation (1.12) can be expressed in terms of the solutions of the corresponding Hamiltonian system

$$\begin{cases}\dot{x}=-\frac{\partial H}{\partial p}=G(x)p-A(x)\\ \dot{p}=\frac{\partial H}{\partial x}=\frac{\partial}{\partial x}(A(x),p)-\frac{1}{2}\frac{\partial}{\partial x}(G(x)p,p)+\frac{\partial V}{\partial x}\end{cases}. \tag{1.15}$$

Step 2. If we had to solve the Cauchy problem for equation (1.1) with a smooth initial function of form (1.6), then clearly in order to get an asymptotic solution in form (1.6) we would have to solve the Cauchy problem for the Hamilton-Jacobi equation (1.11). The question then arises, what solution of

(1.11) (with what initial data) one should take in order to get asymptotics for the Green function. The answer is the following. If the assumptions of Theorem 2.1.1 hold, i.e. there exists a family $\Gamma(x_0)$ of characteristics of (1.15) going out of x_0 and covering some neighbourhood of x_0 for all sufficiently small t, then one should take as the required solution of (1.15) the two-point function $S(t,x,x_0)$ defined in the previous chapter (see formulas (2.1.5), (2.1.6) for local definition and Proposition 2.2.8 for global definition). As was proved in the previous chapter (see Propositions 2.2.8 and 2.3.7), for regular Hamiltonians this function is an almost everywhere solution of the Hamilton-Jacobi equation. One of the reasons for this choice of the solution of (1.12) lies in the fact that for the Gaussian diffusion described in the first chapter this choice of S leads to the exact formula for Green function. Another reason can be obtained considering the Fourier transform of equation (1.1). Yet another explanation is connected with the observation that when considering systematically the idempotent superposition principle on actions as described above one finds that the resolving operator for the Cauchy problem (of generalised solutions) for the nonlinear equation (1.11) is "linear" with respect to this superposition principle and the two-point function $S(t,x,x_0)$ can be interpreted as well as "the Green function" for (1.11) (see details in [KM1],[KM2]). Therefore, by the "correspondence principle", the Green function for (1.1) should correspond to "the Green function" for (1.11). All this reasoning are clearly heuristic, and the rigorous justification of asymptotics constructed in this way needs to be given independently.

Step 3. This is to construct solutions of the transport equation (1.13). The construction is based on the well known (and easily proved) Liouville theorem, which states that if the matrix $\frac{\partial x}{\partial \alpha}$ of derivatives of the solution of an m-dimensional system of ordinary differential equations $\dot{x} = f(x,\alpha)$ with respect to any m-dimensional parameter α is non-degenerate on some time interval, then the determinant J of this matrix satisfies the equation $\dot{J} = Jtr\,\frac{\partial f}{\partial x}$. Let us apply the Liouville theorem to the system

$$\dot{x} = \frac{\partial H}{\partial p}\left(x, \frac{\partial S}{\partial x}(t,x,x_0)\right),$$

which is the first equation of (1.15), whose momentum p is expressed in terms of the derivatives of the two-point function according to Proposition 1.1 of the second chapter. Considering the initial momentum p_0 as the parameter α one gets in this way that on the characteristic $X(t,x_0,p_0)$ the determinant $J = \det \frac{\partial X}{\partial p_0}$ satisfies the equation

$$\dot{J} = J\,tr\,\left(\frac{\partial^2 H}{\partial p \partial x} + \frac{\partial^2 H}{\partial p^2}\frac{\partial^2 S}{\partial x^2}\right),$$

or more explicitly (using (1.4))

$$\dot{J} = J\,tr\,\left(G\frac{\partial^2 S}{\partial x^2} - \frac{\partial A}{\partial x} + \frac{\partial G}{\partial x}p\right),$$

which yields the equation for $J^{-1/2}$

$$(J^{-1/2})^{\cdot} = -\frac{1}{2} J^{-1/2}\, tr \left(\frac{\partial^2 H}{\partial p \partial x} + \frac{\partial^2 H}{\partial p^2} \frac{\partial^2 S}{\partial x^2} \right), \tag{1.16}$$

or more explicitly

$$(J^{-1/2})^{\cdot} = -\frac{1}{2} J^{-1/2} tr \left(G \frac{\partial^2 S}{\partial x^2} - \frac{\partial A}{\partial x} + \frac{\partial G}{\partial x} p \right). \tag{1.16'}$$

Now consider the behaviour of the function ϕ satisfying the transport equation (1.13) along the characteristic $X(t, x_0, p_0)$. The total time derivative is $\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x}$. Consequently denoting this total time derivative by a dot above a function and using (1.15) one can rewrite (1.13) as

$$\dot{\phi} + \frac{1}{2} \phi\, tr \left(\frac{\partial^2 H}{\partial p^2}(x, \frac{\partial S}{\partial x}) \frac{\partial^2 S}{\partial x^2} \right) = 0, \tag{1.17}$$

or more explicitly

$$\dot{\phi} + \frac{1}{2} \phi\, tr \left(G(x) \frac{\partial^2 S}{\partial x^2} \right) = 0. \tag{1.17'}$$

This is a first order linear equation, whose solution is therefore unique up to a constant multiplier. Introducing the new unknown function α by $\phi = J^{-1/2} \alpha$ one gets the following equation for α using (1.16),(1.17):

$$\dot{\alpha} = \frac{1}{2}\, \alpha\, tr\, \frac{\partial^2 H}{\partial p \partial x}(x, \frac{\partial S}{\partial x}),$$

whose solution can be expressed in terms of the solutions of (1.15). Thus one finds a solution for (1.13) in the form

$$\phi(t, x, x_0) = J^{-1/2}(t, x, x_0) \exp \left\{ \frac{1}{2} \int_0^t tr\, \frac{\partial^2 H}{\partial p \partial x}(x(t)) p(t)\, d\tau \right\}, \tag{1.18}$$

or more explicitly

$$\phi(t, x, x_0) = J^{-1/2}(t, x, x_0) \exp \left\{ \frac{1}{2} \int_0^t tr \left(\frac{\partial G}{\partial x}(x(t)) p(t)) - \frac{\partial A}{\partial x}(x(t)) \right) d\tau \right\}, \tag{1.18'}$$

where the integral is taken along the solution $(X, P)(t, x_0, p_0(t, x, x_0))$ of (1.15) joining x_0 and x in time t.

Notice that $J^{-1/2}(t, x, x_0)$ and therefore the whole function (1.18) are well defined only at regular points (see the definitions at the end of Section 2.2), because at these points the minimising characteristic joining x_0 and x in time t is unique and J does not vanish there. This is why in order to get a globally defined function of form (1.6) (even for small t) one should introduce a molyfier.

Namely, let for $t \in (0, t_0]$ and x in some domain $D = D(x_0)$, all points (t, x) are regular (such t_0 and domain $D(x_0)$ exist for regular Hamiltonians again due to the results of the previous chapter) and let χ_D be a smooth function such that χ_D vanishes outside D, is equal to one inside D except for the points in a neighbourhood of the boundary ∂D of D, and takes value in $[0, 1]$ everywhere. Then the function

$$u_G^{as} = C(h)\chi_D(x - x_0)\phi(t, x, x_0)\exp\{-S(t, x, x_0)/h\} \qquad (1.19)$$

with $\phi(t, x; x_0)$ of form (1.18) is globally well defined for $t \le t_0$ and (by (1.14)) satisfies the following equation:

$$h\frac{\partial u_G^{as}}{\partial t} - H\left(x, -h\frac{\partial}{\partial x}\right) u_G^{as} = -h^2 F(t, x, x_0), \qquad (1.20)$$

where F (which also depends on h and D) is equal to

$$F = C(h)\left[\frac{1}{2}tr\left(G(x)\frac{\partial^2\phi}{\partial x^2}\right)\chi_D(x - x_0) - h^{-1}f(x)\right] \times \exp\left\{-\frac{S(t, x, x_0)}{h}\right\}, \qquad (1.21)$$

where $f(x)$ has the form $O(\phi)(1 + O(\frac{\partial S}{\partial x})) + O(\frac{\partial \phi}{\partial x})$ and is non vanishing only in a neighbourhood of the boundary ∂D of the domain D.

Step 4. All the constructions that we have described are correct for regular Hamiltonians by the results of the previous chapter. The only thing that remains in the formal asymptotic construction is to show that u_G^{as} as defined by (1.19) satisfies the Dirac initial condition (1.4). But for regular Hamiltonians this is simple, because, as we have seen in the previous chapter and as we shall show again by another method in the next section, the main term (for small t and $x - x_0$) of the asymptotics of the two-point function is the same as for its quadratic or Gaussian diffusion approximation, and one can simply refer to the results of the first chapter. Alternatively, having the main term of the asymptotics of $S(t, x, x_0)$, one proves the initial condition property of (1.19) (with appropriate coefficient $C(h)$) by means of the asymptotic formula for Laplace integrals, see e.g. Appendix B. Consequently, the function (1.19) is a formal asymptotic solution of the problem given by (1.1), (1.4) in the sense that it satisfies the initial conditions (1.4), and satisfies equation (1.1) approximately up to order $O(h^2)$. Moreover, as we shall see further in Section 4, the exact Green function will have the form (1.9) with the multiplicative remainder $1 + O(h)$ having the form $1 + O(ht)$, which will imply that we have got automatically also the multiplicative asymptotics for the Green function for small times and fixed h, say $h = 1$. The same remark also applies to the next terms of the asymptotics which are described below.

Step 5. Till now we have constructed asymptotic solutions to (1.1) up to terms of the order $O(h^2)$. In order to construct more precise asymptotics (up to order $O(h^k)$ with arbitrary $k > 2$), one should take instead of the ansatz (1.6) the expansion

$$u_G^{as} = C(h)(\phi_0(t, x) + h\phi_1(t, x) + ... + h^k\phi_k(t, x))\exp\{-S(t, x)/h\}. \qquad (1.22)$$

Inserting this in (1.1) and comparing the coefficients of h^j, $j = 0, 1, ..., k+1$, one sees that

$$h\frac{\partial u_G^{as}}{\partial t} - H\left(x, -h\frac{\partial}{\partial x}\right) u_G^{as} = \frac{h^{k+2}}{2} tr\left(G(x)\frac{\partial^2 \phi_k}{\partial x^2}\right) \exp\{-\frac{S(t,x)}{h}\}, \quad (1.23)$$

if (1.12), (1.13) hold for S and ϕ_0 and the following recurrent system of equations (higher order transport equations) hold for the functions ϕ_j, $j = 1, ..., k$:

$$\frac{\partial \phi_j}{\partial t} - \left(A(x), \frac{\partial \phi_j}{\partial x}\right) + \left(G(x)\frac{\partial S}{\partial x}, \frac{\partial \phi}{\partial x}\right) + \frac{1}{2} tr\left(G(x)\frac{\partial^2 S}{\partial x^2}\right)\phi_j$$

$$= \frac{1}{2} tr\left(G(x)\frac{\partial^2 \phi_{j-1}}{\partial x^2}\right),$$

which takes the form

$$\dot{\phi}_j + \frac{1}{2} tr\left(G(x)\frac{\partial^2 S}{\partial x^2}\right)\phi_j = \frac{1}{2} tr\left(G(x)\frac{\partial^2 \phi_{j-1}}{\partial x^2}\right) \quad (1.24)$$

in terms of the total derivative along the characteristics. The change of unknown $\phi_k = \phi_0 \psi_k, k = 1, 2, ...$, yields

$$\dot{\psi}_j = \frac{1}{2}\phi_0^{-1} tr\left(G(x)\frac{\partial^2(\psi_{j-1}\phi_0)}{\partial x^2}\right)$$

and the solution to this equation with vanishing initial data can be found recursively by the integration

$$\psi_j = \frac{1}{2}\int_0^t \phi_0^{-1} tr\left(G(x)\frac{\partial^2(\psi_{j-1}\phi_0)}{\partial x^2}\right)(x(\tau))\, d\tau. \quad (1.25)$$

By this procedure one gets a function of form (1.22), which is a formal asymptotic solution of (1.1) and (1.4) of order $O(h^{k+2})$, i.e. it satisfies the initial condition (1.4) exactly and satisfies equation (1.1) approximately up to order $O(h^{k+2})$, or more precisely, since each ψ_j is obtained by integration, for small times t the remainder is of the form $O(t^k h^{k+2})$.

Example. To conclude this section, let us show how the method works on the simple example of Gaussian diffusions presenting the analytic proof of formula (1.1.4). Consider the Hamiltonian

$$H = -(Ax, p) + \frac{1}{2}(Gp, p) \quad (1.26)$$

with constant matrices A, G, and the corresponding equation (1.2):

$$\frac{\partial u}{\partial t} = \frac{h}{2}\, tr\left(G\frac{\partial^2 u}{\partial x^2}\right) + \left(Ax, \frac{\partial u}{\partial x}\right). \quad (1.27)$$

Let us use formulas (2.1.5), (2.1.4) to calculate the two-point function $S(t,x,x_0)$. For this purpose we need to solve the boundary value problem for the corresponding Hamiltonian system

$$\begin{cases} \dot{x} = -Ax + Gp \\ \dot{p} = A'p. \end{cases} \tag{1.28}$$

The solution of (1.28) with initial data x_0, p_0 has the form

$$\begin{cases} P = e^{A't}p_o \\ X = e^{-At}x_0 + \int_0^t e^{-A(t-\tau)}Ge^{A'\tau}\,d\tau\, p_0, \end{cases}$$

and therefore the function $p_0(t,x,x_0)$ defined by (2.1.2) is (globally) well defined if the matrix $E(t)$ of form (1.3) is non-degenerate, and is given by

$$p_0(t,x,x_0) = E^{-1}(t)(e^{At}x - x_0).$$

Therefore from (2.1.4), (2.1.5) one gets

$$S(t,x,x_0) = \frac{1}{2}\int_0^t (Gp(\tau),p(\tau))\,d\tau = \frac{1}{2}\int_0^t (Ge^{A'\tau}p_0, e^{A'\tau}p_0)\,d\tau$$

$$= \frac{1}{2}(E(t)p_0,p_0) = \frac{1}{2}(E^{-1}(t)(x_0 - e^{At}x), x_0 - e^{At}x)$$

and from (1.18)

$$\phi = (\det \frac{\partial X}{\partial p_0})^{-1/2} e^{t\,tr\,A/2} = (\det E(t))^{-1/2}.$$

It follows from (1.14) that since ϕ does not depend on x, the r.h.s. of (1.14) vanishes, i.e. in the situation under consideration the asymptotic solution of form (1.6) constructed is in fact an exact solution and is defined globally for all t and x. Therefore one gets the formula for the Green function

$$(2\pi h)^{-m/2}(\det E(t))^{-1/2}\exp\{-\frac{1}{2h}(E^{-1}(x_0 - e^{At}x), x_0 - e^{At}x)\}, \tag{1.29}$$

the coefficient $C(h)$ being chosen in the form $(2\pi h)^{-m/2}$ in order to meet the initial condition (1.4), which one verifies directly. Expression (1.29) coincide with (1.1.4) for $h = 1$.

