Chapter 9. PATH INTEGRATION FOR THE SCHRÖDINGER, HEAT AND COMPLEX DIFFUSION EQUATIONS

1. Introduction.

There exist several approaches to the rigorous mathematical construction of the path integral, the most important of them (as well as an extensive literature on this subject) are reviewed briefly in Appendices G and H. Unfortunately, all these methods cover still only a very restrictive class of potentials, which is clearly not sufficient for physical applications, where path integration is widely used without rigorous justification. On the other hand, most of the known approaches define the path integral not as a genuine integral (in the sense of Lebesgue or Riemann), but as a certain generalised functional. In this chapter we give a rigorous construction of the path integral which, on the one hand, covers a wide class of potentials and can be applied in a uniform way to the Schrödinger, heat and complex diffusion equations, and on the other hand, is defined as a genuine integral over a bona fide σ- additive (or even finite) measure on path space. Moreover, in the original papers of Feynman the path integral was defined (heuristically) in such a way that the solutions to the Schrödinger equation was expressed as the integral of the function $\exp\{iS\}$, where S is the classical action along the paths. It seems that the corresponding measure was not constructed rigorously even for the case of the heat equation with sources (notice that in the famous Feynman-Kac formula that gives a rigorous path integral representation for the solutions to the heat equation, a part of the action is actually "hidden" inside the Wiener measure). Here we construct a measure on a path space (actually on the so called Cameron-Martin space of trajectories with L^2 first derivative) such that the solutions to the Schrödinger, heat and complex diffusion equations can be represented as the integrals of the exponential of the action with respect to this measure, which is essentially the same for all these cases (to within certain bounded densities). However, for the case of the Schrödinger equation the integral is usually not absolutely convergent and needs a certain regularisation. This regularisation is of precisely the same kind as is used to define the finite- dimensional integral

$$(U_0 f)(x) = (2\pi t i)^{-d/2} \int_{\mathcal{R}^d} \exp\{-\frac{|x-\xi|^2}{2ti}\} f(\xi)\, d\xi \qquad (1.1)$$

giving the free propagator $e^{it\Delta/2}f$. Namely, this integral is not well defined for general $f \in L^2(\mathcal{R}^d)$. The most natural way to define it is based on the observation that, according to the spectral theorem, for all $t > 0$

$$e^{it\Delta/2} f = \lim_{\epsilon \to 0_+} e^{it(1-i\epsilon)\Delta/2} f \qquad (1.2)$$

in $L^2(\mathcal{R}^d)$ (the operator $e^{it(1-i\epsilon)\Delta/2}$ defines the free Schrödinger evolution in complex time $t(1-i\epsilon)$). Since

$$(e^{it(1-i\epsilon)\Delta/2} f)(x) = (2\pi t(i+\epsilon))^{-d/2} \int_{\mathcal{R}^d} \exp\{-\frac{|x-\xi|^2}{2t(i+\epsilon)}\} f(\xi)\, d\xi$$

($\sqrt{i+\epsilon}$ is defined as the one which tends to $e^{\pi i/4}$ as $\epsilon \to 0$) and the integral on the r.h.s. of this equation is already absolutely convergent for all $f \subset L^2(\mathcal{R}^d)$, one can define the integral (1.1) by the formula

$$(U_0 f)(x) = \lim_{\epsilon \to 0_+} (2\pi t(i+\epsilon))^{-d/2} \int_{\mathcal{R}^d} \exp\{-\frac{|x-\xi|^2}{2t(i+\epsilon)}\} f(\xi)\, d\xi. \tag{1.3}$$

The same regularisation will be used to define the infinite-dimensional integral giving the solutions to the Schrödinger equation with a general potential.

At the end of the Chapter we show that the path integral constructed here has a natural representation in a cetain Fock space, which gives a connection with the Wiener measure and also with non-commutative probability and quantum stochastic calculus.

1.1. The case of potentials which are Fourier transforms of finite measures. The starting point for our construction is a representation of the solutions of the Schrödinger equation whose potential is the Fourier transform of a finite measure, in terms of the expectation of a certain functional over the path space of a certain compound Poisson process. A detailed exposition of this representation, which is due essentially to Chebotarev and Maslov, together with some references on further developments, are given in Appendix G. We begin here with a simple proof of this representation, which clearly indicates the route for the generalisations that are the subject of this chapter.

Let the function $V = V_\mu$ be the Fourier transform

$$V(x) = V_\mu(x) = \int_{\mathcal{R}^d} e^{ipx} \mu(dp) \tag{1.4}$$

of a finite complex Borel measure μ on $\mathcal{R}^d$. Now (see e.g. Appendix G) for any σ-finite complex Borel measure μ there exists a positive σ-finite measure M and a complex-valued measurable function f on $\mathcal{R}^d$ such that

$$\mu(dy) = f(y) M(dy). \tag{1.5}$$

If μ is a finite measure, then M can be chosen to be finite as well. In order to represent Feynman's integral probabilisticly, it is convenient to assume that M has no atom at the origin, i.e. $M(\{0\}) = 0$. This assumption is by no means restrictive, because one can ensure its validity by shifting V by an appropriate constant. Under this assumption, if

$$W(x) = \int_{\mathcal{R}^d} e^{ipx} M(dp), \tag{1.6}$$

then the equation

$$\frac{\partial u}{\partial t} = (W(\frac{1}{i}\frac{\partial}{\partial y}) - \lambda_M) u, \tag{1.7}$$

where $\lambda_M = M(\mathcal{R}^d)$, or equivalently

$$\frac{\partial u}{\partial t} = \int (u(y+\xi) - u(y))\, M(d\xi), \tag{1.8}$$

defines a Feller semigroup, which is the semigroup associated with the compound Poisson process having Lévy measure M, see e.g. [Br] or [Pr] for the necessary background in the theory of Lévy processes (notice only that the condition $M(\{0\}) = 0$ ensures that M is actually a measure on $\mathcal{R}^d \setminus \{0\}$, i.e. it is a finite Lévy measure). As is well known, such a process has almost surely piecewise constant paths. More precisely, a sample path Y of this process on the time interval $[0,t]$ starting at a point y is defined by a finite number, say n, of jump-times $0 < s_1 < ... < s_n \le t$, which are distributed according to the Poisson process N with intensity $\lambda_M = M(\mathcal{R}^d)$, and by independent jumps $\delta_1, ..., \delta_n$ at these times, each of which is a random variable with values in $\mathcal{R}^d \setminus \{0\}$ and with distribution defined by the probability measure M/λ_M. This path has the form

$$Y_y(s) = y + Y^{s_1 \dots s_n}_{\delta_1 \dots \delta_n}(s) = \begin{cases} Y_0 = y, \quad s < s_1, \\ Y_1 = y + \delta_1, \quad s_1 \le s < s_2, \\ \dots \\ Y_n = y + \delta_1 + \delta_2 + ... + \delta_n, \quad s_n \le s \le t \end{cases} \tag{1.9}$$

We shall denote by $E_y^{[0,t]}$ the expectation with respect to this process.

