Chapter 4. INVARIANT DEGENERATE DIFFUSION ON COTANGENT BUNDLES

1. Curvilinear Ornstein-Uhlenbeck process and stochastic geodesic flow

In this chapter we apply the theory developed in the previous chapter to the investigation of invariant degenerate diffusions on manifolds. We confine ourselves to the case of a regular degenerate diffusion of rank one. Since in the conditions of the regularity of a Hamiltonian the linearity of some coefficient in the second variable y is included, one has to suppose when constructing an invariant object that this second variable lives in a linear space. Therefore, an invariant operator ought to be defined on a vector bundle over some manifold: coordinates y in fibres and coordinate x on a base. We reduce ourselves to the most commonly used vector bundle, namely to the cotangent bundle $T^\star M$ of a compact n-dimensional manifold M. In local coordinates, a regular Hamiltonian H of a degenerate diffusion of rank one has form (2.3.4), where the matrix g is positive definite and α is non-degenerate. The corresponding diffusion equation (3.1.2) has the form

$$h\frac{\partial u}{\partial t} = Lu = H\left(x, y, -h\frac{\partial}{\partial x}, -h\frac{\partial}{\partial y}\right) = \frac{h^2}{2} g_{ij}\frac{\partial^2 u}{\partial y_i \partial y_j}$$

$$+h(a^i(x)+\alpha^{ij}(x)y_j)\frac{\partial u}{\partial x^i}+h(b_i(x)+\beta_i^j(x)y_j+\frac{1}{2}\gamma_i^{jl}(x)y_jy_l)\frac{\partial u}{\partial y_i}-V(x,y)u. \quad (1.1)$$

In this section, we give the complete description of the invariant operators of that kind on $T^\star M$. Let us recall that a tensor γ of type (q,p) on a manifold M is by definition a set of n^{p+q} smooth functions $\gamma^{i_1...i_p}_{j_1...j_q}(x)$ on x that under the change of coordinates $x \mapsto \tilde{x}$ changes by the law

$$\tilde{\gamma}^{i_1...i_p}_{j_1...j_q}(\tilde{x}) = \gamma^{k_1...k_p}_{l_1...l_q}(x)\frac{\partial \tilde{x}^{i_1}}{\partial x^{k_1}}...\frac{\partial \tilde{x}^{i_p}}{\partial x^{k_p}}\frac{\partial x^{l_1}}{\partial \tilde{x}^{j_1}}...\frac{\partial x^{l_q}}{\partial \tilde{x}^{j_q}}.$$

To each tensor of the type $(0,p)$ corresponds the polylinear function on the cotangent bundle $T^\star M$ defined by the formula $\gamma(x,y) = \gamma^{i_1...i_p}(x)y_{i_1}...y_{i_p}$.

Theorem 1.1 *Suppose the following objects are given on M:*

(i) Riemanian metric, which in local coordinates x on M is given by a positive definite matrix $g(x)$, $x \in M$;

(ii) non-degenerate tensor $\alpha = \{\alpha^{ij}(x)\}$ of the type $(0,2)$ (non-degeneracy means that the matrix α is non-degenerate everywhere) and a tensor $a = \{a^i(x)\}$ of the type $(0,1)$ (i.e. a vector field); these tensors obviously define a quadratic function $f(x,y) = \alpha^{ij}(x)y_iy_j + a^i(x)y_i$ on $T^\star M$;

(iii) tensors b, β, γ of the types $(1,0),(1,1),(1,2)$ respectively;

(iv) the sum V of tensors of the types $(0,0),(0,1),(0,2),(0,3),(0,4)$, which defines a bounded from below function $V(x,y)$ on $T^\star M$.

Then the second order differential operator

$$L = \frac{1}{2} g_{ij}(x) \frac{\partial^2}{\partial y_i \partial y_j} + \frac{\partial f}{\partial y_i}(x,y)\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}(x,y)\frac{\partial}{\partial y_i}$$

$$+ \left(b_i(x) + \beta_i^j(x) y_j + \frac{1}{2}\gamma_i^{kl}(x) y_k y_l \right) \frac{\partial}{\partial y_i} - V(x,y) \tag{1.2}$$

is an invariant operator on $T^\star M$, which is a regular diffusion of the rank one. Conversely, each such operator has this form.