2. Calculation of the two-point function for regular Hamiltonians

The most important ingredient in the asymptotics of second order parabolic equations is the two-point function. It was investigated in the previous chapter in the case of regular Hamiltonians; it was proved that this function is smooth and satisfies the Hamilton-Jacobi equation almost everywhere, and a method of calculation of its asymptotic for small times and small distances was proposed: by means of the asymptotic solutions of the boundary value problem for corresponding Hamiltonian system. In this section we describe an alternative, more direct method of its calculation for small times and small distances. This method seems to be simpler for calculations but without the rigorous results of the previous chapter, the proof of the correctness of this method seems to be rather difficult problem (especially when the coefficients of the Hamiltonian are not real analytic).

In the case of a non-degenerate matrix G in (1.12), one can represent the two-point function in the form (2.2.17) for small t and $x - x_0$. Substituting (2.2.17) in (1.12) yields recursive formulas, by which the coefficients of this expansion can be calculated to any required order. These calculations are widely represented in the literature, and therefore we omit the details here. We shall deal more carefully with degenerate regular Hamiltonians of form (2.3.1), where the corresponding Hamilton-Jacobi equation has the form

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(g(x)\frac{\partial S}{\partial y}, \frac{\partial S}{\partial y}\right) - a(x,y)\frac{\partial S}{\partial x} - b(x,y)\frac{\partial S}{\partial y} - V(x,y) = 0. \qquad (2.1)$$

A naive attempt to try to solve this equation (following the procedure of the non-degenerate case) by substituting into this equation an expression of the form $Reg(t, x - x_0, y - y_0)/t$ with Reg being a regular expansion with respect to its argument (i.e. as a non-negative power series), or even more generally substituting $t^{-l}Reg$ with some $l > 0$, does not lead to recurrent equations but to a difficult system, for which neither uniqueness nor existence of the solution is clear even at the formal level. In order to get recurrent equations one should chose the arguments of the expansion in a more sophisticated way. Corollary to Proposition 2.1.3 suggests that it is convenient to make the (non-homogeneous) shift of the variables, introducing a new unknown function

$$\sigma(t,x,y) = S(t, x + \tilde{x}(t), y + \tilde{y}(t), x_0, y_0), \qquad (2.2)$$

where $(\tilde{x}, \tilde{y}, \tilde{p}, \tilde{q})(t)$ denote the solution of the corresponding Hamiltonian system with initial conditions $(x_0, y_0, 0, 0)$. In terms of the function σ the Hamilton-Jacobi equation (2.1) takes the form

$$\frac{\partial \sigma}{\partial t} - (a(x+\tilde{x}(t), y+\tilde{y}(t)) - a(\tilde{x}(t), \tilde{y}(t)))\frac{\partial \sigma}{\partial x} - (b(x+\tilde{x}(t), y+\tilde{y}(t)) - b(\tilde{x}(t), \tilde{y}(t)))\frac{\partial \sigma}{\partial y}$$

$$-g(\tilde{x}(t), \tilde{y}(t))\tilde{q}(t)\frac{\partial \sigma}{\partial y}$$

$$+\frac{1}{2}\left(g(x+\tilde{x}(t), y+\tilde{y}(t))\frac{\partial \sigma}{\partial y}, \frac{\partial \sigma}{\partial y}\right) - V(x+\tilde{x}(t), y+\tilde{y}(t)) = 0. \qquad (2.3)$$

The key idea (suggested by Proposition 2.3.7) for the asymptotic solution of this equation in the case of general regular Hamiltonian (2.4.1) is to make the change of variables $(x^0, ..., x^M, y) \mapsto (\xi^0, ..., \xi^M, y)$ defined by the formula $x^I = t^{M-I+1}\xi^I$, $I = 0, ..., M$. Introducing the new unknown function by the formula

$$\Sigma(t, \xi^0, ..., \xi^M, y) = \sigma(t, t^{M+1}\xi^0, ..., t\xi^M, y) \tag{2.4}$$

and noting that

$$\frac{\partial \Sigma}{\partial \xi^I} = \frac{\partial \sigma}{\partial x^I} t^{M-I+1}, \quad I = 0, ..., M,$$

$$\frac{\partial \Sigma}{\partial t} = \frac{\partial \sigma}{\partial t} + (M+1)t^M \xi^0 \frac{\partial \sigma}{\partial x^0} + ... + \xi^M \frac{\partial \sigma}{\partial x^M},$$

one write down equation (2.3) for the case of Hamiltonian (2.4.1) in terms of the function Σ:

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{2}\left(g(t^{M+1}\xi^0 + \tilde{x}^0(t)) \frac{\partial \Sigma}{\partial y}, \frac{\partial \Sigma}{\partial y} \right)$$

$$- \left(g(\tilde{x}^0(t)) \tilde{q}(t), \frac{\partial \Sigma}{\partial y} \right) - R_{2(M+2)}(x(t,\xi) + \tilde{x}(t), y + \tilde{y})$$

$$- \frac{t^M \xi^0 + R_1(x(t,\xi) + \tilde{x}(t), y + \tilde{y}(t)) - R_1(\tilde{x}(t), \tilde{y}(t))}{t^{M+1}} \frac{\partial \Sigma}{\partial \xi^0} - ...$$

$$- \frac{\xi^M + R_{M+1}(x(t,\xi) + \tilde{x}(t), y + \tilde{y}(t)) - R_{M+1}(\tilde{x}(t), \tilde{y}(t))}{t} \frac{\partial \Sigma}{\partial \xi^M}$$

$$-(R_{M+2}(x(t,\xi) + \tilde{x}(t), y + \tilde{y}(t)) - R_{M+2}(\tilde{x}(t), \tilde{y}(t))) \frac{\partial \Sigma}{\partial y}. \tag{2.5}$$

It turns out that by expanding the solution of this equation as a power series in its arguments one does get uniquely solvable recurrent equations. In this procedure lies indeed the source of the main definition of regular Hamiltonians, which may appear to be rather artificial at first sight. This definition insures that after expansion of obtained formal power series solution of (2.5) in terms of initial variables (t, x, y) one gets the expansion of S in the form $t^{-(2M+3)} Reg(t, x - \tilde{x}, y - \tilde{y})$ (where Reg is again a regular, i. e. positive power series, expansion with respect to its arguments), and not in the form of a Laurent type expansion with infinitely many negatives powers. More precisely, the following main result holds.

Theorem 2.1. *Under the assumptions of the main definition of regular Hamiltonians (see Sect. 4 in Chapter 2), there exists a unique solution of equation (2.5) of the form*

$$\Sigma = \frac{\Sigma_{-1}}{t} + \Sigma_0 + t\Sigma_1 + t^2\Sigma_2 + ... \tag{2.6}$$

such that Σ_{-1} and Σ_0 vanish in the origin, Σ_{-1} is strictly convex in a neighbourhood of the origin, and all Σ_j are regular power series in (ξ, y). Moreover, in this solution *(i)* Σ_{-1} *is the quadratic form with the matrix* $\frac{1}{2}(E^0)^{-1}(1)$, *where* $E^0(t)$ *and its inverse are as in Lemmas 1.2.2, 1.2.4,*

(ii) all Σ_j *are polynomials in* (ξ, y) *such that the degrees of* Σ_{-1}, Σ_0,..., Σ_{M-1} *do not exceed 2, the degrees of* Σ_M,..., Σ_{2M} *do not exceed 3 and so on, i.e. the degree of* $\Sigma_{k(M+1)-1+j}$ *does not exceed* $k+2$ *for any* $j = 0, ..., M$ *and* $k = 0, 1, ...$

Corollary. *For the function* σ *corresponding to* Σ *according to (2.4) (and to the two-point function* S *according to (2.2)) one obtains an expansion in the form*

$$\sigma(t, x, y) = t^{-(2M+3)} \sum_{j=0}^{\infty} t^j P_j(x, y), \tag{2.7}$$

where each P_j *is a polynomial in* $(x^1, ..., x^{M+1})$ *of degree* $\deg_{\mathcal{M}} P_j \leq j$ *with the coefficients being regular power series in* x^0. *Moreover,* P_0 *and its derivative vanish at the origin.*

Remark 1. Clearly, in order to have the complete series (2.6) or (2.7) one must assume all coefficients of the Hamiltonian to be infinitely differentiable functions of x^0. More generally, one should understand the result of Theorem 2.1 in the sense that if these coefficients have continuous derivatives up to the order j, then the terms Σ_l of expansion (2.6) can be uniquely defined for $l = -1, 0, 1, ..., j-1$ and the remainder can be estimated by means of the Theorem 3.2 of the previous chapter.

Remark 2. One can see that if the conditions on the degrees of the coefficients of the regular Hamiltonians (RH) 2.4.1 are not satisfied, then after the series expansion of the solution (2.6) with respect to t, x, y one necessarily gets infinitely many negative powers of t and therefore the definition of RH gives a necessary and sufficient condition for a second order parabolic equation to have the asymptotics of the Green function in the form (0.6) with ϕ and S being regular power series in $(x - x_0), t$ (up to a multiplier $t^{-\alpha}$).

This section is devoted to the constructive proof of Theorem 2.1, which is based essentially on Proposition 7.1 obtained in the special section at the end of this chapter. For brevity, we confine ourselves to the simplest nontrivial case, when $M = 0$, i.e. to the case, which was considered in detail in Section 2.3, and we shall use the notations of Section 2.3. The general case is similar, but one should use the canonical coordinate system in a neighbourhood of x_0, y_0, which was described in Chapter 2. In the case of Hamiltonian (2.3.4), equation (2.5) takes the form

$$\frac{\partial \Sigma}{\partial t} - \frac{\xi + (a(t\xi + \tilde{x}) - a(\tilde{x})) + \alpha(t\xi + \tilde{x})(y + \tilde{y}) - \alpha(\tilde{x})\tilde{y}}{t} \frac{\partial \Sigma}{\partial \xi}$$

$$-[b(t\xi + \tilde{x}) - b(\tilde{x})) + (\beta(t\xi + \tilde{x})(y + \tilde{y}) - \beta(\tilde{x})\tilde{y})$$

$$+\frac{1}{2}(\gamma(t\xi + \tilde{x})(y + \tilde{y}), y + \tilde{y}) - \frac{1}{2}(\gamma(\tilde{x})\tilde{y}, \tilde{y})] \frac{\partial \Sigma}{\partial y}$$

$$-\left(g(\tilde{x})\tilde{q}, \frac{\partial \Sigma}{\partial y}\right) + \frac{1}{2}\left(g(t\xi + \tilde{x})\frac{\partial \Sigma}{\partial y}, \frac{\partial \Sigma}{\partial y}\right) - V(t\xi + \tilde{x}, y + \tilde{y}) = 0, \tag{2.8}$$

where

$$\Sigma(t, \xi, y) = \sigma(t, t\xi, y) = S(t, t\xi + \tilde{x}, y + \tilde{y}; x_0, y^0). \tag{2.9}$$

We are looking for the solution of (2.8) of form (2.6) with additional conditions as described in Theorem 2.1. Inserting (2.6) in (2.8) and comparing coefficients of t^{-2} yields

$$-\Sigma_{-1} - (\alpha_0 y + \xi)\frac{\partial \Sigma_{-1}}{\partial \xi} + \frac{1}{2}\left(g_0\frac{\partial \Sigma_{-1}}{\partial y}, \frac{\partial \Sigma_{-1}}{\partial y}\right) = 0. \tag{2.10}$$

Actually we already know the solution. It is the quadratic form (2.3.49) with $t = 1$ and $\delta = 0$. However, let us prove independently in the present setting a stronger result, which includes the uniqueness property.

Proposition 2.1. *Under the additional assumption that Σ_{-1} vanishes and is strictly convex at the origin, equation (2.10) has a unique analytic solution (in fact a unique solution in the class of formal power series) and this solution is the quadratic form (2.3.49), where $t = 1$ and $\delta = 0$.*

Proof. We consider only the case $k = n$, i.e. a quadratic non-degenerate matrix $\alpha(x_0)$. The general case similar, but one needs the decomposition of Y as the direct sum of the kernel of $\alpha(x_0)\sqrt{g(x_0)}$ and its orthogonal complement (as in Section 2.3), which results in more complicated expressions. Thus, let α_0 be quadratic and non-degenerate. It follows then directly that $\Sigma_{-1}(0,0) = 0$ implies that $\frac{\partial \Sigma_{-1}}{\partial y}(0,0) = 0$. Furthermore, differentiating (2.10) with respect to y yields

$$-\frac{\partial \Sigma_{-1}}{\partial y} - \alpha_0\frac{\partial \Sigma_{-1}}{\partial \xi} - (\alpha_0 y + \xi)\frac{\partial^2 \Sigma_{-1}}{\partial y \partial \xi} + \left(g_0\frac{\partial \Sigma_{-1}}{\partial y}, \frac{\partial^2 \Sigma_{-1}}{\partial y^2}\right) = 0,$$

and using the non-degeneracy of α_0 one sees that $\frac{\partial \Sigma_{-1}}{\partial \xi}(0)$ also vanishes.