Consider the Schrödinger equation

$$\frac{\partial \psi}{\partial t} = \frac{i}{2}\Delta\psi - iV(x)\psi, \tag{1.10}$$

where V is a function (possibly complex-valued) of form (1.4). The equation for the inverse Fourier transform

$$u(y) = \tilde{\psi}(y) = (2\pi)^{-d} \int_{\mathcal{R}^d} e^{-iyx}\psi(x)\, dx$$

of ψ (or equation (1.10) in momentum representation) has the form

$$\frac{\partial u}{\partial t} = -\frac{i}{2} y^2 u - iV(\frac{1}{i}\frac{\partial}{\partial y})u. \tag{1.11}$$

Proposition 1.1. *Let u_0 be a bounded continuous function. Then the solution to the Cauchy problem of equation (1.11) with initial function u_0 has the form*

$$u(t,y) = \exp\{t\lambda_M\} E_y^{[0,t]} \left[F(Y(.))u_0(Y(t))\right], \tag{1.12}$$

where, if Y is given by (1.9),

$$F(Y(.)) = \exp\{-\frac{i}{2}\sum_{j=0}^{n}(Y_j, Y_j)(s_{j+1} - s_j)\} \prod_{j=1}^{n}(-if(\delta_j)) \tag{1.13}$$

(here $s_{n+1} = t$, $s_0 = 0$, and the function f is as in (1.5)).

In particular, choosing u_0 to be the exponential function e^{iyx_0}, one obtains a path integral representation for the Green function of equation (1.10) in momentum representation.

Remark. In Appendix G some generalisation of (1.10) is dealt with. We restrict ourselves here to (1.10) for simplicity.

We shall now give an equivalent representation for the integral in (1.12), which we shall prove later on.

1.2. Path integral as a sum of finite-dimensional integrals. One way to visualize the integral (1.12) is by rewriting it as a sum of finite dimensional integrals. To this end, let us introduce some notations. Let $PC_p(s,t)$ (abbreviated to $PC_p(t)$, if $s=0$) denote the set of all right continuous and piecewise-constant paths $[s,t] \mapsto \mathcal{R}^d$ starting from the point p, and let $PC_p^n(s,t)$ denote the subset of paths with exactly n discontinuities. Topologically, PC_p^0 is a point and $PC_p^n = Sim_t^n \times (\mathcal{R}^d \setminus \{0\})^n$, $n = 1, 2, ...$, where

$$Sim_t^n = \{s_1, ..., s_n : 0 < s_1 < s_2 < ... < s_n \leq t\} \tag{1.14}$$

denotes the standard n-dimensional simplex. In fact, the numbers s_j are the jump-times, and the n copies of $\mathcal{R}^d \setminus \{0\}$ represent the magnitudes of these jumps (see (1.9)). To each σ-finite measure M on $\mathcal{R}^d \setminus \{0\}$ (or on $\mathcal{R}^d$, but without an atom at the origin), there corresponds a σ-finite measure $M^{PC} = M^{PC}(t,p)$ on $PC_p(t)$, which is defined as the sum of measures M_n^{PC}, $n = 0, 1, ...$, where each M_n^{PC} is the product-measure on $PC_p^n(t)$ of the Lebesgue measure on Sim_t^n and of n copies of the measure M on $\mathcal{R}^d$. Thus if Y is parametrised as in (1.9), then

$$M_n^{PC}(dY(.)) = ds_1....ds_n M(d\delta_1)...M(d\delta_n).$$

From properties of the Poisson process it follows that (1.12) can be rewritten in the form

$$u(t,y) = \int_{PC_y(t)} M^{PC}(dY(.)) F(Y(.)) u_0(Y(t)), \tag{1.15}$$

or, equivalently, as the sum

$$u(t,y) = \sum_{n=0}^{\infty} u_n(t,y) = \sum_{n=0}^{\infty} \int_{PC_y^n(t)} M_n^{PC}(dY(.)) F(Y(.)) u_0(Y(t)). \tag{1.16}$$

The integrals in this series can be written more explicitly (in terms of the parametrisation (1.9) of the paths Y) as

$$u_n(t,y) = \int_{PC_y^n(t)} M_n^{PC}(dY(.)) F(Y(.)) u_0(Y(t))$$

$$= \int_{Sim_t^n} ds_1...ds_n \int_{\mathcal{R}^d} ... \int_{\mathcal{R}^d} M(d\delta_1)...M(d\delta_n)\, F(y + Y_{\delta_1...\delta_n}^{s_1...s_n}) u_0(y + \delta_1 + ... + \delta_n). \tag{1.17}$$

Notice that the multiplier $\exp\{t\lambda_M\}$ in (1.12) arises because the integral in (1.12) is not over the measure M^{PC}, but over a probability measure obtained from M^{PC} by an appropriate normalisation (namely, $M^{PC}(PC_p^1(t)) = t + O(t^2)$ for small t, and the normalised measure of the corresponding Poisson process is such that the probability of $PC_p^1(t)$ is $\lambda_M(t + O(t^2)) \exp\{-t\lambda_M\}$ and the jumps are distributed according to the normalised measure M/λ_M).

1.3. Connection with perturbation theory and a proof of Proposition 1.1. A simple proof of formula (1.16) can be obtained from non-stationary perturbation theory, which we recall now for use in what follows. First, one can rewrite equation (1.10) in an integral form using the so called interaction representation for the Schrödinger equation, where the evolution of a quantum system is described by the wave function $\phi = e^{-i\Delta t/2}\psi$ rather than the original wave function ψ. From (1.10) one gets that if ψ satisfies the Cauchy problem for equation (1.10) with the initial data ψ_0, then ϕ satisfies the equation

$$\frac{\partial \phi}{\partial t} = -ie^{-i\Delta t/2} V e^{i\Delta t/2} \phi \tag{1.18}$$

(here the symbol V is used to denote the operator of multiplication by the function V), with the same initial data $\phi_0 = \psi_0$. Integrating this equation (which is called the Schrödinger equation in the interaction representation) over t and substituting $\phi = e^{-i\Delta t/2}\psi$ one obtains the equation for ψ

$$\psi(t) = e^{i\Delta t/2}\psi_0 - i \int_0^t e^{i\Delta(t-s)/2} V \psi(s)\, ds, \tag{1.19}$$

which contains not only the information comprised by (1.10), but also the information comprised by the initial data ψ_0. Though, strictly speaking, equation (1.19) is not quite equivalent to (1.10) (because, for example, a solution of (1.19) may not belong to the domain of the operator Δ) under reasonable assumptions on V (for example, if V is bounded, or $V \in L^p + L^\infty$ with $p \geq \max(2, d/2)$, which is quite enough for our purposes) the solutions to (1.19) defines the Schrödinger evolution $e^{it(\Delta/2-V)}$ (see e.g. [Yaj] or the earlier paper [How], where (1.19) is used to prove the existence of the Schrödinger propagator in the more general case of time-dependent potentials).