Proof. Under the change of the variables $x \mapsto \tilde{x}(x)$ the moments change by the rule $\tilde{y} = y\frac{\partial x}{\partial \tilde{x}}$. Therefore,

$$\frac{\partial u}{\partial y_i} = \frac{\partial u}{\partial \tilde{y}_j}\frac{\partial \tilde{y}_j}{\partial y_i}, \quad \frac{\partial^2 u}{\partial y_i \partial y_j} = \frac{\partial^2 u}{\partial \tilde{y}_k \partial \tilde{y}_m}\frac{\partial \tilde{y}_m}{\partial y_j}\frac{\partial \tilde{y}_k}{\partial y_i}, \tag{1.3}$$

$$\frac{\partial u}{\partial x^i} = \frac{\partial u}{\partial \tilde{x}^j}\frac{\partial \tilde{x}^j}{\partial x^i} + \frac{\partial u}{\partial \tilde{y}_j}\frac{\partial \tilde{y}_j}{\partial x^i}, \quad \frac{\partial \tilde{y}_k}{\partial y_j} = \frac{\partial x^j}{\partial \tilde{x}^k}. \tag{1.4}$$

It follows, in particular, that under the change $(x,y) \mapsto (\tilde{x},\tilde{y})$, the second order part of (1.1), the first order part of (1.1), and the zero order part of (1.1) transforms to second order, first order, and zero order operators respectively, and consequently, if the operator (1.1) is invariant, then its second order part, its first order part, and its zero order part must be invariant. In order that the zero order term $V(x,y)u$ was invariant it is necessary and sufficient that $V(x,y)$ is invariant and therefore $V(x,y)$ is a function. From the invariance of the second order part one has

$$g_{ij}(x)\frac{\partial^2 u}{\partial y_i \partial y_j} = g_{ij}(x)\frac{\partial^2 u}{\partial \tilde{y}_k \partial \tilde{y}_m}\frac{\partial \tilde{y}_m}{\partial y_j}\frac{\partial \tilde{y}_k}{\partial y_i}$$

$$= g_{ij}(x)\frac{\partial^2 u}{\partial \tilde{y}_k \partial \tilde{y}_m}\frac{\partial x^j}{\partial \tilde{x}^m}\frac{\partial x^i}{\partial \tilde{x}^k} = \tilde{g}_{km}(\tilde{x})\frac{\partial^2 u}{\partial \tilde{y}_k \partial \tilde{y}_m},$$

and consequently, the invariance of the second order part is equivalent to the requirement that g is a tensor, and therefore defines a riemannian metric. Let us write now the condition of the invariance of the first order part of operator (1.1). Changing the variable $(x,y) \mapsto (\tilde{x},\tilde{y})$ in the first order part of (1,1) one has

$$\left(a^i(x) + \alpha^{ij}(x) y_j\right)\frac{\partial u}{\partial x^i} + \left(b_i(x) + \beta_i^j(x) y_j + \frac{1}{2}\gamma_i^{jl}(x) y_j y_l \right)\frac{\partial u}{\partial y_i}$$

$$= \left(a^i(x) + \alpha^{ij}(x)\tilde{y}_m \frac{\partial \tilde{x}^m}{\partial x^j}\right)\left(\frac{\partial u}{\partial \tilde{x}^l}\frac{\partial \tilde{x}^l}{\partial x^i} + \frac{\partial u}{\partial \tilde{y}_l}\frac{\partial \tilde{y}_l}{\partial x^i}\right)$$

$$+ \left(b_i(x) + \beta_i^j(x)\tilde{y}_m\frac{\partial \tilde{x}^m}{\partial x^j} + \frac{1}{2}\gamma_i^{jl}(x)\tilde{y}_m\tilde{y}_p\frac{\partial \tilde{x}^m}{\partial x^j}\frac{\partial \tilde{x}^p}{\partial x^l}\right)\frac{\partial u}{\partial \tilde{y}_q}\frac{\partial \tilde{y}_q}{\partial y_i}.$$

Therefore, the invariance of this first order part is equivalent to the following two equations:

$$\tilde{a}^i(\tilde{x}) + \tilde{\alpha}^{ij}(\tilde{x})\tilde{y}_j = \left(a^l(x) + \alpha^{lj}(x)\tilde{y}_m \frac{\partial \tilde{x}^m}{\partial x^j} \right) \frac{\partial \tilde{x}^i}{\partial x^l}, \tag{1.5}$$

and

$$\left(\tilde{b}_i(\tilde{x}) + \tilde{\beta}_i^j(\tilde{x})\tilde{y}_j + \frac{1}{2}\tilde{\gamma}_i^{jl}(\tilde{x})\tilde{y}_j\tilde{y}_l \right) = \left(a^l(x) + \alpha^{lj}(x)\tilde{y}_m \frac{\partial \tilde{x}^m}{\partial x^j} \right) \frac{\partial \tilde{y}_i}{\partial x^l}$$

$$+ \left(b_q(x) + \beta_q^j(x)\tilde{y}_m \frac{\partial \tilde{x}^m}{\partial x^j} + \frac{1}{2}\gamma_q^{jl}(x)\tilde{y}_m\tilde{y}_p \frac{\partial \tilde{x}^m}{\partial x^j}\frac{\partial \tilde{x}^p}{\partial x^l} \right) \frac{\partial \tilde{y}_i}{\partial y_q}. \tag{1.6}$$