Let us now find the quadratic part

$$\frac{1}{2}(A\xi, \xi) + (B\xi, y) + \frac{1}{2}(Cy, y) \tag{2.11}$$

of Σ_{-1} supposing that A is non-degenerate. From (2.10) one has

$$-\frac{1}{2}(A\xi, \xi) - (B\xi, y) - \frac{1}{2}(Cy, y)$$

$$-(\alpha_0 y + \xi)(A\xi + B'y) + \frac{1}{2}(g_0(B\xi + Cy), B\xi + Cy) = 0. \tag{2.12}$$

This equation implies the 3 equations for the matrices A, B, C:

$$A = \frac{1}{3}B'g_0B, \quad 2B + \alpha_0'A = Cg_0B, \quad +\frac{1}{2}C + B\alpha_0 = \frac{1}{2}Cg_0C. \tag{2.13}$$

By the first equation of (2.13) and the non-degeneracy of A and g_0, the matrix B is non-degenerate as well. Therefore, inserting the first equation in the second one gets

$$2 + \frac{1}{3}\alpha_0' B' g_0 = C g_0 \quad \text{or} \quad g_0 B \alpha_0 = 3 g_0 C - 6.$$

Inserting this in the third equation (2.13) yields

$$(g_0 C)^2 - 7 g_0 C + 12 = 0,$$

and consequently $g_0 C$ is equal to 3 or 4 (here by 3, for example, we mean the unit matrix multiplied by 3).

Supposing $g_0 C = 3$ gives

$$C = 3 g_0^{-1}, \quad B = 3 g_0^{-1} \alpha_0^{-1}, \quad A = 3 (\alpha_0^{-1})' g_0^{-1} \alpha_0^{-1},$$

and the quadratic form (2.11) is then equal to

$$\frac{3}{2}\left[(g_0^{-1}\alpha_0^{-1}\xi, \alpha_0^{-1}\xi) + 2(g_0^{-1}\alpha_0^{-1}\xi, y) + (g_0^{-1} y, y) \right] = \frac{3}{2}\left[g_0^{-1/2}(\alpha_0^{-1}\xi + y) \right]^2,$$

so that this form would not be strictly positive. Thus, we must take $g_0 C = 4$, and therefore

$$C = 4 g_0^{-1}, \quad B = 6 g_0^{-1} \alpha_0^{-1}, \quad A = 12 (\alpha_0^{-1})' g_0^{-1} \alpha_0^{-1},$$

and the quadratic form (2.11) is

$$6(g_0^{-1}\alpha_0^{-1}\xi, \alpha_0^{-1}\xi) + 6(g_0^{-1}\alpha_0^{-1}\xi, y) + 2(g_0^{-1} y, y). \tag{2.14}$$

Therefore, the quadratic part of Σ_{-1} is as required. It remains for us to prove that all terms σ_j, $j > 2$, vanish in the expansion $\Sigma_{-1} = \sigma_2 + \sigma_3 + ...$ of Σ_{-1} as a series of homogeneous polynomials σ_j. For σ_3 one gets the equation

$$-\sigma_3 - (\alpha_0 y + \xi)\frac{\partial \sigma_3}{\partial \xi} + \left(g_0 \frac{\partial \sigma_2}{\partial \xi}, \frac{\partial \sigma_3}{\partial y} \right) = 0$$

or, by (2.14),

$$-\sigma_3 - (\alpha_0 y + \xi)\frac{\partial \sigma_3}{\partial \xi} + (6\alpha_0^{-1}\xi + 4y)\frac{\partial \sigma_3}{\partial y} = 0.$$

This implies that $\sigma_3 = 0$ by Proposition 7.1, since for the case under consideration the matrix A in (7.1) is

$$\begin{pmatrix} -1 & -\alpha_0 \\ 6\alpha_0^{-1} & 4 \end{pmatrix} \tag{2.15}$$

and has eigenvalues 1 and 2 for any invertible α_0. (The simplest way to prove the last assertion is to use (2.3.41) when calculating the characteristic polynomial for the matrix (2.15).) One proves similarly that all σ_j, $j > 2$, vanish, which completes the proof of Proposition 2.1.

Therefore, Σ_{-1} is given by (2.14). For example, in a particular important case with unit matrices g_0 and α_0, one has

$$\Sigma_{-1} = 6\xi^2 + 6(\xi, y) + 2y^2. \tag{2.16}$$

Furthermore, comparing the coefficients of t^{-1} in (2.8) and using (2.3.24), (2.3.25) yields

$$-(\alpha_0 y + \xi)\frac{\partial \Sigma_0}{\partial \xi} + \left(g_0 \frac{\partial \Sigma_{-1}}{\partial y}, \frac{\partial \Sigma_0}{\partial y}\right) - \left(\beta_0 y + \frac{1}{2}(\gamma_0 y, y), \frac{\partial \Sigma_{-1}}{\partial y}\right)$$

$$-\left(\frac{\partial a}{\partial x}(x_0)\xi + \frac{\partial \alpha}{\partial x}(x_0)\xi(y + y^0) - \frac{\partial \alpha}{\partial x}(x_0)(a_0 + \alpha_0 y^0)y, \frac{\partial \Sigma_{-1}}{\partial \xi}\right) = 0. \tag{2.17}$$

This equation is of the type (7.1) with F being a sum of polynomials of degree 2 and 3, and the matrix A given by (2.15). Thus, by Proposition 7.1, the solution (with the additional condition $\Sigma_0(0,0) = 0$) is defined uniquely and is a polynomial of degree 2 and 3 in ξ, y. The statement of Theorem 2.1 about the other Σ_j can be obtained by induction using Proposition 7.1, because comparing the coefficients of $t^{-(k+1)}$ in (2.8) always yields equation of the type (7.1) on Σ_k with the same matrix A. Thus, we have completed the proof of Theorem 2.1.

3. Asymptotic solution of the transport equation

After the asymptotics of the two-point function has been carried out, the next stage in the construction of the exponential multiplicative asymptotics of the Green function for a second order parabolic equation is the asymptotic solution of the transport equation (1.13). On the one hand, it can be solved using (1.18) and the results of the previous chapter. We shall use this representation of the solution in the next section to get the estimates for the remainder (1.21). On the other hand, when one is interested only in small time and small distances from initial point, one can solve the transport equation by formal expansions similarly to the construction of the two-point function in the previous section. We shall now explain this formal method in some detail. In the non-degenerate case one simply looks for the solution of (1.13) in the form of a regular power series in t and $(x - x_0)$ with a multiplier $(2\pi h t)^{-m/2}$. It is a rather standard procedure and we omit it. Let us consider the case of a degenerate regular Hamiltonian (2.4.1). Proceeding as in the previous section one first make a shift in the variables introducing the function

$$\psi(t, x, y) = \phi(t, x + \tilde{x}(t), y + \tilde{y}(t), x_0, y_0), \tag{3.1}$$

and then one must make the change of the variables $x^I = t^{M-I+1}\xi^I$ as in (2.4). A new feature in comparison with (2.4) consists in the observation that in the case of the transport equation one also needs "the explicit introduction of the normalising constant", i.e. one defines the new unknown function by the formula

$$\Psi(t,\xi^0,...,\xi^M,y)=t^\alpha\psi(t,t^{M+1}\xi^0,...,t\xi^M,y), \tag{3.2}$$

where α is some positive constant (which is to be calculated). For the Hamiltonian (2.4.1) written in terms of the function Ψ equation (1.13) takes the form

$$\frac{\partial\Psi}{\partial t}-\frac{\alpha}{t}\Psi+\frac{1}{2}\Psi\, tr\,\left(g(t^{M+1}\xi^0+\tilde{x}^0(t))\frac{\partial^2\Sigma}{\partial y^2}\right)$$

$$+\left(g(t^{M+1}\xi^0+\tilde{x}^0(t))\frac{\partial\Psi}{\partial y},\frac{\partial\Sigma}{\partial y}\right)-\left(g(\tilde{x}^0(t))\tilde{q}(t),\frac{\partial\Psi}{\partial y}\right)$$

$$-\frac{t^M\xi^0+R_1(x(t,\xi)+\tilde{x}(t),y+\tilde{y}(t))-R_1(\tilde{x}(t),\tilde{y}(t))}{t^{M+1}}\frac{\partial\Psi}{\partial\xi^0}-...$$

$$-\frac{\xi^M+R_{M+1}(x(t,\xi)+\tilde{x}(t),y+\tilde{y}(t))-R_{M+1}(\tilde{x}(t),\tilde{y}(t))}{t}\frac{\partial\Psi}{\partial\xi^M}$$

$$-(R_{M+2}(x(t,\xi)+\tilde{x}(t),y+\tilde{y}(t))-R_{M+2}(\tilde{x}(t),\tilde{y}(t)))\frac{\partial\Psi}{\partial y}. \tag{3.3}$$

The main result of this section is the following.

Theorem 3.1. *There exists a unique $\alpha>0$, in fact this $\alpha$is given by (1.2.11), such that there exists a solution of (3.3) in the form*

$$\Psi=\Psi+t\Psi_1+t^2\Psi_2+... \tag{3.4}$$

with each Ψ_j being a regular power series in (ξ,y) and Ψ_0 being some constant. Moreover, this solution is unique up to a constant multiplier and all Ψ_j turn out to be polynomials in ξ,y such that the degree of $\Psi_{k(M+1)-1-j}$, $j=0,...,M$, does not exceed $k-1$ for any $k=1,2,...$

Proof. Inserting (3.4) in (3.3) and using the condition that Ψ_0 is a constant one gets comparing the coefficients of t^{-1}:

$$\alpha\Psi_0+\frac{1}{2}\Psi_0\, tr\,\left(g(x_0^0)\frac{\partial^2\Sigma_{-1}}{\partial y^2}(x_0,y_0)\right)=0,$$

and therefore

$$\alpha=\frac{1}{2}\, tr\,\left(g(x_0^0)\frac{\partial^2\Sigma_{-1}}{\partial y^2}(x_0,y_0)\right). \tag{3.5}$$

Clearly α is positive. Using the canonical coordinates of Lemmas 1.2.2, 1.2.4, one proves that (3.5) coincides with (1.2.11). The remaining part of the proof of the theorem is the same as the proof of Theorem 2.1. Comparing the coefficients of t^q, $q=0,1,...$, one get a recurrent system of equations for Ψ_q of the form

(2.8) with polynomials F_q of the required degree. Proposition 2.1 completes the proof.

Corollary. *The function $\psi(t,x,y)$ corresponding to the solution ϕ of (1.13) via (3.1) has the form of a regular power expansion in (t,x,y) with multiplier $Ct^{-\alpha}$, where C is a constant.*

This implies in particular that the solution ϕ of the transport equation also has the form of a regular power expansion in $t, x-x_0, y-y_0$ with the same multiplier. Comparing the asymptotic solution constructed with the exact solution for Gaussian approximation, one sees that in order to satisfy the initial condition (1.4) by the function u_G^{as} of form (1.19), where S and ϕ are constructed above, one must take the constant C such that $Ct^{-\alpha}$ is equal to the amplitude (pre-exponential term) in formula (1.2.10) multiplied by $h^{-m/2}$. With this choice of C the dominant term of the asymptotic formula (1.19) will coincide with the dominant term of the asymptotics (1.2.10) for its Gaussian approximation, which in its turn by Theorem 1.2.2 coincides with the exact Green function for the "canonical representative" of the class of Gaussian diffusions defined by the corresponding Young scheme.

4. Local asymptotics of the Green function for regular Hamiltonians

In Sect. 1.1 we have described the construction of the asymptotic solution (1.19) for problem (1.1), (1.4) and in Sect. 1.2, 1.3 we have presented an effective procedure for the calculation of all its elements. In this section we are going to justify this asymtptotical formula, i.e. to prove that the exact Green function can be presented in form (1.9). Roughly speaking, the proof consists in two steps. One should obtain an appropriate estimate for the remainder (1.21) and then use it in performing a rather standard procedure (based on Du Hammel principle) of reconstructing the exact Green function by its approximation. When one is interested only in asymptotics for small times and small distances, it is enough to use only the approximations for S and ϕ obtained in two previous sections (a good exposition of this way of justification for non-degenerate diffusion see e.g. in [Roe]). But in order to be able to justify as well the global "small diffusion" asymptotics, as we intend to do in the next section, one has to use the exact global formulas (2.5), (1.18) for S and ϕ. We shall proceed systematically with this second approach. The starting point for justification is the estimate of the r.h.s. in (1.14), when ϕ is given by (1.18).

Proposition 4.1. *If ϕ is given by (1.18), then the r.h.s. of (1.14) (in a neighbourhood of x_0, where (1.18) is well defined) has the form*

$$O(h^2)C(h)t^{2M+2}\exp\{-\frac{S(t,x,x_0)}{h}\}\phi(t,x;x_0) = O(h^2t^{2M+2}u_G^{as}),$$

where as always $M+1$ is the rank of the regular Hamiltonian under consideration.

Proof. We omit the details concerning the simplest case of non-degenerate diffusion $(M+1=0)$ and reduce ourselves to the degenerate regular case, when

$M \geq 0$ and therefore the Hamiltonian is defined by (2.4.1). Then clearly the first term under the integral in (1.18) vanishes and it is enough to prove that

$$\frac{\partial \nu}{\partial y^j} J^{-1/2}(t,x,x_0) = O(t^{\nu(M+1)}) J^{-1/2}(t,x,x_0), \quad \nu = 1,2, \tag{4.1}$$

$$\frac{\partial \nu}{\partial y^j} \exp\left\{-\frac{1}{2}\int_0^t tr\,\frac{\partial A}{\partial x}(X(\tau))\,d\tau\right\}$$
$$= O(t^{\nu(M+1)} \exp\left\{-\frac{1}{2}\int_0^t tr\,\frac{\partial A}{\partial x}(X(\tau))\,d\tau\right\}, \quad \nu = 1,2, \tag{4.2}$$

where $X(\tau) = X(\tau, x_0, p_0(t,x;x_0))$. We have

$$\frac{\partial}{\partial x^I} J^{-1/2} = -\frac{1}{2} J^{-1/2}\left(J^{-1}\frac{\partial J}{\partial x^I}\right) = -\frac{1}{2} J^{-1/2}\, tr\left(\frac{\partial}{\partial x^I}\left(\frac{\partial X}{\partial p_0}\right)\left(\frac{\partial X}{\partial p_0}\right)^{-1}\right)$$

$$= -\frac{1}{2} J^{-1/2} \frac{\partial^2 X^K}{\partial p_0^L \partial p_0^N}\left(\frac{\partial X}{\partial p_0}\right)^{-1}_{LI}\left(\frac{\partial X}{\partial p_0}\right)^{-1}_{NK}, \tag{4.3}$$

and by estimates (2.4.3)-(2.4.5) it can be presented in the form

$$J^{-1/2} O\left(t^{6+4M-K-L-N} t^{-(2M+3-L-I)} t^{-(2M+3-N-K)}\right) = J^{-1/2} O(t^I).$$

For the derivatives with respect to $y = x^{M+1}$, one has $I = M+1$ and one gets (4.1) with $\nu = 1$. Differentiating the r.h.s. in (4.3) once more and again using (2.4.3)-(2.4.5) one gets (4.1) for $\nu = 2$. Let us turn now to (4.2). Let us prove only one of these formula, namely that with $\nu = 1$, the other being proved similarly. Note that due to the main definition of RH, the function under the integral in (4.2) depends only on x^0 and x^1, because it is a polynomial $Q_1(x)$ in $x^1, ..., x^{M+1}$ of $\mathcal{M}$-degree ≤ 1. Therefore, one should prove that

$$\frac{\partial}{\partial y} \exp\left\{-\frac{1}{2}\int_0^t Q_1(X(\tau))\,d\tau\right\} = O(t^{M+1}) \exp\left\{-\frac{1}{2}\int_0^t Q_1(X(\tau))\,d\tau\right\}. \tag{4.4}$$

One has

$$\frac{\partial}{\partial y} \exp\left\{-\frac{1}{2}\int_0^t Q_1(X(\tau))\,d\tau\right\}$$
$$= \int_0^t \sum_{I=0}^{1} \frac{\partial Q_1}{\partial x^I}\frac{\partial X^I}{\partial p_0^K}(\tau)\left(\frac{\partial X}{\partial p_0}\right)^{-1}_{K,M+1}(t)\,d\tau \exp\left\{-\frac{1}{2}\int_0^t Q_1(X(\tau))\,d\tau\right\},$$

and using again (2.4.3)-(2.4.5) one sees that the coefficient before the exponential in the r.h.s. of this expression has the form

$$\int_0^t O\left(\tau^{2M+3-1-K}\right) O\left(t^{-(2M+3-K-M-1)}\right)\,d\tau = O(t^{M+1}),$$

which proves (4.4) and thus completes the proof of Proposition 4.1.