Substituting expression (1.19) for ψ in the r.h.s. of (1.19) and iterating this procedure one obtains the standard perturbation theory expansion for ψ

$$\psi(t) = \big[e^{i\Delta t/2} - i \int_0^t e^{i\Delta(t-s)/2} V e^{i\Delta s/2}\, ds$$

$$+(-i)^2 \int_0^t ds \int_0^s d\tau\, e^{i\Delta(t-s)/2} V e^{i\Delta(s-\tau)/2} V e^{i\Delta \tau/2} + ...\big]\psi_0. \tag{1.20}$$

More precisely, from this procedure one obtains the following: if series (1.20) is convergent, say in L^2-sense, then its sum defines a solution to equation (1.19), and this solution is unique. Clearly, this is the case for bounded functions V, but actually holds also for more general V, see [Yaj].

Clearly in momentum representation (1.19) has the form

$$u(t, y) = e^{-ity^2/2} u_0(y) - i \int_0^t e^{-i(t-s)y^2/2} \left(V(-i\frac{\partial}{\partial y}) u_0 \right)(y)\, ds. \tag{1.21}$$

Since the operator $V(-i(\partial/\partial y))$ is that of convolution with the measure μ, in momentum representation the series (1.20) takes the form

$$u(t,y) = \sum_{j=0}^{\infty} I_j(t,y) = I_0(t,y) + (\mathcal{F}I_0)(t,y) + (\mathcal{F}^2 I_0)(t,y) + ..., \tag{1.22}$$

where $\mathcal{F}$ is the integral operator given by

$$(\mathcal{F}\phi)(t,y) = -i \int_0^t ds \int_{\mathcal{R}^d} M(dv - y) g(t-s, y) f(v-y) \phi(s,v) \tag{1.23}$$

and

$$g(t,y) = \exp\{-it(y,y)/2\}, \quad I_0 = g(t,y)u(y).$$

It is convenient to consider this series in the Banach space $C_0(\mathcal{R}^d)$ of continuous functions vanishing at infinity. The terms of the series (1.22) can now be obtained from the corresponding terms of the series (1.16), (1.17) by linear change of integration variables. Consequently, if either the series (1.22) or the series (1.16)-(1.17) is absolutely convergent and all its terms are absolutely convergent integrals, as is clearly the case under the assumptions of Proposition 1.1, one obtains the representation (1.15) (and hence also (1.12)) for the solution $u(t,x)$ of the Cauchy problem for equation (1.11).

1.4. Regularization by introducing complex times or continuous non-demolition measurement. In this chapter we are going to generalise the representations (1.12) or (1.16) to a wide class of potentials. In Section 2 we begin with a class of potentials that have form (1.4) with measure μ having support in a convex cone and being of polynomial growth. In this case (1.15) still holds without any change, even though the measure μ is not finite. However, this case is rather artificial from the physical point of view, because it does not include real potentials. In general, the terms of the series (1.22) would not be absolutely convergent integrals, or, even worse, (1.22) would not be convergent at all. To deal with this situation, one has to use some regularisations of the Schrödinger equation. As we mentioned, this regularisation will be of the same kind as is used to define the standard finite-dimensional (but not absolutely convergent) integral (1.1). Namely, if the operator $-\Delta/2 + V(x)$ is self-adjoint and bounded from below, by the spectral theorem,

$$\exp\{it(\Delta/2 - V(x))\}f = \lim_{\epsilon \to 0_+} \exp\{it(1-i\epsilon)(\Delta/2 - V(x))\}f. \tag{1.24}$$

In other words, solutions to equation (1.10) can be approximated by the solutions to the equation

$$\frac{\partial \psi}{\partial t} = \frac{1}{2}(i+\epsilon)\Delta\psi - (i+\epsilon)V(x)\psi, \tag{1.25}$$

which describes the Schrödinger evolution in complex time. The corresponding integral equation (the analogue of (1.19)) can be obtained from (1.19) by replacing i by $i+\epsilon$ everywhere. It has the form

$$\psi(t) = e^{(i+\epsilon)\Delta t/2}\psi_0 - (i+\epsilon)\int_0^t e^{(i+\epsilon)\Delta(t-s)/2} V\psi(s)\, ds. \tag{1.26}$$

If ψ satisfies (1.25), its Fourier transform u satisfies the equation

$$\frac{\partial u}{\partial t} = -\frac{1}{2}(i+\epsilon)y^2 u - (i+\epsilon)V(\frac{1}{i}\frac{\partial}{\partial y})u. \tag{1.27}$$

We shall define a measure on a path space such that for arbitrary $\epsilon > 0$ and for a rather general class of potentials V, the solution $\exp\{it(1-i\epsilon)(\Delta/2 - V(x))\}u_0$ to the Cauchy problem of equation (1.25) can be expressed as the Lebesgue (or even Riemann) integral of some functional F_ϵ with respect to this measure, which gives a rigorous definition (analogous to (1.3)) of an improper Riemann integral corresponding to the case $\epsilon = 0$, i.e. to equation (1.10). Thus, unlike the usual method of analytical continuation often used for defining Feynman integrals, where the rigorous integral is defined only for purely imaginary Planck's constant h, and for real h the integral is defined as the analytical continuation by rotating h through a right angle, in our approach, the measure is defined rigorously and is the same for all complex h with non-negative real part. Only on the boundary $Im\, h = 0$ does the corresponding integral usually become an improper Riemann integral.

Of course, the idea of using equation (1.25) as an appropriate regularisation for defining Feynman integrals is not new and goes back at least to the paper [GY]. However, this was not carried out there, because, as was noted in [Ca], there exists no direct generalisation of Wiener measure that could be used to define Feynman integral for equation (1.25) for any real ϵ. Here we shall carry out this regularization using a measure which differs essentially from Wiener measure. The connection with Wiener measure will be discussed in the last section of this chapter.