From (1.5) one obtains that a and α are tensors, as is required. Next,

$$\frac{\partial \tilde{y}_i}{\partial x^l} = -\tilde{y}_p \frac{\partial \tilde{x}^p}{\partial x^l \partial x^m} \frac{\partial x^m}{\partial \tilde{x}^i},$$

Therefore, equating in (1.6) the terms which do not depend on $\tilde{y}$, the terms depending on $\tilde{y}$ linearly, and the terms depending on $\tilde{y}$ quadratically, one gets that b is a tensor of the type $(1,0)$, and that the law of the transformation of β and γ has the form

$$\tilde{\gamma}_i^{mp}(\tilde{x}) = \gamma_q^{jl}(x) \frac{\partial \tilde{x}^m}{\partial x^j}\frac{\partial \tilde{x}^p}{\partial x^l}\frac{\partial x^q}{\partial \tilde{x}^i} - 2\alpha^{lj}(x)\frac{\partial \tilde{x}^m}{\partial x^j}\frac{\partial^2 \tilde{x}^p}{\partial x^l \partial x^q}\frac{\partial x^q}{\partial \tilde{x}^i},$$

$$\tilde{\beta}_i^p(\tilde{x}) = \beta_q^j(x)\frac{\partial \tilde{x}^p}{\partial x^j}\frac{\partial x^q}{\partial \tilde{x}^i} - a^l(x)\frac{\partial^2 \tilde{x}^p}{\partial x^l \partial x^m}\frac{\partial x^m}{\partial \tilde{x}^i}.$$

Since

$$\frac{\partial \tilde{a}^p}{\partial \tilde{x}^i}(\tilde{x}) = \frac{\partial a^j}{\partial x^q}\frac{\partial x^q}{\partial \tilde{x}^i}\frac{\partial \tilde{x}^p}{\partial x^j} + a^l(x)\frac{\partial^2 \tilde{x}^p}{\partial x^l \partial x^m}\frac{\partial x^m}{\partial \tilde{x}^i},$$

$$\frac{\partial \tilde{\alpha}^{mp}}{\partial \tilde{x}^i}(\tilde{x}) = \frac{\partial \alpha^{lj}}{\partial x^q}\frac{\partial x^q}{\partial \tilde{x}^i}\frac{\partial \tilde{x}^m}{\partial x^l}\frac{\partial \tilde{x}^p}{\partial x^j} + 2\alpha^{lj}(x)\frac{\partial^2 \tilde{x}^p}{\partial x^q \partial x^l}\frac{\partial x^q}{\partial \tilde{x}^i}\frac{\partial \tilde{x}^m}{\partial x^j},$$

it follows that $\{\gamma_i^{mp} + \frac{\partial \alpha^{mp}}{\partial x^i}\}$ and $\{\beta_i^p + \frac{\partial a^p}{\partial x^i}\}$ are tensors of the types $(1,2)$ and $(1,1)$ respectively. Denoting these tensors again by γ and β respectively, yields representation (1.2). The proof is complete.

Let us write the stochastic differential equation for the diffusion process corresponding to the operator (1.2) with vanishing V. Let $r : M \mapsto \mathcal{R}^N$ be an embedding of the Riemanian manifold M in the Euclidean space (as is well known, such embedding always exists). The operator (1.2) stands for the diffusion on $T^\star M$ defined by the stochastic system

$$\begin{cases} dx = \frac{\partial f}{\partial y}\, dt \\ dy_i = -\frac{\partial f}{\partial x^i}\, dt + (b_i(x) + \beta_i^j(x)y_j + \frac{1}{2}\gamma_i^{kl}(x)y_k y_l)\, dt + \frac{\partial r^j}{\partial x^i}\, dw_j, \end{cases} \tag{1.7}$$

where w is the standard N-dimensional Wiener process. This statement follows from the well known formula for the Riemanian metric

$$g_{ij}(x) = \sum_{k=1}^{N} \frac{\partial r^k}{\partial x^i} \frac{\partial r^k}{\partial x^j}$$

and the Ito formula. It is interesting to note that though system (1.7) depends explicitly on the embedding r, the corresponding operator L defining the transition probabilities for diffusion process (1.7) depends only on the Riemanian structure.