Consider now the globally defined function (1.19). For RH (2.4.1) it is convenient to take the polydisc $D_t^r = B_r(x_0^0)\times B_{r/t}(x_0^1)\times ...\times B_{r^{M+1}/t^{M+1}}(x_0^{M+1})$ as the domain D. The following is the direct consequence of the previous result.

Proposition 4.2. *For the remainder F in (1.21) one has the estimate*

$$F = O(t^{M+1})u_G^{as} + O\left((\exp\{-\frac{\Omega}{ht^{2M+3}}\}\right)$$

with some positive Ω.

Now, in order to prove the representation (1.9) for the exact solution of (1.1),(1.4) we shall use the following classical general method. Due to the Du Hammel principle (the presentation of the solutions of a non-homogeneous linear equation in terms of the general solution of the corresponding homogeneous one), the solution u_G^{as} of problem (1.21), (1.4) is equal to

$$u_G^{as}(t,x;x_0) = u_G(t,x;x_0) - h\int_0^t \int_{\mathcal{R}^m} u_G(t-\tau,x,\eta)F(\tau,\eta,x_0)\,d\eta d\tau, \quad (4.6)$$

where u_G is the exact Green function for equation (1.1). It is convenient to rewrite (4.6) in the abstract operator form

$$u_G^{as} = (1-h\mathcal{F}_t)u_G, \quad (4.7)$$

with $\mathcal{F}_t$ being the integral operator

$$(\mathcal{F}_t\phi)(t,x,\xi) = (\phi\otimes F)(t,x,\xi) \equiv \int_0^t \int_{\mathcal{R}^m} \phi(t-\tau,x,\eta)F(\tau,\eta,\xi)\,d\eta d\tau, \quad (4.8)$$

where we have denoted by $\phi\star F$ the (convolution type) integral in the r.h.s. of (4.8). It follows from (4.7) that

$$u_G = (1-h\mathcal{F}_t)^{-1}u_G^{as} = (1+h\mathcal{F}_t+h^2\mathcal{F}_t^2+...)u_G^{as}$$
$$= u_G^{as} + hu_G^{as}\otimes F + h^2u_G^{as}\otimes F\otimes F + ... \quad (4.9)$$

Therefore, in order to prove the representation (1.9) for u_G one ought to show the convergence of series (4.9) and its presentation in form (1.9). This is done in the following main Theorem of this section.

Theorem 4.1. *For small t, the Green function of equation (1.2) with regular Hamiltonian (2.4.1), whose coefficients can increase polynomially as $x\to\infty$, has the form*

$$u_G = u_G^{as}(1+O(ht)) + O(\exp\{-\frac{\Omega}{ht}\}), \quad (4.10)$$

where u_G^{as} is given by (1.19) with the domain D defined in Proposition 4.2 above, the functions S and ϕ defined by formulas (2.1.5) and (1.18), and calculated *asymptotically in Sect. 1.2, 1.3. Moreover, the last term in (4.10) is an integrable function of x, which is exponentially small as $x \to \infty$.*

Remark 1. The result of this theorem is essentially known for the case of non-degenerate diffusion. Let us note however that usually in the literature one obtains separately and by different methods the small time and small distance asymptotics, either without a small parameter h (see e.g. a completely analytical exposition in [Roe], [CFKS]), or with a small parameter (see e.g [Var1],[MC1],[Mol], which are essentially based on the probabilistic approach), and global estimates often given for bounded coefficients and without a small parameter. (see e.g. [PE], [Da1], where completely different technique is used). Therefore, the uniform analytic exposition of all these facts together as given here can be perhaps of interest even in non-degenerate situation.

Remark 2. In our proof of the Theorem we obtain first for the case of bounded coefficients the estimate for the additive remainder in (4.10) in the form $O(e^{-|x|})$, which allows afterwards to extend the result to the case of polynomially increasing coefficients. More elaborate estimate of the series (4.9) in the case of bounded coefficients gives for the additive remainder in (4.10) more exact estimate $O(\exp\{-\Omega|x|^2/ht\})$, which allows to generalise the result of the Theorem to the case of the unbounded coefficients increasing exponentially as $x \to \infty$.

Remark 3. In the previous arguments, namely in formula (4.6), we have supposed the existence of the Green function for (1.1), which follows surely from general results on parabolic second order equations, see e.g. [IK]. But this assumption proves to be a consequence of our construction. In fact, when the convergence of series (4.9) and its representation in form (1.9) is proved, one verifies by simple direct calculations that the sum of series (4.9) satisfies equation (1.1).

Remark 4. When the Theorem is proved, the justification of more exact asymptotics as constructed in Sect. 1, Step 5, can be now carried out automatically. In fact, if

$$h\frac{\partial u_G^{as}}{\partial t} - H(x, -h\frac{\partial}{\partial x})u_G^{as} = O(t^j h^k)u_G^{as},$$

then from (4.6)

$$u_G = u_G^{as} + \int_0^t \int_{\mathcal{R}^m} u_G(t-\tau, x, \eta) O(t^j h^k) u_G^{as}(\tau, \eta, x_0)\, d\eta d\tau,$$

and due to (4.10) and the semigroup property of u_G one concludes that

$$u_G = u_G^{as}(1 + O(t^{j+1}h^k)) + O(\exp\{-\frac{\Omega}{ht}\}).$$

Proof. Though in principle the convergence of (4.9) is rather clear from the estimate of the first nontrivial term by the Laplace method using Proposition

2.1.4, the rigorous estimate of the whole series involves the application of the Laplace method infinitely many times, where one should keep control over the growth of the remainder in this procedure, which requires a "good organisation" of the recursive estimates of the terms of (4.9). Let us present the complete proof in the simplest case of the non-degenerate diffusion, the general case being carried out similarly due to Proposition 4.2, but requires the consideration of polydisks instead of the disks, which makes all expressions much longer. Consider first the diffusion with bounded coefficients. In non-degenerate case one can take the ball $B_r(x_0)$ as the domain D for the molifier χ_D. Let

$$\Omega = \min\{tS(t,x,\xi) : |x-\xi| = r-\epsilon\}. \tag{4.11}$$

For given $\delta > 0, h_0 > 0$ one can take t_0 such that for $t \le t_0$, $h \le h_0$

$$(2\pi ht)^{-m/2} \le \exp\{\frac{\delta}{th}\}, \quad \frac{1}{ht} \le \exp\{\frac{\delta}{th}\}. \tag{4.12}$$

To write the formulas in a compact form let us introduce additional notations. Let

$$f(t,x,x_0) = \Theta_r(|x-x_0|)(2\pi ht)^{-m/2}\exp\{-S(t,x,x_0)/h\}, \tag{4.13}$$

$$g_k(t,x,x_0) = \Theta_{kr}(|x-x_0|)\exp\{-\frac{\Omega-\delta}{ht}\}. \tag{4.14}$$

From Proposition 4.2

$$F(t,x,x_0) = O(u_G^{as}(t,x,x_0)) + O((ht)^{-1}\exp\{-S(t,x,x_0)/h\}),$$

where the second function has a support in the ring $B_{x_0}(r) \setminus B_{x_0}(r-\epsilon)$. Using these formulas and the estimates for the solution ϕ of the transport equation one can choose a constant $C > 1$ such that

$$\frac{1}{C}\chi_D(x-x_0)f(t,x,x_0) \le u_G^{as}(t,x,x_0) \le Cf(t,x,x_0), \tag{4.15}$$

$$F(t,x,x_0) \le C(f(t,x,x_0) + g_1(t,x,x_0)). \tag{4.16}$$

To estimate the terms $u_G^{as} \otimes F \otimes F...$ in series (4.9) we shall systematically use the simple estimate of the Laplace integral with convex phase, namely the formula (B3) from Appendix B. Let us choose d such that

$$\frac{\partial^2 S}{\partial x^2}(t,x,\xi) \ge \frac{d}{t}, \quad \frac{\partial^2 S}{\partial \xi^2}(t,x,\xi) \ge \frac{d}{t} \tag{4.17}$$

for $|x-\xi| \le r$ and $t \le t_0$. Such d exists due to the asymptotic formula for S given in Sect. 2.2 or 3.3. We claim that the following inequalities hold (perhaps, for a smaller t_0):

$$f \otimes f \le td^{-m/2}(f+g_2), \quad f \otimes g_k \le td^{-m/2} g_{k+1}, \quad g_k \otimes g_1 \le td^{-m/2} g_{k+1}. \tag{4.18}$$

In fact, since

$$\min_{\eta}(S(t-\tau,x,\eta)+S(\tau,\eta,x_0))=S(t,x,x_0) \qquad (4.19)$$

and the minimum point η_0 lies on the minimal extremal joining x_0 and x in time t, it follows from (B3) that

$$(f\otimes f)(t,x,x_0)\le \theta_{2r}(x-x_0)\int_0^t(2\pi h(t-\tau))^{-m/2}(2\pi h\tau)^{-m/2}$$

$$\times\exp\{-S(t,x,x_0)/h\}\left(\frac{d}{t}+\frac{d}{t-\tau}\right)^{-m/2}(2\pi h)^{m/2}\,d\tau$$

$$=\Theta_{2r}(|x-x_0|)d^{-m/2}t(2\pi ht)^{-m/2}\exp\{-S(t,x,x_0)/h\}.$$

To estimate this function outside $B_r(x_0)$ we use (4.12) and thus get the first inequality in (4.18). Furthermore,

$$(f\otimes g_k)(t,x,x_0)$$

$$\le g_{k+1}(x-x_0)\int_0^t\int_{\mathcal{R}^d}(2\pi h(t-\tau))^{-m/2}\exp\{-S(t-\tau,x,\eta)/h\}d\eta d\tau.$$

Since $S\ge 0$ and due to (B3), this implies the second inequality in (4.18). At last, obviously, $g_k\star g_1\le tb_mr^mg_{k+1}^2$, where b_m denotes the volume of the unit ball in $\mathcal{R}^m$. This implies the last inequality in (4.18) for small enough t_0.

It is easy now to estimate the terms of series (4.18):

$$hu_G^{as}\otimes F\le hC^2(f\otimes f+f\otimes g_1)=hC^2td^{-m/2}(f+2g_2),$$

$$h^2(u_G^{as}\otimes F)\otimes f\le h^2C^3td^{-m/2}(f+2g_2)(f+g_1)\le C(Chtd^{-m/2})^2(f+2g_2+4g_3),$$

and by induction

$$h^{k-1}\mathcal{F}^{k-1}u_G^{as}\le C[Chtd^{-m/2}]^{k-1}(f+2g_2+4g_3+...+2^{k-1}g_k).$$

Since $2g_2+4g_3+...+2^{k-1}g_k\le 2^kg_k$, one has for $k>1$

$$h^{k-1}\mathcal{F}^{k-1}u_G^{as}\le 2C(2Cthd^{-m/2})^{k-1}(f+g_k).$$

Therefore, series (4.9) is convergent (uniformly on the compacts); outside the ball $B_{rk}, k\ge 1$, it can be estimated by

$$2C\sum_{l=k}^{\infty}(2Cthd^{-m/2})^l\exp\{-\frac{\Omega-\delta}{th}\}\le\frac{2C(2Cthd^{-m/2})^k}{1-2Ct_0h_0d^{-m/2}}\exp\{-\frac{\Omega-\delta}{th}\}, \quad (4.20)$$

and inside the ball B_r, $|u_G-u_G^{as}|$ does not exceed

$$\frac{4C^2thd^{-m/2}}{1-2Ct_0h_0d^{-m/2}}\left[(2\pi ht)^{-m/2}\exp\{-S(t,x,x_0)/h\}+\exp\{-\frac{\Omega-\delta}{th}\}\right]. \quad (4.21)$$

Notice that the number k in (4.20) is of the order $|x-x_0|/r$ and therefore the coefficient t^k can be estimated by a function of the form $e^{-\kappa|x-x_0|}$ with some $\kappa > 0$. The statement of the theorem follows now from (4.16),(4.20),(4.21).