Equation (1.25) is certainly only one of many different ways to regularise the Feynman integral. However, this method is one of the simplest method, because the limit (1.24) follows directly from the spectral theorem, and other methods may require additional work to obtain the corresponding convergence result. As another regularisation to equation (1.10), one can take, for example, the equation

$$\frac{\partial \psi}{\partial t} = \frac{1}{2}(i+\epsilon)\Delta\psi - iV(x)\psi. \tag{1.28}$$

A more physically motivated regularisation can be obtained from the quantum theory of continuous measurement. Though the work with this regularisation is technically more difficult than with the regularisation based on equation (1.25), we shall describe it, because, firstly, Feynman's integral representation for continuously obserbed quantum system is a matter of independent interest, and secondly, the idea to use the theory of continuous observation for regularisation of Feynman's integral was already discussed in physical literature (see e.g. [Me2]) and it is interesting to give to this idea a rigorous mathematical justification. The idea behind this approach lies in the observation that in the process of continuous non-demolition quantum measurement a spontaneous collapse of quantum states occurs (see e.g. [Di2], [Be2], or Section 1.4), which gives a sort of regularisation for large x (or large momenta p) divergences of Feynman's integral.

As is well known, the standard Schrödinger equation describes an isolated quantum system. In quantum theory of open systems one considers a quantum system under observation in a quantum environment (reservoir). This leads to a generalisation of the

Schrödinger equation, which is called stochastic Schrödinger equation (SSE), or quantum state diffusion model, or Belavkin's quantum filtering equation (see Appendix A and Chapter 7). In the case of a non-demolition measurement of diffusion type, the SSE has the form

$$du + (iH + \frac{1}{2}\lambda^2 R^\star R)u\,dt = \lambda Ru\,dW, \tag{1.29}$$

where u is the unknown a posterior (non-normalised) wave function of the given continuously observed quantum system in a Hilbert space $\mathcal{H}$, the selfadjoint operator $H = H^\star$ in $\mathcal{H}$ is the Hamiltonian of a free (unobserved) quantum system, the vector-valued operator $R = (R^1, ..., R^d)$ in $\mathcal{H}$ represents the observed physical values, W is the standard d-dimensional Brownian motion, and the positive constant λ represents the precision of measurement. The simplest natural examples of (1.29) concern the case when H is the standard quantum mechanical Hamiltonian and the observed physical value R is either the position or momentum of the particle. The first case was considered in detail in Chapter 7. Here we are going to use mostly the second case when R represents the momentum of the particle (and therefore one models a continuous non-demolition observation of the momentum of a quantum particle). In this case the SSE (1.29) takes the form

$$d\psi = \left(\frac{1}{2}(i + \frac{\lambda}{2})\Delta\psi - iV(x)\psi\right)\,dt + \frac{1}{i}\sqrt{\frac{\lambda}{2}}\frac{\partial}{\partial x}\psi\,dW. \tag{1.30}$$

As $\lambda \to 0$, equation (1.30) approaches the standard Schrödinger equation (1.10). If ψ satisfies the SSE (1.30), the equation on the Fourier transform $u(y)$ of ψ clearly has the form

$$du = \left(-\frac{1}{2}(i + \frac{\lambda}{2})y^2 u - iV(\frac{1}{i}\frac{\partial}{\partial y})u\right)\,dt + \sqrt{\frac{\lambda}{2}}yu\,dW. \tag{1.31}$$

By Ito's formula, the solution to this equation with initial function u_0 and with vanishing potential V equals $g_\lambda^W(t,y)u_0(t,y)$ with

$$g_\lambda^W(t,y) = g_\lambda^{W(t)}(t,y) = \exp\{-\frac{1}{2}(i+\lambda)y^2 t + \sqrt{\frac{\lambda}{2}}yW(t)\}, \tag{1.32}$$

and therefore the analog of equation (1.21) corresponding to (1.31) has the form

$$u(t,y) = g_\lambda^{W(t)}(t,y)u_0 - i\int_0^t g_\lambda^{W(t)-W(s)}(t-s,.)V(-\frac{1}{i}\frac{\partial}{\partial y})g_\lambda^{W(s)}(t,.)u(s,.)\,ds. \tag{1.33}$$

As we shall see, equation (1.31) can be used to regularise Feynman's integral for equation (1.10) as an alternative to equation (1.25).

For conclusion, let us sketch the content of this chapter. In Section 2 we will obtain the path integral representation for the solutions of equations (1.11), (1.27), (1.31) for rather general scattering potentials V, including the Coulomb potential.

The momentum representation for wave functions is known to be usually convenient for the study of interacting quantum fields (see e.g. [BSch]). In quantum mechanics, however, one usually deals with the Schrödinger equation in x-representation. Therefore, it is desirable to write down Feynman's integral representation directly for equation (1.10). Since in p-representation our measure is concentrated on the space PC of piecewise constant paths, and since classically trajectories $x(t)$ and momenta $p(t)$ are connected by the equation $\dot{x} = p$, one can expect that in x-representation the correspondonding measure is concentrated on the set of continuous piecewise linear paths. In Sections 3 and 4 we shall construct this measure and the corresponding Feynman integral for equation (1.10) with bounded potentials and also for a class of singular potentials. In Section 5 we discuss the connection with the semiclassical asymptotics giving a different path integral representation for the solutions of the Schrödinger or heat equation, which is an integral of the exponential of the classical action. In the last section, we give a repersentation of our measures in Fock space and make some other remarks.

2. Momentum representation and occupation number representation

If a is a unit vector in $\mathcal{R}^d$, $b \in \mathcal{R}$, $\theta \in (0, \pi/2)$, denote

$$Con_a^\theta = \{y : (y,a) \geq |y| \cos\theta\}, \quad \Pi_a(b) = \{y : (y,a) \leq b\}.$$

Let M be a (positive) measure with support in Con_a^θ. Suppose also that M is of polynomial growth, i.e. $M(\Pi_a(b)) \leq Cb^N$ for some positive constants C, N and all positive b. Let V be given by (1.4), (1.5) in the sense of distributions, so that V is the Fourier transform of the measure fM considered as a distribution over the Schwarz space $S(\mathcal{R}^d)$.