One sees that system (1.7) describes a curvilinear version of the classical Ornstein-Uhlenbeck process (see e.g. [Joe] for an invariant definition) defined originally (see, e.g.[Nel1]) by the system ($x, y \in \mathcal{R}^n$)

$$\begin{cases} \dot{x} = y \\ dy = -\frac{\partial V}{\partial x}\, dt - \beta y\, dt + dw(t) \end{cases} \tag{1.8}$$

as a model of Brownian motion, where $\beta \geq 0$ is some constant and $V(x)$ is some (usually bounded from below) function (potential). System (1.8) defines a Newton particle (Hamiltonian system with the Hamiltonian $V(x) + y^2/2$) disturbed by the friction force βy and by the white noise random force dw. System (1.7) describes a Hamiltonian system (defined by the Hamiltonian function f which is quadratic in momentum but with varying coefficients) with additional deterministic force (defined by the 1-form b), the friction $\beta_i^j(x)y_j + \frac{1}{2}\gamma_i^{kl}(x)y_k y_l$ (which can depend on the first and second degree of the velocity) and the white noise force depending on the position of the particle.

In the case of vanishing b, β, γ system (1.7) is a stochastic Hamiltonian system with non-homogeneous singular random Hamiltonian $f(x, y) + r(x)\dot{w}$, which describes the deterministic Hamiltonian flow disturbed by the white noise force:

$$\begin{cases} dx = \frac{\partial f}{\partial y}\, dt \\ dy = -\frac{\partial f}{\partial x}\, dt + \frac{\partial}{\partial x}(r, dw). \end{cases} \tag{1.9}$$

The "plane" stochastic Hamiltonian systems, i.e. (1.9) for $M = \mathcal{R}^n$, were investigated recently in connection with their application to the theory of stochastic partial differential equation, see [K1], [TZ1], [TZ2].

The mostly used example of the Hamiltonian system on the cotangent bundle $T^\star M$ of a Riemanian manifold is of course the geodesic flow, which stands for the Hamiltonian function $f = (G(x)y, y)/2$, where $G(x) = g^{-1}(x)$. For this f, system (1.9) takes the form

$$\begin{cases} \dot{x} = G(x)y \\ dy = -\frac{1}{2}\frac{\partial}{\partial x}(G(x)y, y)\, dt + \frac{\partial}{\partial x}(r, dw), \end{cases} \tag{1.10}$$

This system was called in [K1] the stochastic geodesic flow. The investigation of its small time asymptotics was begun in [AHK2]. Corresponding Hamiltonian (2.3.4) of the stochastic geodesic flow is

$$H = \frac{1}{2}(g(x)q, q) - (G(x)y, p) + \frac{1}{2}\left(\frac{\partial}{\partial x}(G(x)y, y), q\right) \tag{1.11}$$

and the invariant diffusion equation is

$$\frac{\partial u}{\partial t} = Lu = \frac{h}{2}\, tr\left(g(x)\frac{\partial^2 u}{\partial y_i \partial y_j}\right) + \left(G(x)y, \frac{\partial u}{\partial x}\right) - \frac{1}{2}\left(\frac{\partial}{\partial x}(G(x)y, y), \frac{\partial u}{\partial y}\right). \tag{1.12}$$

It depends only on the Riemanian structure and therefore its property should reflect the geometry of M, which explain more explicitly in the next sections.

2. Small time asymptotics for stochastic geodesic flow

The stochastic geodesic flow is a good example for performing the general results of the previous chapter. Using these results we present now the calculation of the main terms of the small time asymptotics for the Green function of equation (1.12), i. e. its solution with the initial data

$$u_G(0, x, y; x_0, y^0) = \delta(x - x_0)\delta(y - y^0) \tag{2.1}$$

in a neighbourhood of the point $(x_0, y^0) \in T^\star M$.

All calculations will be carried out in normal coordinates around x_0 (see, e.g. [CFKS]), in which $x_0 = 0$,

$$g_{ij}(x) = \delta_i^j + \frac{1}{2} g_{ij}^{kl} x^k x^l + O(|x|^3), \tag{2.2}$$

and $\det g(x) = 1$ identically. These conditions imply that

$$\sum_{i=1}^{n} g_{ii}^{kl} = 0 \quad \forall k, l \tag{2.3}$$

and that the Gaussian (or scalar) curvature in x_0 is equal to

$$R = \sum_{i,k} g_{ik}^{ik}. \tag{2.4}$$

Remark. Some authors do not include the requirement $\det g = 1$ in the definition of normal coordinates. Notice however that if a system of coordinates x on a n-dimensional riemanian manifold M satisfies all other conditions of normality but for the condition $\det g = 1$, then the coordinates $\tilde{x}$ defined by the formula

$$\tilde{x}^1 = \int_0^{x^1} \sqrt{g}(s, x^2, ..., x^n)\, ds, \quad \tilde{x}^i = x^i,\, i \geq 2,$$

satisfies all the conditions of normality given above, as one checks easily (see [CFKS]).