Now let the functions A, G, V are not bounded but can increase polynomially as $x \to \infty$ (the uniform boundedness of V from below is supposed always). This case can be reduced to the case of bounded coefficients in following way. Let x_0 be given and let $\tilde{G}(x), \tilde{A}(x), \tilde{V}(x)$ be the functions which are uniformly bounded and coincide with $G(x), A(x), V(x)$ respectively in a neighbourhood of x_0. Taking the Green function $\tilde{u}_G$ of the diffusion equation with coefficients $\tilde{G}, \tilde{A}, \tilde{V}$ as the first approximation to the Green function $u_G(t, x, x_0)$ yields for u_G the series representation

$$u_G = \tilde{u}_G + \tilde{u}_G \otimes F + \tilde{u}_G \otimes F \otimes F + ...$$

with

$$F = \frac{1}{2} tr(\tilde{G} - G) \frac{\partial^2 \tilde{u}_G}{\partial x^2} - (\tilde{A} - A) \frac{\partial \tilde{u}}{\partial x} - (\tilde{v} - V)\tilde{u}.$$

Due to the exponential decrease (of type $e^{-\kappa|x-x_0|}$ with some $\kappa > 0$) of $\tilde{u}$ (and, as one shows similarly, the same rate of decrease holds for the first and second derivatives of $\tilde{u}$) all terms of this series are well defined and it is convergent, which completes the proof.

Let us note that by passing we have proved a convergent series representation for the Green function, i.e. the following result, which we shall use in Chapter 9.

Proposition 4.3. *Under the assumptions of Theorem 4.1, the Green function of equation (1.2) can be presented in the form of absolutely convergent series (4.9), where*

$$u_G^{as} = C(h)\chi_D(x - x_0)\phi(t, x, x_0) \exp\{-S(t, x, x_0)/h\},$$

F is defined by (1.21) and the operation $\otimes$ is defined by (4.8).

Theorem 4.1 gives for the Green function of certain diffusion equations the multiplicative asymptotics for small times and small distances, but only a rough estimate for large distances. In the next section we shall show how to modify this asymptotics in order to have an asymptotic representation valid for all (finite) distances. There is however a special case when the global asymptotic formula is as simple as the local one (at least for small times), this is the case of diffusion equations with constant drift and diffusion coefficients. This special case is important for the study of the semi-classical spectral asymptotics of the Shrödinger equation and therefore we formulate here the corresponding result obtained in [DKM1].

Theorem 4.2. *Let $V(x)$ be a positive smooth function in $\mathcal{R}^m$ with uniformly bounded derivatives of the second, third and fourth order. Then there exists $t_0 > 0$ such that for $t \leq t_0$ the boundary value problem for the corresponding Hamiltonian system with the Hamiltonian $H = \frac{1}{2}p^2 - V(x)$ is uniquely* *solvable for all x, x_0 and t_0 and the Green function for the equation*

$$h\frac{\partial u}{\partial t} = \frac{h^2}{2}\Delta u - V(x)u \tag{4.22}$$

has the form

$$u_G(t,x,x_0,h) = (2\pi h)^{-m/2} J(t,x,x_0)^{-1/2} \exp\{-S(t,x,x_0)/h\}(1+O(ht^3)) \tag{4.23}$$

with $O(ht^3)$ being uniform with respect to all x, x_0.

Remark. Notice the remainder is of the order t^3, which holds only in the described situation.

5. Global small diffusion asymptotics and large deviations

In the previous section we have written the exponential asymptotics of the Green function of equation (1.1) for small times. Moreover, for large distances it states only that the Green function is exponentially small. In order to obtain the asymptotics for any finite t and also give a more precise formula for large $x - x_0$, one can use the semigroup property of the Green function. Namely, for any $n \in \mathcal{N}$, $\tau > 0$, and $x \in \mathcal{R}^m$

$$u_G(n\tau, x, x_0)$$

$$= \int_{\mathcal{R}^m} ... \int_{\mathcal{R}^m} u_G(\tau, x, \eta_1) u_G(\tau, \eta_1, \eta_2) ... u_G(\tau, \eta_{n-1}, x_0)\, d\eta_1 ... d\eta_{n-1}. \tag{5.1}$$

Notice that according to the result of sect. 2.2, 2.3, the set $Reg(x_0)$ of regular pairs (t,x), i.e such pairs for which there exists a unique minimising extremal joining x_0 and x in time t, is open and everywhere dense in $\mathcal{R}_+ \times \mathcal{R}^m$ for any regular Hamiltonian. Therefore, formula (1.19) without the multiplier χ_D is correctly defined globally (for all x and $t > 0$) almost everywhere, because $S(t,x,x_0)$ is defined globally for all t, x as the minimum of the functional (2.1.12) and for regular points S, ϕ are given by (2.24),(2.2.5), and (1.18) with integrals in both formulas taken along the unique minimising extremal. It turns out that though this function does not give a correct asymptotics to the Green function in a neighbourhood of a non-regular point, its convolution with itself already does. We restrict ourselves to the approximations of the first order. Next orders can be obtained by formulas (1.22)- (1.25).

Theorem 5.1. *For any t, x and $\tau < t$*

$$u_G(t,x;x_0) = (2\pi h)^{-m}(1+O(h)) \int_{\mathcal{R}^m} \phi(t-\tau, x, \eta)\phi(\tau, \eta, x_0)$$

$$\times \exp\left\{-\frac{S(t-\tau,x,\eta) + S(\tau,\eta,x_0)}{h}\right\} d\eta. \tag{5.2}$$

In particular, for any $(t,x) \in Reg(x_0)$

$$u_G(t,x,x_0) = (2\pi h)^{-m/2}\phi(t,x,x_0)(1+O(h))\exp\{-S(t,x,x_0)/h\}, \qquad (5.3)$$

Proof. First let us show that the integral in (5.2) is well defined. To this end, we use the Cauchy inequality and (1.18) to estimate (5.2) by

$$\int_{\mathcal{R}^m} J^{-1}(t-\tau,x,\eta)\exp\left\{-\frac{2S(t-\tau,x,\eta)}{h}\right\} d\eta$$

$$\times \int_{\mathcal{R}^m} J^{-1}(\tau,\eta,x_0)\exp\left\{-\frac{2S(\tau,\eta,x_0)}{h}\right\} d\eta$$

We shall estimate the second integral in this product, the first one being dealt in the completely similar way. Let us make in this second integral the change of the variable of integration $\eta \mapsto p_0(\tau,\eta,x_0)$, where p_0 is the initial momentum of the minimising extremal joining x_0 and η in time τ. The mapping $\eta \mapsto p_0$ is well defined for regular, and thus for almost all η. Thus the second integral can be estimated by the integral

$$\int_{\mathcal{R}^m} \exp\left\{-\frac{2S(\tau,\eta(\tau,x_0,p_0),x_0)}{h}\right\} dp_0,$$

which already does not contain any singularities. To see that this integral converges one only need to observe that due to the estimates of the two-point function S, the function under the integral here is decreasing exponentially as $p_0 \to \infty$.

We have shown that integral (5.2) is well defined. It follows from the Laplace method and Proposition 2.1.4 that that for regular t,x formula (5.2) can be written in form (5.3). To show that (5.2) presents the global asymptotics for the Green function, let us start first with the simplest case of equation (4.22), where the asymptotics of the Green function for small times is proved to be given by (4.23). For any t there exists $n \in \mathcal{N}$ such that $t/n < t_0$ and one can present the Green function for the time t in form (5.1) with function (4.23) instead of u_G. If the point (t,x) is regular, then there is only one and non-degenerate minimum point $(\eta_1,...,\eta_{n-1})^{min}$ of the "compound action"

$$S(\tau,x,\eta_1) + S(\tau,\eta_1,\eta_2) + ... + S(\tau,\eta_{n-1},x_0)$$

given by the formula

$$\eta_j^{min} = X(j\tau,x_0,p_0(t,x,x_0)),$$

i.e. all η_j^{min} lie on the unique minimising extremal. Using Proposition 2.1.4 and the Laplace method one gets (5.3). Alternatively, one can use the induction in n. Thus we have proved (5.3) for all regular points. Now, for any non-regular (t,x) let us use (5.1) with $n=2$. It follows from the Laplace method that only those

η contribute to the first order asymptotics of this integral that lie on minimising extremals (which may not be unique now) joining x_0 and x in time t. But by the Jacobi theory these points η are regular with respect to x_0 and x and hence around this point one can use formula (5.3) for u_G. Therefore the asymptotics of this integral is the same as that of (5.2).

In the case of a general regular Hamiltonian, there is only one additional difficulty. Namely, if one calculates the asymptotic of the Green function for a regular point using (5.1) with the function (4.10) instead of u_G, then for sufficiently large n and $x-x_0$ the exponentially small remainders in (4.10) begin to spoil the correct phase of the corresponding Laplace integral. Hence, in order to get the correct asymptotics for (t,x) from any fixed compact one should improve (4.10) respectively. Namely, in proving theorem 4.1, one must take instead of the approximation (1.19), its convolution with itself of the form

$$\tilde u_G^{as}(n\tau, x, x_0)$$

$$= \int_{\mathcal{R}^m} ... \int_{\mathcal{R}^m} u_G^{as}(\tau, x, \eta_1) u_G^{as}(\tau, \eta_1, \eta_2) ... u_G^{as}(\tau, \eta_{n-1}, x_0)\, d\eta_1 ... d\eta_{n-1}.$$

Formulas (4.9) and (4.10) are then modified respectively. Increasing n to infinity, one increases to infinity the range of x for which (5.2) is valid. This argumentation completes the proof.

The global integral formula (valid for regular and non-regular points) for the asymptotics of the Green function for non-degenerate diffusion was first written by Maslov [M2] by means of his tunnel canonical operator. We have given here an equivalent but essentially more simple formula (5.2) thus avoiding the beautiful but rather sophisticated definition of the tunnel canonical operator (see [M2], [DKM1]). Moreover, we have presented the rigorous proof including a large class of degenerate diffusions.

For some non-regular points, the integral (5.2) can still be calculated explicitly. The two important cases are the following.

(i) There exist a finite number of non-degenerate extremals joining x_0 and x in time t. Then (5.2) is equal to the sum of expressions (5.3) corresponding to each extremal.

(ii) There is a (non-degenerate) closed manifold of extremals (as for instance is often the case for geodesics on symmetric spaces) joining x_0 and x in time t. Then one integrates (5.2) by means of the modified Laplace method (see, e.g. [Fed1], [K3]) standing for the case of the whole manifold of minimal points of the phase.

For general non-regular points one can write down explicitly only the logarithmic asymptotics of the solution. To this end, let us recall first a general result on logarithmic asymptotics of Laplace integrals.

Proposition 5.1 [MF2],[Fed1]. *Let the functions f and ϕ in the integral*

$$F(h) = \int_{\mathcal{R}^m} \phi(x) \exp\{-\frac{S(x)}{h}\}\, dx \qquad (5.4)$$

be continuous, ϕ having finite support supp ϕ. Let $M(S)$ denote the set of all $x \in$ supp ϕ, where S is equal to its minimum M, and let $M_c(S) \subset$ supp ϕ be the set of x, where $S(x) \leq M + c$. Denote by $V(c)$ the volume of the set $M_c(S)$. Suppose $\phi \geq \delta$ in a neighbourhood of $M(S)$ for some positive constant δ.

(i) Then

$$\lim_{h\to 0} h \log F(h) = -M. \tag{5.5}$$

(ii) If

$$\lim_{c\to 0} \frac{\log V(c)}{\log c} = \alpha > 0, \tag{5.6}$$

then

$$\log F(h) = -\frac{M}{h} + \alpha \log h + o(\log h). \tag{5.7}$$

(iii) If $V(0) = 0$ and (5.7) holds, then (5.6) holds as well.

(iv) If $M(S)$ consists of a unique point and S is real analytic in a neighbourhood of this point, then the limit (5.6) exists and (5.7) holds.

The last statement is in fact a consequence of a theorem from [BG], see e.g. [At].

Theorem 5.1 and Proposition 5.1 imply the principle of large deviation for the Green function of regular diffusions.

Proposition 5.2. *For all t, x*

$$\lim_{h\to 0} h\, u_G(t, x, x_0) = -S(t, x, x_0). \tag{5.8}$$

This principle for regular points of non-degenerate diffusion was obtained by Varadhan, see [Var1]-[Var4]. In some cases, one can calculate the logarithmic asymptotic more precisely. For instance, Theorem 5.1 and Proposition 5.1 imply the following result.

Proposition 5.3. *If there exists a unique minimising extremal joining x_0 and x in time t (generally speaking, degenerate, i.e. the points x_0 and x can be conjugate along this extremal) and if $S(t, y, x_0)$ is real analytic in a neighbourhood of this extremal, then there exists $\alpha > 0$ such that for small h*

$$\log h\, u_G(t, x, x_0) = -\frac{S(t, x, x_0)}{h} + \alpha \log h + o(\log h). \tag{5.9}$$

Formula (5.9) for the case of non-degenerate diffusion was first written in [MC1] (and proved there under some additional assumptions), where α was called the invariant of the degeneracy of the extremal.

Let us present the solution of the general large deviation problem for regular diffusions. If in (1.1) the last term $V(x)$ vanishes, then the corresponding second order equation describes the evolution of the expectations (and its adjoint operator - the probability density) of the diffusion process defined by the stochastic equation

$$dX = A(X)\, dt + h\sqrt{G(x)}\, dW. \tag{5.10}$$

One is especially interested in the solution of (1.1) with the discontinuous initial function

$$u_0(x) = \begin{cases} 1, & \text{if} \quad x \in \mathrm{D} \\ 0, & \text{otherwise}, \end{cases} \tag{5.11}$$

where D is some closed bounded domain in $\mathcal{R}^m$. This solution corresponds to the diffusion starting in D. The problem of large deviation is to find the small h asymptotics of this solution on large distances from D. The solution of (1.1), (5.11) is given by formula (1.5),(5.3). To simplify it, one can use the Laplace method. As in the case of the Green function, for a general non-regular point only the logarithmic limit can be found explicitly. Namely, the following result is the direct consequence of formulas (1.5), (5.2), (5.5).

Proposition 5.4 (Large deviation principle for regular diffusions). *For the solution $u(t,x)$ of the problem (1.2), (5.11) with a regular Hamiltonian, one has*

$$\lim_{h\to 0} h \log u(t,x) = -S(t,x),$$

where $S(t,x)$ is the generalised solution of the Cauchy problem for Hamilton-Jacobi equation (1.11) with initial data

$$S(0,x) = \begin{cases} 0, & \textit{if} \quad x \in D \\ +\infty, & \textit{otherwise}, \end{cases}$$

i.e.. $S(t,x)$ is given by the formula

$$S(t,x) = \min_{\xi}(S(t,x,\xi) + S(0,\xi)) = \min_{\xi\in D} S(t,x,\xi), \tag{5.12}$$

where $S(t,x;\xi)$ denotes as always the corresponding two-point function.