Proposition 2.1. *For any continuous function u_0 such that $supp\, u_0 \subset \Pi_a(b)$ for some real b, all terms of the series (1.16) are absolutely convergent integrals representing continuous functions, and this series is absolutely convergent uniformly on compact sets. In particular, there exists a solution to the Cauchy problem of equation (1.11) that has support in $\Pi_a(b)$, has at most polynomial growth, and is represented by means of the Feynman integral (1.15).*

Proof. Clearly, if y does not belong to $\Pi_a(b)$, then all u_n given by (1.17) vanish. Hence, $supp\, u_n \subset \Pi_a(b)$ for all n. If $y \in \Pi_a(b)$, then

$$|u_n(t,y)| \leq \frac{t^n}{n!} \int_{Con_a^\theta} \dots \int_{Con_a^\theta} M(d\delta_1) \dots M(d\delta_n) u_0(y + \delta_1 + \dots + \delta_n).$$

Since $supp\, u_0 \subset \Pi_a(b)$, the integrand in this formula vanishes whenever $(\delta_1 + \dots + \delta_n, a) > b - (y,a)$, and, in particular, if $(\delta_j, a) > b - (y,a)$ for at least one $j = 1, \dots, n$. Hence, denoting $K = \sup\{|u_0(x)|\}$, one has

$$|u_n(t,y)| \leq \frac{t^n}{n!} K \int_{\Pi_a(b-(y,a))} \dots \int_{\Pi_a(b-(y,a))} M(d\delta_1) \dots M(d\delta_n)$$

$$\leq \frac{t^n}{n!} K C^n (b - (y,a))^{Nn},$$

which for fixed b and large y does not exceed $t^n K C^n 2^{Nn} (y,a))^{Nn}/n!$. Consequently, the series (1.16) consists of absolutely convergent integrals, is convergent, and is bounded by $K \exp\{2^N (y,a)^N Ct\}$, which implies the statement of the Proposition.

As we mentioned in the introduction, the potentials of the form occurring in Proposition 2.1 seem to have no physical relevance. As an example of a physically meaningful situation, we consider now the case of scattering potentials using the regularisation (1.27) or (1.31). Let V have form (1.4), (1.5) (again in the sense of distributions) with M being the Lebesgue measure M_{Leb} and $f \in L^1 + L^q$, i.e. $f = f_1 + f_2$ with $f_1 \in L^1$, $f_2 \in L^q$, with q in the interval $(1, d/(d-2))$, $d > 2$. Notice that this class of potentials includes the Coulomb case $V(x) = |x|^{-1}$ in $\mathcal{R}^3$, because for this case $f(y) = |y|^{-2}$.

Proposition 2.2. *Under the given assumptions on V there exists a (strong) solution $u(t,y)$ to the Cauchy problem of equations (1.27) and (1.31) with initial data u_0, which is given in terms of the Feynman integral of type (1.15). More precisely*

$$u(t,y) = \int_{PC_y(t)} M_{Leb}^{PC}(dY(.))F(Y(.))u_0(Y(t)), \tag{2.1}$$

where, if Y is parametrised as in (1.9),

$$F(Y(.)) = F_\epsilon(Y(.)) = \exp\{-\frac{1}{2}(i+\epsilon)\sum_{j=0}^{n} Y_j^2(s_{j+1}-s_j)\} \prod_{j=1}^{n}(-i(1-i\epsilon)f(\delta_j)) \tag{2.2}$$

for the case of equation (1.27), and $F(Y(.))$ equals

$$F_\lambda^W(Y(.)) = \prod_{j=1}^{n}(-if(\delta_j))$$

$$\times \exp\{-\sum_{j=0}^{n}\left[\frac{i+\lambda}{2}Y_j^2(s_{j+1}-s_j) - \sqrt{\frac{\lambda}{2}}Y_j(W(s_{j+1})-W(s_j))\right]\} \tag{2.3}$$

for the case of equation (1.31).

Proof. Since the proofs for equations (1.27) and (1.31) are quite similar, let us consider only the case of equation (1.31). As is explained in the introduction, it is sufficient to prove that for any bounded continuous function ϕ the integral (1.23), with $g = g_\lambda^W$ as in (1.32), is absolutely convergent (almost surely), and that furthermore, the corresponding series (1.22) is absolutely convergent. To this end, consider the integral

$$J = \int_{\mathcal{R}^d} |f(v-y)| g_\lambda^W(t,y)\, dy.$$

Clearly, the function g_λ^W is bounded (for a.a. W) for times in an arbitrary finite closed subinterval of the positive halfline, and for small t

$$\sup_y\{|g(t,y)|\} = \exp\{W^2(t)/4t\} \le \exp\{\log|\log t|/2\} = \sqrt{|\log t|}, \tag{2.4}$$

due to the well known law of the iterated logarithm for the Brownian motion W. Hence, by the assumptions on f and the Hölder inequality

$$J = O(\sqrt{|\log t|}) + O(1)\|g_\lambda^W(t,.)\|_{L^p},$$

where $p^{-1}+q^{-1}=1$. Since

$$\|g_\lambda^W(t,.)\|_{L^p}^p = \left(\frac{2\pi}{p\lambda t}\right)^{d/2} \exp\{\frac{pW^2(t)}{4t}\},$$

it follows that J is bounded for t in any finite interval of the positive halfline, and $J = O(\lambda t)^{-d/2p}\sqrt{|\log t|}$ for small t. Since the condition $q < d/(d-2)$ is equivalent to the condition $p > d/2$, there exists $\epsilon \in (0,1)$ such that $J \le C((\lambda t)^{-(1-\epsilon)})$. Moreover, clearly $I_0(t,y) = g(t,y)u_0(y)$ does not exceed $Kt^{-\epsilon}$ for some constant K. We can now easily estimate the terms of the series (1.22). Namely, we have

$$|\mathcal{F}I_0(t,y)| \le KC\lambda^{-(1-\epsilon)} \int_0^t (t-s)^{-(1-\epsilon)} s^{-\epsilon}\, ds = KC\lambda^{-(1-\epsilon)} B(\epsilon, 1-\epsilon),$$

where B denotes the Euler β-function. Similarly,

$$|\mathcal{F}^2 I_0(t,y)| \le \lambda^{-2(1-\epsilon)} B(\epsilon,1-\epsilon) KC^2 \int_0^t (t-s)^{-(1-\epsilon)} ds$$

$$= B(\epsilon,1-\epsilon)B(\epsilon,1)KC^2 t^\epsilon.$$

By induction, we obtain the estimate

$$|\mathcal{F}^k I_0(t,y)| \le KC^k \lambda^{-k(1-\epsilon)} t^{(k-1)\epsilon} B(\epsilon,1-\epsilon)B(\epsilon,1)...B(\epsilon,1+(k-2)\epsilon).$$

Using the representation of the β-function in terms of the Γ-function, one obtainss that the terms of series (1.22) are of order $t^{k\epsilon}/\Gamma(1+k\epsilon)$, which implies the convergence of this series for all t. Since we have estimated all functions by their magnitude, we have proved also that all terms of series (1.22) are absolutely convergent integrals, and that this series converges absolutely.

The following is a direct consequence of (1.24) and Proposition 2.2.