Moreover, from (2.2) one gets obviously the expansions

$$G_{ij} = \delta_i^j - \frac{1}{2} g_{ij}^{kl} x^k x^l + O(|x|^3), \tag{2.5}$$

for the inverse matrix $G(x) = g^{-1}(x)$, and also

$$\frac{\partial G_{ij}}{\partial x^k}(x) = -g_{ij}^{kl} + O(|x|^2). \tag{2.6}$$

To find the asymptotics of the two-point function one should solve the main equation (3.2.12), which for the case of Hamiltonian (1.11) takes the form

$$\frac{\partial \Sigma}{\partial t} - \frac{\xi + G(t\xi + \tilde{x})(y + \tilde{y}) - G(\tilde{x})\tilde{y}}{t}\frac{\partial \Sigma}{\partial \xi} - \left(g(\tilde{x})\tilde{q}, \frac{\partial \Sigma}{\partial y}\right)$$

$$+\frac{1}{2}\left[\left(\frac{\partial G}{\partial x}(t\xi + \tilde{x})(y + \tilde{y}), y + \tilde{y}\right) - \left(\frac{\partial G}{\partial x}(\tilde{x})\tilde{y}, \tilde{y}\right)\right]\frac{\partial \Sigma}{\partial y}$$

$$+\frac{1}{2}\left(g(t\xi + \tilde{x})\frac{\partial \Sigma}{\partial y}, \frac{\partial \Sigma}{\partial y}\right) = 0. \tag{2.7}$$

Using (2.2), (2.3) one concludes that

$$\tilde{x} = x_0 - ty^0 + O(t^3), \quad \tilde{y} = y^0 + O(t^2), \quad \tilde{q} = O(t^2), \tag{2.8}$$

and then one rewrites (2.7) in the coordinate form (using now low indices for both ξ and y:

$$\frac{\partial \Sigma}{\partial t} - \frac{(\xi + y)_i - \frac{t^2}{2} g_{ij}^{kl}[(\xi_k - y_k^0)(\xi_l - y_l^0)(y_j + y_j^0) - y_k^0 y_l^0 y_j^0] + O(t^3)}{t}\frac{\partial \Sigma}{\partial \xi_i}$$

$$-\left[\frac{t}{2} g_{ij}^{kl}[(\xi_l - y_l^0)(y_i + y_i^0)(y_j + y_j^0) + y_i^0 y_j^0 y_l^0] + O(t^2)\right]\frac{\partial \Sigma}{\partial y_k}$$

$$+\frac{1}{2}(1 + \frac{t^2}{2} g_{ij}^{kl}(\xi_k - y_k^0)(\xi_l - y_l^0) + O(t^3))\frac{\partial \Sigma}{\partial y_i}\frac{\partial \Sigma}{\partial y_j} = 0. \tag{2.9}$$

Following the arguments of Sect.2 of the previous chapter one looks for the solution of this equation in form (3.2.6), where Σ_{-1} is a positive quadratic form and $\Sigma_0(0,0) = 0$. For Σ_{-1} one gets equation (3.2.10) with α_0 and g_0 being unit matrices. Its solution is given by (3.2.16). For Σ_0 one finds then the equation

$$-(y + \xi)\frac{\partial \Sigma_0}{\partial \xi} + (6\xi + 4y)\frac{\partial \Sigma_0}{\partial y} = 0,$$

whose solution vanishes, due to Proposition 3.7.1. Furthermore, for Σ_1 one obtains the equation

$$\Sigma_1 - (y + \xi)_i \frac{\partial \Sigma_1}{\partial \xi_i} + (6\xi + 4y)_i \frac{\partial \Sigma_1}{\partial y_i}$$

$$+g_{ij}^{kl}[(\xi_k - y_k^0)(\xi_l - y_l^0)(y_j + y_j^0) - y_k^0 y_l^0 y_j^0](6\xi + 3y)_i$$

$$-g_{ij}^{kl}[(\xi_l - y_l^0)(y_i + y_i^0)(y_j + y_j^0) + y_i^0 y_j^0 y_l^0](3\xi + 2y)_k$$
$$+g_{ij}^{kl}(\xi_k - y_k^0)(\xi_l - y_l^0)(3\xi + 2y)_i(3\xi + 2y)_j = 0.$$