The explicit formula (without integration) for the asymptotics of $u(t,x)$ exists on the open everywhere dense set of regular (with respect to the domain D) points, where the critical point of the phase used in the integration of (1.5) by the Laplace method is unique and non-degenerate. On the complement to this set the asymptotics can be only written in the integral form similar to (5.3). Let us give the precise results, which follow more or less straightforwardly from (1.5), (5.2) and the Laplace method. For the case of the equation of form (4.22) this result was proved in [DKM1]. Let Hamiltonian (1.3) be regular with vanishing V, i.e for some Young scheme $\mathcal{M}$ it has the form (2.4.1) with vanishing $R_{2(M+1)}$. Let $Y(t,y_0)$ denote the solution of the system

$$\dot{y} = -A(y) \tag{5.13}$$

with initial value $y(0) = y_0$. Note that the solution of (5.13) is in fact the characteristic of the Hamiltonian system on which the momentum vanishes identically (the insertion of the vanishing momentum in the Hamiltonian system does not lead to a contradiction due to the assumption of vanishing V). Let D_t denote the smooth manifold with boundary, which is the image of $D = D_0$ with respect to the mapping $y_0 \mapsto Y(t,y_0)$.

Proposition 5.5. *On the set $Int\, D_t$ of the internal points of the domain D_t the solution $u(t,x)$ of problem (1.1), (5.11) can be presented in the form of regular series in h. More precisely, if $x \in Int\, D_t$, then*

$$u(t,x) = \left(\det \frac{\partial Y}{\partial y_0}(t, y_0)\right)^{-1/2} (1 + h\phi_1 + ... + h^k \phi_k + O(h^{k+1})),$$

where ϕ_j can be found by the formulas similar to (1.24), (1.25).

As we have mentioned, the most interesting is the problem of calculating the solution far away from D, in particular outside D_t. To formulate the result of the calculation of the Laplace integral (1.5) in that case, we need some other notations. Consider the m-dimensional manifold $\Lambda_0 = \partial D \times \mathcal{R}_+$ with coordinates $(\alpha_1, ..., \alpha_{m-1}, s)$, where $s \geq 0$ and $\alpha = (\alpha_1, ..., \alpha_{m-1})$ are some orthogonal coordinates on ∂D. Let $n(\alpha)$ denote the unit vector of the external normal to ∂D at the point α and let $\Gamma(D)$ be the family of characteristics $X(t, \alpha, sn(\alpha))$ with initial conditions $x_0 = \alpha, p_0 = sn(\alpha)$. For any t and $x \in \mathcal{R}^m \setminus D_t$, there exists a pair $(\alpha, s) = (\alpha, s)(t, x)$ (perhaps not unique) such that the characteristic $X(t, \alpha, sn(\alpha))$ comes to x at the time t and $S(t, x)$ (as defined in (5.12)) is equal to the action (2.1.4) along this characteristic. In fact, α is the coordinate of the point $\xi \in D$ that furnishes minimum in (5.8). Clear that p_0 is perpendicular to ∂D at ξ and thus has the form $p_0 = sn(\alpha)$ for some $s > 0$. Let Reg_D denote the set of pairs (t, x) such that $(\alpha, s)(t, x)$ is unique and moreover, the Jacobian $J(t, x) = \det \frac{\partial X}{\partial(\alpha, s)}$ does not vanish. Similarly to the proof of Proposition 2.2.7, 2.3.7 one shows that the set Reg_D is open and everywhere dense in the outside of the set $\{(t, x \in D_t)\}$.

Proposition 5.6. *For $(t, x) \in Reg_D$ the solution $u(t, x)$ of problem (1.2), (5.11) has the following asymptotics for small h:*

$$u(t,x) = \left(\det \frac{\partial X}{\partial(\alpha, s)}|_{(\alpha,s)=(\alpha,s)(t,x)}\right)^{-1/2}$$

$$\times \exp\{-\frac{S(t,x)}{h}\}(1 + ... + h^k \phi_k + O(h^{k+1})).$$

The asymptotics of the global representation (1.5), (5.3) can be also calculated explicitly for some classes of non-regular points described similarly to the case of the Green function (see (i), (ii) after Theorem 5.2 and Theorem 5.4).

6. Non-regular degenerate diffusions: an example

In this Chapter we have constructed the theory of global semi-classical asymptotics and large deviations for a class of degenerate diffusions that were called regular. This class is characterised in particular by regular asymptotic representation (0.6) of the Green function. It seems however that the global small h asymptotics are valid actually for a larger class of degenerate diffusions. We present here an example of a non-regular diffusion for which small h and small time asymptotics can be calculated explicitly, so to say, by hands, and shall see that the small h asymptotics can be obtained as well by a formal application of the formulas of Section 6. We consider the equation

$$h\frac{\partial u}{\partial t} = \frac{h^2}{2}\frac{\partial^2 u}{\partial y^2} + \frac{h}{2}y^2\frac{\partial u}{\partial x}, \tag{6.1}$$

which corresponds to the simple non-regular Hamiltonian $H = (q^2 - y^2 p)/2$ discussed at the end of Section 2.3, where we have noted that for this Hamiltonians the boundary value problem is not solvable if $x > x_0$ for any time. We are going to construct the Green function u_G for this equation corresponding to the initial point $(0, 0)$, i.e. the solution with initial condition

$$u_G(t, x, y)|_{t=0} = \delta(x)\delta(y).$$

Proposition 6.1. *(i) The Green function u_G vanishes for $x \geq 0$,*
(ii) If $x < 0$ and $y = 0$, then

$$u_G(t, x, 0) = \frac{1}{\sqrt{|x|}ht^{3/2}}\exp\{-\frac{\pi^2|x|}{ht^2}\}(1 + O(\frac{ht^2}{|x|})), \tag{6.2}$$

(iii) if $x < 0$ and $y < 0$, then there exists a unique real solution $\lambda(t, x, y) > -\pi^2$ of the equation

$$\frac{4x}{ty^2} = \frac{1}{\sinh^2\sqrt{\lambda}} - \frac{\cot\sqrt{\lambda}}{\sqrt{\lambda}}, \tag{6.3}$$

and

$$u_G = \frac{1}{2\pi ht^2\sqrt{|S''(\lambda(t,x,y))|}}\sqrt{\frac{\sqrt{\lambda(t,x,y)}}{\sinh\sqrt{\lambda(t,x,y)}}}$$

$$\times\exp\{-\frac{1}{ht}S(\lambda(t,x,y))\}(1 + O(ht)), \tag{6.4}$$

where the function S is defined by the formula

$$S(\lambda; t, x, y) = \frac{\lambda x}{t} + \frac{\sqrt{\lambda}}{2}y^2\coth\sqrt{\lambda}; \tag{6.5}$$

moreover $\lambda(t, x, y) \in (-\pi^2, 0)$, $\lambda = 0$, $\lambda > 0$ respectively for $x < -ty^2/6$, $x = -ty^2/6$, $x > -ty^2/6$ and in the first case $\sqrt{\sqrt{\lambda}/\sinh\sqrt{\lambda}}$ should be understood as $\sqrt{\sqrt{|\lambda|}/\sin\sqrt{|\lambda|}}$; at last, for small

$$\epsilon = \frac{x}{ty^2} + \frac{1}{6}$$

the function $\lambda(t,x,y)$ can be presented as the convergent power series in ϵ: $\lambda = 45\epsilon + O(\epsilon^2)$, so that for small ϵ

$$u_G = \frac{3\sqrt{5}}{2\pi h t^2 |y|}(1+O(\epsilon))(1+O(th))\exp\{-\frac{y^2}{2ht}(1+O(\epsilon))\}. \qquad (6.6)$$

Sketch of the proof. It is done essentially by direct calculations using the Fourier transform and the saddle-point method. Namely, carrying out the h-Fourier transform F_h of equation (6.1) with respect to the variable y one finds for $\tilde{u}(t,x,p) = (F_h u)(t,x,p)$ the equation

$$h\frac{\partial \tilde{u}}{\partial t} = \frac{h^2}{2}\frac{\partial^2 \tilde{u}}{\partial y^2} + \frac{i}{2} y^2 \tilde{u},$$

which is actually the equation of the evolution of the quantum oscillator in imaginary time and with the complex frequency $\sqrt{-ip} = \sqrt{|p|}\exp\{-i\pi\, sgnp/4\}$. Since the Green function for such equation is well known, one obtains for u_G the following integral representation

$$u_G = -\frac{i}{(2\pi h)^{3/2} t^{5/2}} \int_{-i\infty}^{i\infty} \sqrt{\frac{\sqrt{\lambda}}{\sinh\sqrt{\lambda}}} \exp\{-\frac{1}{ht} S(\lambda(t,x,y))\}\, d\lambda. \qquad (6.7)$$

Notice that the function under the integral in this representation is regular everywhere except for singularities at points $-k^2\pi^2$, $k = 1,2,...$, and is a one-valued analytic function on the complex plane cut along the line $(-\infty,\pi^2)$. For $\lambda = Re^{i\phi}$ with $|\phi| < \pi$, one has

$$|\sinh\sqrt{\lambda}|^2 = \sinh^2(\sqrt{R}\cos\frac{\phi}{2}) + \sin^2(\sqrt{R}\sin\frac{\phi}{2}),$$

$$ReS(\lambda;t,x,y)) = \frac{x}{t} R\cos\phi$$

$$+\frac{y^2\sqrt{R}\left[\cos\frac{\phi}{2}\sinh(2\sqrt{R}\cos\frac{\phi}{2}) + \sin\frac{\phi}{2}\sin(2\sqrt{R}\sin\frac{\phi}{2})\right]}{2\left[\cosh(2\sqrt{R}\cos\frac{\phi}{2}) - \cos(2\sqrt{R}\sin\frac{\phi}{2})\right]}.$$

It implies that for $x \geq 0$ one can close the contour of integration by a semicircle on the right half of the complex plane, which by Cauchy theorem gives the statement (i) of the Proposition. Let $x < 0$ and $y = 0$. Then one can transform the contour of integration in (6.7) to the contour C which goes from $-\infty$ to $-\pi^2$ along the lower edge of the half-line $(-\infty, -\pi^2)$ and then returns to the $-\infty$ along the upper edge of this half-line (notice that all singularities at $\lambda = -k^2\pi^2$ are of the type $z^{-1/2}$ and are therefore integrable). The simple analysis of the argument of $\sqrt{\sinh\sqrt{\lambda}}$ shows that the values of the integrand in (6.7) on the upper edge of the cut coincides (respectively differs by the sign) with its corresponding values on the lower edge on the intervals $((2k\pi)^2, (2k+1)^2\pi^2)$

(resp. on the intervals $((2k-1)^2\pi^2,(2k\pi)^2)$), which yields (after the change $\lambda=v^2$) the formula

$$u_G=\frac{4}{(2\pi h)^{3/2}t^{5/2}}\sum_{k=1}^{\infty}\int_{\pi(2k-1)}^{2k\pi}(-1)^{k-1}v\exp\{-\frac{v^2|x|}{ht^2}\}\sqrt{\frac{v}{|\sin v|}}\,dv$$

$$=\frac{\sqrt{2}}{(\pi h)^{3/2}t^{5/2}}\int_0^{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{(u+\pi(2k-1))^{3/2}}{\sqrt{\sin u}}\exp\{-\frac{|x|(u+\pi(2k-1))^2}{ht^2}\}\,dv$$

For large $|x|/(ht^2)$ all terms in this sum are exponentially small as compared with the first one. Calculating this first term for large $|x|/(ht^2)$ by the Laplace method yields (6.2).

Consider now the main case $x<0, y<0$. To calculate this integral asymptotically for small h one can use the saddle-point method. The equation $S'(\lambda)=0$ for saddle points is just equation (6.3), and simple manipulations show the properties of the solution $\lambda(t,x,y)$ given in the formulation of the Proposition. Now the application of the saddle-point method to the integral in (6.7) amounts to the shift of the contour of integration on $\lambda(t,x,y)$ and the following calculation of thus obtained integral by means of the Laplace method, which yields (6.4), (6.6) and thus completes the proof.

Notice now that as we have mentioned above, the formal application of semi-classical formulas of Sections 1 or 5, i.e. of representation (1.19) with S being the two point function corresponding to Hamiltonian $H=(q^2-y^2p)/2$ and ϕ being given by (1.18), would give the same result as we have obtained above using the explicit expression for the h-Fourier transform of u_G. In fact, the Cauchy problem for the Hamiltonian system with the Hamiltonian H and initial conditions $(0,0,p_0,q_0)$ has the explicit solution

$$y=\frac{q_0}{\sqrt{p_0}}\sinh(\sqrt{p_0}t),\quad x=\frac{q_0^2}{4p_0}(t-\frac{\sinh(2\sqrt{p_0}t)}{2\sqrt{p_0}}),$$

and the problem of finding the solution to the boundary value problem with $x(t)=x, y(t)=y$ reduces to the solution of equation (6.3) for $\lambda=pt^2$. For $x>0$ there is no solution to this boundary value problem, i.e. S is infinity and the u_G should vanish. Similarly for $x<0$ one finds that semi-classical formulas (5.2), (5.1) yield (6.2) and (6.4). That is where the natural question arises, which we pose for the conclusion. For what class of non-regular Hamiltonians, to begin with those given by (2.4.7), one can justify asymptotic representations of type (1.19) or (5.2) for the Green function with the two-point function as the phase? Notice that unless $f=y^2$ as in the example before, exact representation of type (6.7) does not exist, and since these Hamiltonians are not regular (unless $f=y$) the machinery presented above in Sections 3-5 also does not work.

7. Analytic solutions to some linear PDE

In this short section we collect some general facts on analytic (or even formal power series) solutions to linear first order partial differential equations of the form

$$\lambda S + \left(Ax, \frac{\partial S}{\partial x} \right) = F(x), \tag{7.1}$$

where $x \in \mathcal{R}^d$, $F(x)$ is a polynomial, λ is a constant and A is a matrix with strictly positive eigenvalues $a_1 \le ... \le a_d$ which are simple, i.e. there exists an invertible matrix C such that $C^{-1}AC = D$ is the diagonal matrix $diag(a_1, ..., a_d)$. These facts are used in the asymptotic calculations of the two point function and of the solutions to the transport equation, which are carried out in Sections 3.2, 3.3 and Chapter 4.

Let p be the smallest non-negative integer such that $\lambda + pa_1 > 0$. In the most of examples $\lambda > 0$, and therefore $p = 0$.