Proposition 2.3. *Assume the assumptions of Proposition 2.2 hold. If the operator $-\Delta/2 + V(x)$ is self-adjoint and bounded from below, then for any $u_0 \in L^\infty \cap L^2$, the solution to equation (1.10) is given by the improper Feynman integral (1.15), which should be understood as*

$$u(t,y) = \lim_{\epsilon\to 0} \int_{PC_y(t)} M_{Leb}^{PC}(dY(.)) F_\epsilon(Y(.)) u_0(Y(t)). \tag{2.5}$$

Since the convergence of solutions of equation (1.31) to solutions of the ordinary Schrödinger equation seems to be unknown, the use of equation (1.31) to obtain a regularisation for the Feynman integral for the Schrödinger equation similar to (2.5) requires some additional work. It seems that this can be done under the assumptions of Proposition 2.2 using the technique from [Yaj]. But we shall restrict ourselves here to the case of a bounded potential, which will be used also in the next Section. Notice that we prove now this result using p-representation, but it automatically implies the same fact for the Schrödinger equation in x-representation.

Proosition 2.4. *Let V be a bounded measurable function. Then for any $u_0 \in L^2(\mathcal{R}^d)$ the solution u_λ^W of equation (1.33) (which exists and is unique, see details in Section 3) tends (almost surely) as $\lambda \to 0$ to the solution of this equation with $\lambda = 0$.*

Proof. Using the boundedness of all operators on the r.h.s. of (1.33), and (2.4) one obtains that

$$\|u_\lambda^W - u\| \leq \|g_\lambda^W(t,y)u_0 - g_0^W(t,y)u_0\| + O(t)|\log t|\|u_\lambda^W - u\| + o(\lambda)$$
$$= O(t)|\log t|\|u_\lambda^W - u\| + o(\lambda), \quad (2.6)$$

where $o(\lambda)$ depends on u_0 but is uniform with respect to finite times t. It follows that $\|u_\lambda^W - u\| = o(\lambda)$ for small t, which proves the Proposition.

Corollary. *If V is a bounded function, and the assumptions of Proposition 2.2 holds, then the solution to (1.11) can be presented in the form*

$$u(t,y) = \lim_{\lambda \to 0} \int_{PC_y(t)} M_{Leb}^{PC}(dY(.))F_\lambda^W(Y(.))u_0(Y(t)). \quad (2.7)$$

To conclude, let us point out that different kind of interactions can be treated similarly if one changes p- repersentation into occupation number representation. Namely, consider the Schrödinger equation for the anharmonic oscillator

$$i\frac{\partial \psi}{\partial t} = \left(-\frac{\Delta}{2} + \frac{x^2}{2} + V(x)\right)\psi.$$

If V is a smooth function, a path integral representation for its solutions can be obtained as in Section 5. If V is a bounded function, one can also proceed as in Section 3 using the explicit Green function for the quantum harmonic oscillator instead of the "free" Green function. Alternatively, one can use the spectral representation for the harmonic oscillator, i.e. for the operator $\Delta + x^2$. In that approach the corresponding measure on path space will be concentrated on the right continuous piecewise-constant trajectories $q : [0,t] \mapsto \{0, 1, ...\}$ with values in the countable number of eigenstates ψ_j of the harmonic oscillator, and the analog of formula (1.16) will hold, where the integrand will have the form $\exp\{-i\int_0^t E(q(s)\, ds\}$, where $E(n)$ denotes the energy of the state ψ_n. If the infinite dimensional matrix $(V\psi_j, \psi_k)$ defines finite transition probabilities, we find ourselves in the situation, where the corresponding path integral is expressed as an expectation with respect to a generalised Poisson process in the sense of Combe et al [Com1] (see e.g. [BGR] where this is done in a more general infinite-dimensional model describing a particle interacting with a boson reservoir). If not, we can as usual use the regularisation based on (1.24).

3. Path integral for the Schrödinger equation in x-representation

As we mentioned in introduction, we are going to deal here with measures on paths that are concentrated on the set of continuous piecewise linear paths. Denote this set by

CPL. Let $CPL^{x,y}(0,t)$ denote the class of paths $q:[0,t]\mapsto \mathcal{R}^d$ from CPL joining x and y in time t, i.e. such that $q(0)=x$, $q(t)=y$. By $CPL_n^{x,y}(0,t)$ we denote the subclass consisting of all paths from $CPL^{x,y}(0,t)$ that have exactly n jumps of their derivative. Obviously,

$$CPL^{x,y}(0,t)=\cup_{n=0}^{\infty}CPL_n^{x,y}(0,t).$$

Notice also that the set $CPL^{x,y}(0,t)$ belongs to the Cameron-Martin space of curves that have derivatives in $L^2([0,t])$.

To any σ-finite measure M on $\mathcal{R}^d$ there corresponds a unique σ-finite measure M^{CPL} on $CPL^{x,y}(0,t)$, which is the sum of the measures M_n^{CPL} on $CPL_n^{x,y}(0,t)$, where M_0^{CPL} is just the unit measure on the one-point set $CPL_0^{x,y}(0,t)$ and each M_n^{CPL}, $n>0$, is the direct product of the Lebesgue measure on the simplex (1.14) of the jump times $0<s_1<...<s_n<t$ of the derivatives of the paths $q(.)$ and of n copies of the measure M on the values $q(s_j)$ of the paths at these times. In other words, if

$$q(s)=q_{\eta_1...\eta_n}^{s_1...s_n}(s)=\eta_j+(s-s_j)\frac{\eta_{j+1}-\eta_j}{s_{j+1}-s_j},\quad s\in[s_j,s_{j+1}] \tag{3.1}$$

(where $s_0=0,s_{n+1}=t,\eta_0=x,\eta_{n+1}=y$) is a typical path in $CPL_n^{x,y}(0,t)$ and Φ is a functional on $CPL^{x,y}(0,t)$, then

$$\int_{CPL^{x,y}(0,t)}\Phi(q(.))M^{CPL}(dq(.))=\sum_{n=0}^{\infty}\int_{CPL_n^{x,y}(0,t)}\Phi(q(.))M_n^{CPL}(dq(.))$$

$$=\sum_{n=0}^{\infty}\int_{Sim_t^n}ds_1...ds_n\int_{\mathcal{R}^d}...\int_{\mathcal{R}^d}M(d\eta_1)...M(d\eta_n)\Phi(q(.)). \tag{3.2}$$

Remark. Nothing is changed if $CPL_n^{x,y}(0,t)$ is defined as the set of paths with at most n jumps in their derivative. In fact, the M_n^{CPL}-measure of the subset $CPL_{n-1}^{x,y}(0,t)\subset CPL_n^{x,y}(0,t)$ vanishes, because if the jump, say, at the time s_j vanishes then $(\eta_j-\eta_{j-1})(s_{j+1}-s_{j-1})=(\eta_{j+1}-\eta_{j-1})(s_j-s_{j-1})$, therefore s_j can be only one point, and the Lebesgue measure has no atoms.