Opening the brackets one presents this equation in the form

$$\Sigma_1 - (y + \xi)_i \frac{\partial \Sigma_1}{\partial \xi_i} + (6\xi + 4y)_i \frac{\partial \Sigma_1}{\partial y_i} = F(\xi, y), \tag{2.10}$$

where F is the sum $F_2 + F_3 + F_4$ of the homogeneous polynomials of degree 2,3,4 given by the formulas

$$F_2 = g_{ij}^{kl}[(12\xi_i\xi_k - 4y_iy_k)y_j^0 y_l^0 - (18\xi_i y_j + 7y_iy_j + 9\xi_i\xi_j)y_k^0 y_l^0 + (3\xi_k\xi_l + 2\xi_k y_l)y_i^0 y_j^0], \tag{2.11}$$

$$F_3 = g_{ij}^{kl}[(-6\xi_i\xi_k\xi_l + 3\xi_k\xi_l y_i + 4\xi_k y_i y_l)y_j^0$$
$$+(36\xi_i\xi_k y_j + 11\xi_k y_i y_j - 2y_i y_j y_k + 18\xi_i\xi_j\xi_k)y_l^0], \tag{2.12}$$

$$F_4 = g_{ij}^{kl}[2\xi_k y_i y_j y_l - 4\xi_k\xi_l y_i y_j - 18\xi_i\xi_k\xi_l y_j - 9\xi_i\xi_j\xi_k\xi_l]. \tag{2.13}$$

The solution of this equation is the sum of the solutions Σ_1^2, Σ_1^3, Σ_1^4 corresponding to F_2, F_3, and F_4 in the r.h.s. of (2.10). These solutions can be calculated by formula (3.7.10). For instance, Σ_1^2 is given by (3.7.15) with F_2 being equal to (2.11). These calculations are rather long, but the form of the solution is clear:

$$\Sigma_1 = g_{ij}^{kl} R_{ijkl}(\xi, y, y^0), \tag{2.14}$$

where R_{ijkl} are homogeneous polynomials of degree 4 in the variables ξ, y, y^0. Similarly one sees that the other terms Σ_j are homogeneous polynomials in ξ, y, y^0 of degree $j + 3$, which is important to know when making the estimates uniform in y^0.

Let us find now the first nontrivial term of the asymptotic solution of the transport equation. In the case of Hamiltonian (1.11), the general equation (3.3.3) takes the form

$$\frac{\partial \Psi}{\partial t} - \frac{\alpha}{t}\Psi - \frac{\xi + G(\xi t + \tilde{x})(y + \tilde{y}) - G(\tilde{x})\tilde{y}}{t}\frac{\partial \Psi}{\partial \xi}$$
$$+\frac{1}{2}\left[\left(\frac{\partial G}{\partial x}(t\xi + \tilde{x})(y + \tilde{y}), y + \tilde{y}\right) - \left(\frac{\partial G}{\partial x}(\tilde{x})\tilde{y}, \tilde{y}\right)\right]\frac{\partial \Psi}{\partial y}$$
$$-\left(g(\tilde{x})\tilde{q}, \frac{\partial \Psi}{\partial y}\right) + \left(g(t\xi + \tilde{x})\frac{\partial \Sigma}{\partial y}, \frac{\partial \Psi}{\partial y}\right) + \frac{1}{2}\Psi\, tr\left(g(t\xi + \tilde{x})\frac{\partial^2 \Sigma}{\partial y^2}\right) = 0, \tag{2.15}$$

where

$$\Psi(t, \xi, y) = t^\alpha \phi(t, t\xi + \tilde{x}, y + \tilde{y}; 0, y^0). \tag{2.16}$$

From (3.3.5) one finds $\alpha = 2n$. Looking for the solution of (2.15) in the form

$$\Psi = 1 + t\Psi_1 + t^2\Psi_2 + ...$$

one gets comparing the terms at t^0 the following equation (since $\Sigma_0 = 0$):

$$\Psi_1 - \left(\xi + y, \frac{\partial \Psi_1}{\partial \xi}\right) + \left(6\xi + 4y, \frac{\partial \Psi_1}{\partial y}\right) = 0.$$

Due to Proposition 3.7.1, Ψ_1 vanishes. Comparing the coefficients at t yields

$$\Psi_2 - \left(\xi + y, \frac{\partial \Psi_2}{\partial \xi}\right) + \left(6\xi + 4y, \frac{\partial \Psi_2}{\partial y}\right)$$

$$+tr\left(\frac{1}{2}\frac{\partial^2 \Sigma_1}{\partial y^2} + g^{kl}(\xi_k - y_k^0)(\xi_l - y_l^0)\right) = 0. \tag{2.17}$$

It is again the equation of type (2.10) with the polynomials of degree 0,1,2 in the r.h.s. The solution of this equation is therefore given by Proposition 3.7.3. Again the calculations are rather long but the form of the solution is clear:

$$\Psi_2 = \sum_i g_{kl}^{ii} P_{kl} + g_{il}^{ik} Q_{kl} + G_{ii}^{kl} R_{kl}, \tag{2.18}$$

where P_{kl}, Q_{kl}, R_{kl} are some homogeneous polynomials in ξ, y, y^0 of degree 2.