Proposition 7.1.(i) *Let $F(x)$ be a homogeneous polynomial of degree $q \ge p$. Then there exists a solution S of (7.1) which is a polynomial of degree q with coefficients defined by:*

$$\frac{\partial^q S}{\partial x_{i_1} ... \partial x_{i_q}} = \frac{1}{\lambda + a_{j_1} + ... + a_{j_q}} (C^{-1})_{j_1 i_1} ... (C^{-1})_{j_q i_q} \frac{\partial^q F}{\partial x_{l_1} ... \partial x_{l_q}} C_{l_1 j_1} ... C_{l_q j_q}. \tag{7.2}$$

This solution is unique in the class of real analytic functions (in fact, even in the class of formal power series) under the additional assumption that all its derivatives at the origin up to order $p-1$ vanish (this additional assumption is void in the main case $p = 0$).

(ii) *Let F be a sum $F = \sum_{q=p}^m F_q$ of homogeneous polynomials F_q of degree q. If $m = \infty$, let us suppose that this sum is absolutely convergent in a ball B_R (R may be finite or not). Then the analytic solution of (7.2) exists and is again unique under the condition above, and is given by the sum $\sum_{q=p}^m S_q$ of the solutions corresponding to each F_q. If $m = \infty$, this sum is convergent in the same ball B_R, as the sum presenting the function F.*

Proof. The change of variables $x = Cy$ transforms (7.1) to

$$\lambda \tilde{S} + \sum_{m=1}^{d} a_m y_m \frac{\partial \tilde{S}}{\partial y_m} = \tilde{F}(y), \tag{7.3}$$

where $\tilde{S}(y) = S(Cy)$ and $\tilde{F}(y) = F(Cy)$. Differentiating this equation $k \ge p$ times yields

$$(\lambda + a_{i_1} + ... + a_{i_k}) \frac{\partial^k \tilde{S}}{\partial y_{i_1} ... \partial y_{i_k}} + \sum a_m y_m \frac{\partial^{k+1} \tilde{S}}{\partial y_m \partial y_{i_1} ... \partial y_{i_k}} = \frac{\partial^k \tilde{F}}{\partial y_{i_1} ... \partial y_{i_k}}.$$

It follows that under the conditions of (i) only the derivatives of order q at the origin do not vanish, and for the derivatives of order q one gets

$$\frac{\partial^q \tilde{S}}{\partial y_{i_1} ... \partial y_{i_q}}(0) = \frac{1}{\lambda + a_{i_1} + ... + a_{i_q}} \frac{\partial^q \tilde{F}}{\partial y_{i_1} ... \partial y_{i_q}}(0). \tag{7.4}$$

Returning to the original variables x yields (7.2), because

$$\frac{\partial^q S}{\partial x_{i_1}...\partial x_{i_q}}(0) = \frac{\partial^q \tilde{S}}{\partial y_{j_1}...\partial y_{j_q}}(0)(C^{-1})_{j_1 i_1}...(C^{-1})_{j_q i_q},$$

$$\frac{\partial^q \tilde{F}}{\partial y_{j_1}...\partial y_{j_q}}(0) = \frac{\partial^q F}{\partial x_{l_1}...\partial x_{l_q}}(0) C_{l_1 j_1}...C_{l_q j_q}.$$

Similar arguments prove (ii) for finite m. If $m = \infty$, the convergence of the series representing the solution S (and thus the analyticity of S in the ball B_R) follows from (7.4), because this equations imply that

$$\left| \frac{\partial^q \tilde{S}}{\partial y_{i_1}...\partial y_{i_q}}(0) \right| = \frac{O(1)}{q} \left| \frac{\partial^k \tilde{F}}{\partial y_{i_1}...\partial y_{i_q}}(0) \right|.$$

We are going now to present an equivalent form of formula (7.2), which is more convenient for calculations. This formula will be used only in Section 4.3.

Cosider the graph Γ_q with vertices of two kinds such that there are exactly d^q vertices of each kind, and the vertices of the first kind (resp. second kind) are labeled by the sequences $(l_1, ..., l_q)_x$ (resp. $(l_1, ..., l_q)_y$) with each $l_j \in \{1, ..., d\}$. The graph Γ_q is considered to be a complete oriented bipartite graph, which means that any pair of the vertices of different kind are connected by a (unique) arc, and the vertices of the same kind are not connected. Let us define the weights of the arcs by the formulas

$$W[(l_1, ..., l_q)_x \to (j_1, ..., j_q)_y] = C_{l_1 j_1}...C_{l_q j_q},$$

$$W[(j_1, ..., j_q)_y \to (l_1, ..., l_q)_x] = (C^{-1})_{j_1 l_1}...(C^{-1})_{j_q l_q}.$$

Furthermore, let us consider the weight of any vertex of the first kind to be one, and the weights of the vertices $(j_1, ..., j_q)_y$ of the second kind being equal to $(\lambda + a_{j_1} + ... + a_{j_q})^{-1}$. The weight of any path in the graph Γ_q will be equal (by definition) to the product of the weights of its arcs and vertices. In particular, the weight of a two-step path is given by the formula

$$W[(l_1, ..., l_q)_x \to (j_1, ..., j_q)_y \to (i_1...i_q)_x]$$

$$= W[(l_1, ..., l_q)_x \to (j_1, ..., j_q)_y] \frac{1}{\lambda + a_{j_1} + ... + a_{j_q}} W[(j_1, ..., j_q)_y \to (i_1, ..., i_q)_x]. \tag{7.5}$$

The foolowing statement is a direct consequence of Proposition 7.1 and the definition of the bipartite weighted graph Γ_q.

Coroolary. *Formula (7.2) can be written in the following geometric form*

$$\frac{\partial^q S}{\partial x_{i_1}...\partial x_{i_q}}$$

$$= \sum_{(l_1,...,l_q)_x} \sum_{(j_1,...,j_q)_y} \frac{\partial^q F}{\partial x_{l_1} ... \partial x_{l_q}} W[(l_1, ..., l_q)_x \to (j_1, ..., j_q)_y \to (i_1 ... i_q)_x]. \quad (7.6)$$

Let us discuss now the computational aspects of this formula for a special type of equation (7.1), which appears in the calculation (in Chapter 4) of the trace of the Green function of regular invariant diffusions corresponding to the stochastic geodesic flows. This equation has additional symmetries, which allow to reduce a large number of calculations encoded in formulas (7.2) or (7.6). The equation we are going to discuss, has the form

$$\lambda f - \left(\xi + y, \frac{\partial f}{\partial \xi}\right) + \left(6\xi + 4y, \frac{\partial f}{\partial y}\right) = F, \quad (7.7)$$

where ξ and y belong to $\mathcal{R}^k$, λ is a positive integer, and F is a polynomial in ξ, y. By Proposition 2.1 it is enough to be able to calculate the solutions corresponding to homogeneous polynomials F of each degree q. The solution is then given by the polynomial of degree q whose coefficients are calculated from (7.6). In the case of equation (7.7), A is the block-diagonal $2k \times 2k$-matrix with 2×2-blocks $\begin{pmatrix} -1 & -1 \\ 6 & 4 \end{pmatrix}$ on its diagonal. The corresponding matrix C is then also block-diagonal with 2×2-blocks $\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$ on its diagonal. It means that the change of the variables, which was used in the proof of Proposition 7.1 is now $(\xi, y)^i \mapsto (\eta, z)^i$, $i = 1, ..., k$, with

$$\begin{pmatrix} \xi^i \\ y^i \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} \eta^i \\ z^i \end{pmatrix} = C \begin{pmatrix} \eta^i \\ z^i \end{pmatrix},$$

$$\begin{pmatrix} \eta^i \\ z^i \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \xi^i \\ y^i \end{pmatrix} = C^{-1} \begin{pmatrix} \xi^i \\ y^i \end{pmatrix}, \quad (7.8)$$

and thus equation (7.3) is

$$\tilde{f} + \eta^i \frac{\partial \tilde{f}}{\partial \eta^i} + 2z^i \frac{\partial \tilde{f}}{\partial z^i} = \tilde{F}(\eta, z),$$

which implies

$$\frac{\partial^{q+p} \tilde{f}}{\partial \eta^{i_1} ... \partial \eta^{i_q} \partial z^{j_1} ... \partial z^{j_p}} = \frac{1}{\lambda + q + 2p} \frac{\partial^{q+p} \tilde{F}}{\partial \eta^{i_1} ... \partial \eta^{i_q} \partial z^{j_1} ... \partial z^{j_p}}. \quad (7.9)$$

Due to the special block-diagonal form of C one sees that in the sum (7.2) consisting of $(2k)^{2q}$ terms only 2^{2q} terms do not necessarily vanish. Moreover, there is a large amount of symmetry, since A has only two different eigenvalues. Using simple combinatorial considerations we shall obtain now the following result.

Proposition 7.2. *If F is a polynomial of degree q and λ is positive number, then the unique analytic solution of (7.5) is the homogeneous polynomial of degree q in ξ, y with derivatives of the order q at the origin given by the formula*

$$\frac{\partial^q f}{\partial \xi^I \partial y^J} = \sum_{\tilde{I}\subset I}\sum_{\tilde{J}\subset J} \mathcal{A}^{|I|,|J|}_{|\tilde{I}|,|\tilde{J}|} \frac{\partial^q F}{\partial \xi^{\tilde{I}} \partial y^{I\setminus\tilde{I}} \partial \xi^{J\setminus\tilde{J}} \partial y^{\tilde{J}}}, \tag{7.10}$$

where I and J are arbitrary sequences of indices from $\{1, ..., k\}$ such that $|I| + |J| = q$, and the coefficients $\mathcal{A}$ are given by the formula

$$\mathcal{A}^{\sigma\kappa}_{\mu\nu} = \sum_{l=0}^{\mu}\sum_{m=0}^{\sigma-\mu}\sum_{n=0}^{\kappa-\nu}\sum_{p=0}^{\nu} C^l_\mu C^m_{\sigma-\mu} C^n_{\kappa-\nu} C^p_\nu$$

$$\times \frac{1}{\lambda + 2q - l - m - n - p}(-1)^{m-n+\kappa} 2^{p-l} (-3)^{l-p+\nu-\mu} 6^\sigma, \tag{7.11}$$

where C^j_i are the binomial coefficients.

Proof. In the case under consideration the vertices of the first kind (resp. of the second) of the graph Γ_q can be labeled by sequences ξ^I, y^J (resp. η^I, z^J), where both I and J are sequences of numbers from the set $1, ..., k$ of the lengths $|I|$ and $|J|$ respectively with $|I| + |J| = q$. Consequently, formula (7.6) takes the form

$$\frac{\partial^q f}{\partial \xi^I \partial y^J} = \sum_{\tilde{I}\subset I}\sum_{\tilde{J}\subset J} \frac{\partial^q F}{\partial \xi^{\tilde{I}} \partial y^{I\setminus\tilde{I}} \partial \xi^{J\setminus\tilde{J}} \partial y^{\tilde{J}}}$$

$$\times \sum_\omega W[(\xi^{\tilde{I}}, y^{I\setminus\tilde{I}}, \xi^{J\setminus\tilde{J}}, y^{\tilde{J}}) \to (\omega^{\tilde{I}}, \omega^{I\setminus\tilde{I}}, \omega^{J\setminus\tilde{J}}, \omega^{\tilde{J}}) \to (\xi^{\tilde{I}}, \xi^{I\setminus\tilde{I}}, y^{J\setminus\tilde{J}}, y^{\tilde{J}})], \tag{7.12}$$

where each ω can be either η or z with the corresponding index. Therefore we obtained (7.10), and it remains only to obtain formula for the weights in (7.12). To this end, we denote by l m, n and p the number of varibles η in $\omega^{\tilde{I}}$, $\omega^{I\setminus\tilde{I}}$, $\omega^{J\setminus\tilde{J}}$ and $\omega^{\tilde{J}}$ respectively, and we have

$$\mathcal{A}^{\sigma\kappa}_{\mu\nu} = \sum_{l=0}^{\mu}\sum_{m=0}^{\sigma-\mu}\sum_{n=0}^{\kappa-\nu}\sum_{p=0}^{\nu} C^l_\mu C^m_{\sigma-\mu} C^n_{\kappa-\nu} C^p_\nu W^{\mu,\sigma-\mu,k-\nu,\nu}_{l,m,n,p},$$

where $W^{\mu,\sigma-\mu,k-\nu,\nu}_{l,m,n,p}$ is the weight of an arc having l transactions of the type $\xi \to \eta \to \xi$, $\mu - l$ transactions of the type $\xi \to z \to \xi$, m transactions of the type $y \to \eta \to \xi$, $\sigma - \mu - m$ transactions of the type $y \to z \to \xi$, n transactions of the type $\xi \to \eta \to y$, $k - \nu - n$ transactions of the type $\xi \to z \to y$, p transactions of the type $y \to \eta \to y$, and $\nu - p$ transactions of the type $y \to z \to y$. Due to (7.8), the weights of the transactions $\eta \to \xi$, $\eta \to y$, $z \to \xi$, $z \to y$ equal to $1, -2, -1, 3$ respectively, and the weights of the transactions $\xi \to \eta$, $\xi \to z$, $y \to \eta$, $y \to z$ equal to $3, 2, 1, 1$ respectively. Multiplying the corresponding weights yields

$$W^{\mu,\sigma-\mu,k-\nu,\nu}_{l,m,n,p} = \frac{1}{\lambda + 2q - l - m - n - p}$$

$$\times 3^l 2^{\mu-l}(-1)^{\mu-l}(-2)^m(-1)^{\sigma-\mu-m}3^n(-2)^n 2^{k-\nu-n}3^{k-\nu-n}(-2)^p 3^{\nu-p},$$

which implies (7.11), and the Proposition is proved.

Notice that it follows from (7.9) that

$$\mathcal{A}_{00}^{\sigma\kappa} = (-6)^{\sigma-\kappa}\mathcal{A}_{00}^{\kappa\sigma}. \tag{7.13}$$

In the next chapter, we shall need to solve equation (7.7) for the polynomials F of the fourth order. Actually, we shall need the full solution for the case of polynomials of order 2, and only a part of it for orders 3 and 4. We obtain now the neccesary formulas as an example of the application of Proposition 7.2.