To express the solutions to the Schrödinger equation in terms of a path integral we shall use the following functionals on $CPL^{x,y}(0,t)$, depending on a given measurable function V on $\mathcal{R}^d$:

$$\Phi_\epsilon(q(.))=\prod_{j=1}^{n+1}(2\pi(s_j-s_{j-1})(i+\epsilon))^{-d/2}$$

$$\times\exp\{-\sum_{j=1}^{n+1}\frac{|\eta_j-\eta_{j-1}|^2}{2(i+\epsilon)(s_j-s_{j-1})}\}\prod_{j=1}^{n}(-(i+\epsilon)V(\eta_j))$$

$$=\prod_{j=1}^{n+1}(2\pi(s_j-s_{j-1})(i+\epsilon))^{-d/2}\prod_{j=1}^{n}(-(i+\epsilon)V(\eta_j))\exp\{-\frac{1}{2(i+\epsilon)}\int_0^t\dot{q}^2(s)\,ds\}, \tag{3.3}$$

and

$$\Phi_\lambda^W(q(.)) = \prod_{j=1}^{n+1}(2\pi(s_j - s_{j-1})(i+\lambda))^{-d/2}$$

$$\times \exp\{-\sum_{j=1}^{n+1}\frac{(\eta_j - \eta_{j-1} - i\sqrt{\frac{\lambda}{2}}(W(s_j) - W(s_{j-1}))^2}{2(i+\lambda)(s_j - s_{j-1})}\}\prod_{j=1}^{n}(-iV(\eta_j)). \tag{3.4}$$

As in Section 2, we shall denote Lebesgue measure on $\mathcal{R}^d$ by M_{Leb}.

Theorem 3.1. *Let V be a bounded measurable function on $\mathcal{R}^d$. Then for arbitrary $\epsilon > 0$ or $\lambda > 0$ and a.a. Wiener trajectories W, there exists a unique solution $G_\epsilon(t, x, x_0)$ or $G_\lambda^W(t, x, x_0)$ to the Cauchy problem of equations (1.25) or (1.30) respectively with Dirac initial data $\delta(x - x_0)$. These solutions (i.e. the Green functions for these equations) are uniformly bounded for all (x, x_0) and t in any compact interval of the open half-line, and they are expressed in terms of path integrals as follows:*

$$G_\epsilon(t, x, x_0) = \int_{CPL^{x,y}(0,t)}\Phi_\epsilon(q(.))M_{Leb}^{CPL}(dq(.)), \tag{3.5}$$

$$G_\lambda^W(t, x, x_0) = \int_{CPL^{x,y}(0,t)}\Phi_\lambda^W(q(.))M_{Leb}^{CPL}(dq(.)), \tag{3.6}$$

with Φ_ϵ and Φ_λ^W given by (3.3), (3.4). For arbitrary $\psi_0 \in L^2(\mathcal{R}^d)$ the solution $\psi_0(t, s)$ of the Cauchy problem for equation (1.10) with the initial data ψ_0 has the form of an improper (not absolutely convergent) path integral that can be understood rigorously as either

$$\psi(t, x) = \lim_{\epsilon\to 0_+}\int_{CPL^{x,y}(0,t)}\int_{\mathcal{R}^d}\psi_0(y)\Phi_\epsilon(q(.))M_{Leb}^{CPL}(dq(.))dy, \tag{3.7}$$

or (almost surely) as

$$\psi(t, x) = \lim_{\lambda\to 0_+}\int_{CPL^{x,y}(0,t)}\int_{\mathcal{R}^d}\psi_0(y)\Phi_\lambda^W(q(.))M_{Leb}^{CPL}(dq(.))dy. \tag{3.8}$$

Proof. Formulas (3.7), (3.8) follow from (3.5), (3.6), (1.24) and Proposition 2.4. The proofs of (3.5), (3.6) are similar and we shall prove only (3.5). To this end, notice that the analogue of series (1.20) for the case of equation (1.25) has form (1.20) with i replaced by $(i + \epsilon)$ everywhere. In particular, for the Green function one has the representation

$$G_\epsilon(t, x, x_0) = G_\epsilon^{free}(t, x, x_0)$$

$$-(i+\epsilon)\int_0^t\int_{\mathcal{R}^d}G_\epsilon^{free}(t-s, x-\eta)V(\eta)G_\epsilon^{free}(s, \eta - x_0)\,d\eta ds + ..., \tag{3.9}$$

where G_ϵ^{free} is the Green function of the "free" equation (1.25) (i.e. with $V = 0$):

$$G_\epsilon^{free}(t, x - x_0) = (2\pi t(i+\epsilon))^{-d/2}\exp\{-\frac{(x-x_0)^2}{2(i+\epsilon)t}\}.$$

To prove (3.5) one needs to prove that the terms of this series are absolutely convergent integrals and the series is absolutely convergent with a bounded sum. This is more or less straightforward. Namely, to prove that the second integral in this series is absolutely convergent, we must estimate the integral

$$\int_0^t \int_{\mathcal{R}^d} |2\pi(i+\epsilon)|^{-d}((t-s)s)^{-d/2}| \exp\{-\frac{(x-\eta)^2}{2(t-s)(i+\epsilon)} - \frac{(\eta-x_0)^2}{2s(i+\epsilon)}\}| \, ds d\eta$$

$$= \int_0^t \int_{\mathcal{R}^d} (2\pi\sqrt{1+\epsilon^2})^{-d}((t-s)s)^{-d/2}$$

$$\times \exp\{-\frac{\epsilon}{2(1+\epsilon^2)} \left[\frac{(x-\eta)^2}{t-s} + \frac{(\eta-x_0)^2}{s}\right]\} \, ds d\eta.$$

This is a Gaussian integral in η which can be explicitly evaluated using standard integrals to be

$$t(2\pi\epsilon t)^{-d/2} \exp\{-\frac{\epsilon(x-x_0)^2}{2(1+\epsilon^2)t}\}.$$

By induction and similar calculations we obtain the estimate for the n-th term of the series (3.9)

$$C(2\pi t)^{-d/2}(t\epsilon^{-d/2})^k/k!$$

for some real number C. This completes the proof of the Theorem.