3. The trace of the Green function and geometric invariants

It turns out that similarly to the case of non-degenerate diffusion on a compact manifold (see, e.g. [Gr],[Roe]), the resolving operator for the Cauchy problem for equation (1.12) belongs to the trace class, i.e. the trace

$$tre^{-tL} = \int_{T^\star M} u_G(t, x, y; x, y)\, dxdy \tag{3.1}$$

exists. Moreover, this integral can be developed in asymptotic power series in t with coefficients being the invariants of the Riemanian manifold. For brevity, let us put $h = 1$. The following result was announced in [AHK2] and its complete proof will be published elsewhere. We shall sketch here only the main line of necessary calculations using the technique developed in Section 3.7.

Theorem 3.1. *Integral (3.1) exists and has the asymptotical expansion for small time in the form*

$$(2\pi t^3)^{-n/2}(Vol\, M + a_3 t^3 + a_4 t^4 + ...),$$

the first nontrivial coefficient a_3 being proportional to the Gaussian curvature $G(M) = \int_M R\, dx$ of M and $Vol\, M = \int_M dx$ being the Riemanian volume.

Sketch of the Proof. The existence of the expansion follows from the asymptotic formula for the Green function obtained above. Let us show how to prove the last statement, indicating as well the main steps of the exact calculation of a_3. From (3.2.2),(3.2.4) it follows that

$$S(t, x_0, y^0; x_0, y^0) = \Sigma(t, \frac{x_0 - \tilde{x}}{t}, y^0 - \tilde{y}).$$

Therefore in normal coordinate around the point $x_0 = 0$ one has

$$S = \frac{1}{t}\left(6\frac{\tilde{x}^2}{t^2} - 6\frac{\tilde{x}}{t}(y^0 - \tilde{y}) + 2(y^0 - \tilde{y})^2\right) + t\Sigma_1\left(-\frac{\tilde{x}}{t}, y^0 - \tilde{y}\right) + O(t^2).$$

Using (2.3.5), (2.3.14) and expansion (2.2), (2.5),(2.6) let us make formulas (2.8) more precise:

$$\begin{cases} \tilde{x}^i = -ty_i^0 + \frac{1}{6}t^3(g_{ij}^{kl} - \frac{1}{2}g_{kl}^{ij})y_j^0 y_k^0 y_l^0 + O(t^4) \\ \tilde{y}_i = y_i^0 + \frac{1}{4}t^2 g_{kl}^{ij} y_j^0 y_k^0 y_l^0 + O(t^3). \end{cases} \tag{3.3}$$

Therefore

$$S = \frac{6}{t}\sum_i (y_i^0 - \frac{1}{6}t^2(g_{ij}^{kl} - \frac{1}{2}g_{kl}^{ij})y_j^0 y_k^0 y_l^0)^2 - \frac{3t}{2}y_i^0 g_{kl}^{ij} y_k^0 y_l^0 y_j^0 + t\Sigma_1(y^0, 0) + O(t^2).$$

Consequently,

$$S = \frac{6}{t}(y^0, y^0) - \frac{5}{2}tg_{ij}^{kl}y_i^0 y_j^0 y_k^0 y_l^0 + t\Sigma_1(y^0, 0) + O(t^2). \tag{3.4}$$

Therefore, to get the first nontrivial term of the expansion of S one needs the solution of (2.10) at $y = 0, \xi = y^0$.

Similarly, we have

$$\phi(t, 0, y^0; 0, y^0) = t^{-2n}\Psi(t, -\frac{\tilde{x}}{t}, y^0 - \tilde{y})$$

$$= t^{-2n}(1 + t^2\Psi_2(-\frac{\tilde{x}}{t}, y^0 - \tilde{y}) + O(t^3) = t^{-2n}(1 + t^2\Psi_2(y^0, 0) + O(t^3)), \tag{3.5}$$

and therefore we need the solution of (2.17) also only at $y = 0, \xi = y^0$. From (2.14) and (2.18) it follows that

$$\Sigma_1(y^0, 0) = \sigma g_{ij}^{kl} y_i^0 y_j^0 y_k^0 y_l^0, \tag{3.6}$$

$$\Psi_2(y^0, 0) = \sum_k (\beta g_{kk}^{ij} + \gamma g_{ij}^{kk} + \delta g_{jk}^{ik}) y_i^0 y_j^0 \tag{3.7}$$

with some constants $\sigma, \beta, \gamma, \delta$.