Proposition 7.3. *Let f_j be the solution of equation (7.7) with $\lambda = 1$ and the r.h.s. F_j being a homogeneous polynomials in ξ, y of degree j. Then $f_0 = F_0$ and*

$$f_1 = \left(\frac{5}{6}\frac{\partial F_1}{\partial \xi^i} - \frac{\partial F_1}{\partial y_i}\right)\xi^i + \frac{1}{6}\frac{\partial F_1}{\partial \xi^i}y^i, \tag{7.14}$$

$$f_2 = \frac{1}{2}\left[\frac{6}{5}\frac{\partial^2 F_2}{\partial y^i \partial y^j} - \frac{9}{10}\left(\frac{\partial^2 F_2}{\partial y^i \partial \xi^j} + \frac{\partial^2 F_2}{\partial y^j \partial \xi^i}\right) + \frac{4}{5}\frac{\partial^2 F_2}{\partial \xi^i \partial \xi^j}\right]\xi^i\xi^j$$
$$+\left[-\frac{1}{5}\frac{\partial^2 F_2}{\partial y^i \partial \xi^j} + \frac{3}{20}\frac{\partial^2 F_2}{\partial \xi^i \partial \xi^j} + \frac{1}{10}\frac{\partial^2 F_2}{\partial y^i \partial y^j} + \frac{1}{20}\frac{\partial^2 F_2}{\partial \xi^i \partial y^j}\right]\xi^i y^j$$
$$+\frac{1}{2}\left[\frac{1}{30}\frac{\partial^2 F_2}{\partial \xi^i \partial \xi^j} - \frac{1}{60}\left(\frac{\partial^2 F_2}{\partial \xi^i \partial y^j} + \frac{\partial^2 F_2}{\partial \xi^j \partial y^i}\right) + \frac{2}{15}\frac{\partial^2 F_2}{\partial y^i \partial y^j}\right]y^i y^j. \tag{7.15}$$

Moreover,

$$\frac{\partial^3 f_3}{\partial \xi_i \partial \xi_j \partial \xi_k} = -\frac{54}{35}\frac{\partial^3 F_3}{\partial y_i \partial y_j \partial y_k} + \frac{39}{35}\left(\frac{\partial^3 F_3}{\partial \xi_i \partial y_j \partial y_k} + \frac{\partial^3 F_3}{\partial y_i \partial \xi_j \partial y_k} + \frac{\partial^3 F_3}{\partial y_i \partial y_j \partial \xi_k}\right)$$
$$-\frac{61}{70}\left(\frac{\partial^3 F_3}{\partial \xi_i \partial \xi_j \partial y_k} + \frac{\partial^3 F_3}{\partial \xi_i \partial y_j \partial \xi_k} + \frac{\partial^3 F_3}{\partial y_i \partial \xi_j \partial \xi_k}\right) + \frac{113}{140}\frac{\partial^3 F_3}{\partial \xi_i \partial \xi_j \partial \xi_k}, \tag{7.16}$$

and

$$\frac{\partial^4 f_4}{\partial \xi_i \partial \xi_j \partial \xi_k \partial \xi_l} = \frac{72}{35}\frac{\partial^4 F_4}{\partial y_i \partial y_j \partial y_k \partial y_l} + \frac{263}{315}\frac{\partial^4 F_4}{\partial \xi_i \partial \xi_j \partial \xi_k \partial \xi_l}$$
$$-\frac{51}{35}\left(\frac{\partial^4 F_4}{\partial \xi_i \partial y_j \partial y_k \partial y_l} + \frac{\partial^4 F_4}{\partial y_i \partial \xi_j \partial y_k \partial y_l} + \frac{\partial^4 F_4}{\partial y_i \partial y_j \partial \xi_k \partial y_l} + \frac{\partial^4 F_4}{\partial y_i \partial y_j \partial y_k \partial \xi_l}\right)$$
$$-\frac{92}{105}\left(\frac{\partial^4 F_4}{\partial y_i \partial \xi_j \partial \xi_k \partial \xi_l} + \frac{\partial^4 F_4}{\partial \xi_i \partial y_j \partial \xi_k \partial \xi_l} + \frac{\partial^4 F_4}{\partial \xi_i \partial \xi_j \partial y_k \partial \xi_l} + \frac{\partial^4 F_4}{\partial \xi_i \partial \xi_j \partial \xi_k \partial y_l}\right)$$
$$+\frac{38}{35}\sum_{I\subset\{i,j,k,l\}:|I|=2}\frac{\partial^4 F_4}{\xi^I \partial y^{\{i,j,k,l\}\setminus I}}. \tag{7.17}$$

Proof. Using (7.10) yields

$$\frac{\partial f}{\partial \xi_i} = A_{00}^{10}\frac{\partial F}{\partial y_i} + A_{10}^{10}\frac{\partial F}{\partial \xi_i},$$

$$\frac{\partial f}{\partial y_i} = A_{01}^{00}\frac{\partial F}{\partial \xi_i} + A_{01}^{01}\frac{\partial F}{\partial y_i},$$

and from (7.11) one obtains

$$A_{00}^{10} = \sum_{m=0}^{1} \frac{6}{3-m}(-1)^m = 6\left(\frac{1}{3} - \frac{1}{2}\right) = -1,$$

$$A_{10}^{10} = \sum_{l=0}^{1} \frac{6}{3-l}2^{-l}(-3)^{l-1} = 6\left(-\frac{1}{9} + \frac{1}{4}\right) = \frac{5}{6},$$

$$A_{00}^{01} = \sum_{n=0}^{1} \frac{1}{3-n}(-1)^{1-n} = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6},$$

$$A_{01}^{01} = \sum_{p=0}^{1} \frac{1}{3-p}(-1)2^p(-3)^{1-p} = 1 - 1 = 0,$$

which implies (7.14).

To get (7.15) one first uses (7.10) to obtain the formulas

$$\frac{\partial^2 f}{\partial \xi_i \partial \xi_j} = A_{00}^{20}\frac{\partial^2 F}{\partial y_i \partial y_j} + A_{10}^{20}\left(\frac{\partial^2 F}{\partial \xi_i \partial y_j} + \frac{\partial^2 f}{\partial y_i \partial \xi_j}\right) + A_{20}^{20}\frac{\partial^2 F}{\partial \xi_i \partial \xi_j},$$

$$\frac{\partial^2 f}{\partial \xi_i \partial y_j} = A_{00}^{11}\frac{\partial^2 F}{\partial y_i \partial \xi_j} + A_{10}^{11}\frac{\partial^2 F}{\partial \xi_i \partial \xi_j} + A_{01}^{11}\frac{\partial^2 F}{\partial y_i \partial y_j} + A_{11}^{11}\frac{\partial^2 F}{\partial \xi_i \partial y_j},$$

$$\frac{\partial^2 f}{\partial y_i \partial y_j} = A_{00}^{02}\frac{\partial^2 F}{\partial \xi_i \partial \xi_j} + A_{01}^{02}\left(\frac{\partial^2 F}{\partial \xi_i \partial y_j} + \frac{\partial^2 F}{\partial y_i \partial \xi_j}\right) + A_{02}^{02}\frac{\partial^2 F}{\partial y_i \partial y_j}. \tag{7.18}$$

Using (7.11) one calculates

$$A_{00}^{20} = \sum_{m=0}^{2} C_2^m 6^2 \frac{1}{5-m}(-1)^m = 36(\frac{1}{5} - \frac{1}{2} + \frac{1}{3}) = \frac{6}{5},$$

$$A_{10}^{20} = \sum_{l=0}^{1}\sum_{m=0}^{1} 6^2 \frac{1}{5-l-m}(-1)^{m+l-1}2^{-l}3^{l-1} = 36(-\frac{1}{15} + \frac{1}{12} + \frac{1}{8} - \frac{1}{6}) = -\frac{9}{10},$$

$$A_{20}^{20} = \sum_{l=0}^{2} C_2^l 6^2 \frac{1}{5-l}2^{-l}(-3)^{l-2} = 36(\frac{1}{45} - \frac{1}{12} + \frac{1}{12}) = \frac{4}{5},$$

$$A_{01}^{02} = \sum_{n=0}^{1}\sum_{p=0}^{1} \frac{1}{5-n-p}(-1)^{-n-p+3}2^p3^{-p+1} = -\frac{3}{5} + \frac{3}{4} + \frac{2}{4} - \frac{2}{3} = -\frac{1}{60},$$

$$A_{02}^{02} = \sum_{p=0}^{2} C_2^p \frac{1}{5-p}(-1)^p 2^p 3^{2-p} = \frac{9}{5} - 3 + \frac{4}{3} = \frac{2}{15},$$

$$A_{00}^{02} = \sum_{n=0}^{2} C_2^n \frac{1}{5-n}(-1)^{-n+2} = \frac{1}{5} - \frac{1}{2} + \frac{1}{3} = \frac{1}{30},$$

the last coefficient could be also obtain using the previous calculations and formula (7.13). Furthermore,

$$A_{00}^{11} = 6\sum_{m=0}^{1}\sum_{n=0}^{1} \frac{1}{5-m-n}(-1)^{m-n+1} = 6\left(-\frac{1}{5} + \frac{1}{2} - \frac{1}{3}\right) = -\frac{1}{5},$$

$$A_{01}^{11} = 6\sum_{m=0}^{1}\sum_{p=0}^{1} \frac{1}{5-m-p}(-1)^{m-p}2^p 3^{-p+1} = 6\left(\frac{3}{5} - \frac{3}{4} - \frac{2}{4} + \frac{2}{3}\right) = \frac{1}{10},$$

$$A_{10}^{11} = 6\sum_{l=0}^{1}\sum_{n=0}^{1} \frac{1}{5-l-n}(-1)^{l-n}2^{-l}3^{l-1} = 6\left(\frac{1}{15} - \frac{1}{12} - \frac{1}{8} + \frac{1}{6}\right) = \frac{3}{20},$$

$$A_{11}^{11} = 6\sum_{l=0}^{1}\sum_{p=0}^{1} \frac{1}{5-l-p}(-1)^{l-p+1}2^{p-l}3^{l-p} = 6\left(-\frac{1}{5} + \frac{1}{6} + \frac{3}{8} - \frac{1}{3}\right) = \frac{1}{20}.$$

Substituting these formulas in (7.18) yields (7.15).

To obtain (7.16) one uses (7.10) to write

$$\frac{\partial^3 f_3}{\partial\xi_i\partial\xi_j\partial\xi_k} = A_{00}^{30}\frac{\partial^3 F_3}{\partial y_i\partial y_j\partial y_k} + A_{10}^{30}\left(\frac{\partial^3 F_3}{\partial\xi_i\partial y_j\partial y_k} + \frac{\partial^3 F_3}{\partial y_i\partial\xi_j\partial y_k} + \frac{\partial^3 F_3}{\partial y_i\partial y_j\partial\xi_k}\right)$$

$$+A_{20}^{30}\left(\frac{\partial^3 F_3}{\partial\xi_i\partial\xi_j\partial y_k} + \frac{\partial^3 F_3}{\partial\xi_i\partial y_j\partial\xi_k} + \frac{\partial^3 F_3}{\partial y_i\partial\xi_j\partial\xi_k}\right) + A_{30}^{30}\frac{\partial^3 F_3}{\partial\xi_i\partial\xi_j\partial\xi_k}. \qquad (7.19)$$

Then one uses formula (7.11) to calculate

$$A_{00}^{30} = 6^3\sum_{m=0}^{3} C_3^m \frac{1}{7-m}(-1)^m = 6^3\left(\frac{1}{7} - \frac{1}{2} + \frac{3}{5} - \frac{1}{4}\right) = -\frac{54}{35},$$

$$A_{10}^{30} = 6^3\sum_{l=0}^{1}\sum_{m=0}^{2} C_2^m \frac{1}{7-l-m}(-1)^{m+l-1}2^{-l}3^{l-1}$$

$$= 6^3\left(-\frac{1}{21} + \frac{1}{9} - \frac{1}{15} + \frac{1}{12} - \frac{1}{5} + \frac{1}{8}\right) = \frac{39}{35},$$

$$A_{20}^{30} = 6^3\sum_{l=0}^{2}\sum_{m=0}^{1} C_2^l \frac{1}{7-l-m}(-1)^{m+l}2^{-l}3^{l-2}$$

$$= 6^3\left(\frac{1}{9}\left(\frac{1}{7} - \frac{1}{6}\right) + \frac{1}{3}\left(\frac{1}{5} - \frac{1}{6}\right) + \frac{1}{4}\left(\frac{1}{5} - \frac{1}{4}\right)\right) = -\frac{61}{70},$$

$$A_{30}^{30} = 6^3 \sum_{l=0}^{3} C_3^l \frac{1}{7-l} 2^{-l} (-3)^{l-3} = 6^3 \left(-\frac{1}{189} + \frac{1}{36} - \frac{1}{20} + \frac{1}{32} \right) = \frac{113}{140}.$$

Substituting these coefficients in (7.19) yields (7.16). Similarly, to get (7.17) one needs the following coefficients (which are again obtained from the general formula (7.11):

$$A_{40}^{40} = 6^4 \sum_{l=0}^{4} C_4^l \frac{1}{9-l} 2^{-l} (-3)^{l-4} = 6^4 \left(\frac{1}{9} \frac{1}{3^4} - \frac{1}{4} \frac{1}{27} + \frac{1}{42} - \frac{1}{36} + \frac{1}{80} \right) = \frac{263}{315},$$

$$A_{30}^{40} = 6^4 \sum_{l=0}^{3} \sum_{m=0}^{1} C_3^l \frac{1}{9-l-m} (-1)^{m+l-3} 2^{-l} 3^{l-3} = -\frac{92}{105},$$

$$A_{20}^{40} = 6^4 \sum_{l=0}^{2} \sum_{m=0}^{2} C_2^l C_2^m \frac{1}{9-l-m} (-1)^{m+l-2} 2^{-l} 3^{l-2} = \frac{38}{35},$$

$$A_{10}^{40} = 6^4 \sum_{l=0}^{1} \sum_{m=0}^{3} C_3^m \frac{1}{9-l-m} (-1)^{m+l-1} 2^{-l} 3^{l-1} = -\frac{51}{35},$$

$$A_{00}^{40} = 6^4 \sum_{m=0}^{4} C_4^m \frac{1}{9-m} (-1)^m = 6^4 \left(\frac{1}{9} - \frac{1}{2} + \frac{6}{7} - \frac{2}{3} + \frac{1}{5} \right) = \frac{72}{35}.$$

Proposition is proved.