The key point in the proof of the theorem is the following fact.

Lemma 3.1. *In formula (3.6), one has* $\sigma = \frac{5}{2}$.

Proof. To simplify calculations let us first note that formula (7.31) will not change if we take instead of the tensor g_{ij}^{kl} its symmetrisation, and therefore, when calculating $\Sigma_1(y^0, 0)$ from equation (7.18) we can consider the coefficients g_{ij}^{kl} in the expression for F to be completely symmetric (with respect to any change of the order of its indices i, j, k, l. In particular, it means that instead of F_2 and F_3 from (2.11), (2.12) we can take

$$\tilde{F}_2 = (6\xi_i\xi_j - 16\xi_i y_j - 11 y_i y_j) g_{ij}^{kl} y_k^0 y_l^0, \tag{3.8}$$

$$\tilde{F}_3 = (12\xi_i\xi_j\xi_k + 39\xi_i\xi_k y_j + 15\xi_i y_j y_k - 2 y_i y_j y_k) g_{ij}^{kl} y_l^0. \tag{3.9}$$

Next, clearly

$$\Sigma_1(y^0, 0) = \frac{1}{2}\frac{\partial^2 \Sigma_1^2}{\partial\xi_i\partial\xi_j} y_0^i y_0^j + \frac{1}{3!}\frac{\partial^3 \Sigma_1^3}{\partial\xi_i\partial\xi_j\partial\xi_k} y_0^i y_0^j y_0^k + \frac{1}{4!}\frac{\partial^4 \Sigma_1^4}{\partial\xi_i\partial\xi_j\partial\xi_k\partial\xi_l} y_0^i y_0^j y_0^k y_0^l, \tag{3.10}$$

where Σ_1^p, $p = 2, 3, 4$, denote the corresponding homogeneous part of Σ_1. Now taking into consideration the assumed symmetricity of the coefficients of g_{ij}^{kl} one gets from (3.7.15) and (3.8) that

$$\frac{1}{2}\frac{\partial^2 \Sigma_1^2}{\partial\xi_i\partial\xi_j}(y^0, 0) = \left(-\frac{6}{5} \times 11 + \frac{9}{10} \times 16 + \frac{4}{5} \times 6\right) g_{ij}^{kl} y_k^0 y_l^0, = 6 g_{ij}^{kl} y_k^0 y_l^0,$$

from (3.7.16) and (3.9) that

$$\frac{1}{3!}\frac{\partial^3 \Sigma_1^3}{\partial\xi_i\partial\xi_j\partial\xi_k} = \left(\frac{54}{35} \times 2 + \frac{39}{35} \times 15 - \frac{61}{70} \times 39 + \frac{113}{140} \times 12\right) g_{ij}^{kl} y_l^0 = -\frac{9}{2} g_{ij}^{kl} y_l^0,$$

and from (3.7.17) and (2.13) that

$$\frac{1}{4!}\frac{\partial^4 \Sigma_1^4}{\partial\xi_i\partial\xi_j\partial\xi_k\partial\xi_l} = \left(-\frac{263}{315} \times 9 - \frac{51}{35} \times 2 + \frac{92}{105} \times 18 - \frac{38}{35} \times 4\right) g_{ij}^{kl} = g_{ij}^{kl}.$$

Subsituting these formulas to (3.10) yields

$$\Sigma_1(y^0, 0) = (6 - \frac{9}{2} + 1) g_{ij}^{kl} y_i^0 y_j^0 y_k^0 y_l^0 = \frac{5}{2} g_{ij}^{kl} y_i^0 y_j^0 y_k^0 y_l^0,$$

and the Lemma is proved.

End of the proof of the Theorem. Due to the Lemma, the sum of the second and third terms in the expression (3.4) for S vanishes. Therefore, due to (3.4), (3.6), (3.7), and to the fact that the odd degrees of y^0 do not contribute to the integral, one concludes that the integral $\int u(t, 0, y; 0, y)\, dy$ is equal to

$$\left(\frac{\sqrt{3}}{\pi t^2}\right)^n \int e^{-6y^2/t}[1 + t^2 \sum_k (\beta g_{kk}^{ij} + \gamma g_{ij}^{kk} + \delta g_{jk}^{ik}) y_i y_j + O(t^3|y|^6) + O(t^4|y|^4)]\, dy.$$

Due to (2.3), (2.4), this is equal to

$$= \left(\frac{\sqrt{3}}{\pi t^2}\right)^n \left(\frac{t\pi}{6}\right)^{n/2} [1 + \frac{1}{12} t^3 \delta R + O(t^4)].$$

Integrating this expression over M obviously gives (3.2) with $a_3 = \delta G(M)/12$.