The measure M^{CPl} in (3.2) may well be not finite, for example M^{CPL}_{Leb} is not finite. But every Hilbert space can be represented as an L^2 over a probability space. For example, the obvious isomorphism of $L^2(\mathcal{R}, dx)$ with $L^2(\mathcal{R}, e^{-x^2/2}dx)$ is very useful in many situation. In the same way, an integral over M^{CPL}_{Leb} can be rewritten as an integtral over the probability space (up to a normalisation) $(e^{-x^2/2}dx)^{CPL}$. Thus one can always rewrite the integral from (3.2) as an expectation of a ceratin stochastic process, which can be taken to be an integral of the compound Poisson process that stands for the path integral formula for the solutions to the Schrödinger equation in momentum representation. More systematic way of obtaining probabilistic interpertations of path integral constructed is discussed later in Section 6.

4. Singular potentials

There exists an extensive literature devoted to the study of the Schrödinger equations with singular potentials, and more precisely with potentials being Radon measures supported by null sets. As most important examples of such null sets one should mention discrete sets (point interaction), smooth surfaces (surface delta interactions), Brownian paths and more general fractals, see e.g. [BF], [ABD], [AFHL], [AHKS], [AntGS], [DaSh], [Kosh], [Metz], [DeO], [Pav] and references therein for different mathematical techniques used for the study of these models and for physical motivations. We are going to show now that the path integral constructed above can be successfully applied to the construction of solutions to these models.

The one-dimensional situation turns out to be particularly simple in our approach, because in this case no regularisation is needed to express the solutions to the corresponding Schrödinger equation and its propagator in terms of path integral.

Proposition 4.1. *Let V be a bounded (complex) measure on $\mathcal{R}$. Then the solution ψ_G to equation (1.25)), where ϵ is any complex number with $\epsilon \neq i$ and non-negative real part, with initial function $\psi_0(x) = \delta(x - x_0)$ (i.e. the propagator or the Green function of (1.25)) exists and is a continuous function of the form*

$$\psi_G(t, x) = (2\pi(i + \epsilon)t)^{-1/2} \exp\{-\frac{|x - x_0|^2}{2t(i + \epsilon)}\} + O(1) \tag{4.1}$$

uniformly for finite times. Moreover, one has the path integral representation for ψ_G of the form

$$\psi_G(t, x) = \int_{CPL^{x,y}(0,t)} \Phi(q(.))V^{CPL}(dq(.)), \tag{4.2}$$

where V^{CPL} is related to V as M^{CPL} is to M in Section 2, and

$$\Phi(q(.)) = \prod_{j=1}^{n+1} (2\pi(s_j - s_{j-1})(i + \epsilon))^{-1/2} \exp\{-\frac{1}{2(i + \epsilon)} \int_0^t \dot{q}^2(s)\, ds\}.$$

Remark. The cases $\epsilon = 0$, i.e. the Schrödinger equation, and $\epsilon = 1 - i$, i.e. the heat equation, are particular cases of the situation considered in the proposition.

Proof. Since V is a finite measure, in order to prove that the terms of series (3.9) with $\epsilon = 0$ (which expresses the Green function) are absolutely convergent, one needs to estimate the integrals

$$\int_0^t ds_1\, (2\pi(t - s_1))^{-1/2} \int_0^{s_1} ds_2\, (2\pi(s_1 - s_2))^{-1/2} ... \int_0^{s_{n-1}} ds_n (2\pi s_n)^{-1/2},$$

which clearly exist (one-dimensional effect!) and can be expressed explicitly in terms of the Euler β-function. One sees directly that the corresponding series is convergent, which completes the proof.

For the Schrödinger equation in finite-dimensional case one needs a regularisation, say (1.25) with $\epsilon > 0$ or (1.27) with $\lambda > 0$. For simplicity we consider here only the regularisation given by (1.25).

Following essentially [AFHK] (see also [Pus], [KZPS]) we shall say that a number $\alpha \geq 0$ is *admissible* for a finite Borel measure V on $\mathcal{R}^d$, if there exists a constant $C = C(\alpha)$ such that

$$V(B_r(x)) \leq Cr^\alpha \tag{4.3}$$

for all $x \in \mathcal{R}^d$ and all $r > 0$. The least upper bound of all admissible numbers for V is called *dimensionality* of V. It will be denoted by $dim(V)$.

Proposition 4.2. *Let V be a finite Borel measure on $\mathcal{R}^d$ with $dim(V) > d-2$. Then for any $\epsilon > 0$ and any bounded initial function $\psi_0 \in L^2(\mathcal{R}^d)$ there exists a unique solution*

$\psi_\epsilon(t,x)$ to the Cauchy problem of equation (1.26) with the initial data $\psi_0(x)$. This solution has the form

$$\psi_\epsilon(t,x)=\int_{CPL^{x,y}(0,t)}\int_{\mathcal{R}^d}\psi_0(y)\Phi_\epsilon(q(.))V^{CPL}(dq(.))dy, \tag{4.4}$$

where

$$\Phi_\epsilon=\prod_{j=1}^{n+1}(2\pi(s_j-s_{j-1})(i+\epsilon))^{-d/2}(-(\epsilon+i))^n\exp\{-\frac{1}{2(i+\epsilon)}\int_0^t\dot{q}^2(s)\,ds\}.$$

Proof. One needs to prove that the terms of the series (1.20), in which i has been replaced by $(i+\epsilon)$, are absolutely convergent integrals and then to estimate the corresponding series. Starting with the first non-trivial term one needs to estimate the integral

$$J=K\int_0^t\int_{\mathcal{R}^{2d}}|2\pi(i+\epsilon)|^{-d}((t-s)s)^{-d/2}$$

$$\times|\exp\{-\frac{(x-\xi)^2}{2(t-s)(i+\epsilon)}-\frac{(\xi-\eta)^2}{2s(i+\epsilon)}\}|\,dsd\eta|V|(d\xi).$$

$$=K\int_0^t\int_{\mathcal{R}^{2d}}(2\pi\sqrt{1+\epsilon^2})^{-d}((t-s)s)^{-d/2}\exp\{-\epsilon\frac{(x-\xi)^2}{2(t-s)(1+\epsilon)^2}\}$$

$$\times\exp\{-\epsilon\frac{(\xi-\eta)^2}{2s(1+\epsilon)^2}\}\,ds\,|V|(d\xi)d\eta,$$

where $K=\sup\{|\psi_0(\eta)|\}$. Integrating over η yields

$$J\le K\int_0^t(2\pi\sqrt{1+\epsilon^2})^{-d/2}(t-s)^{-d/2}\exp\{-\epsilon\frac{(x-\xi)^2}{2(t-s)(1+\epsilon)^2}\}\epsilon^{-d/2}\,ds\,|V|(d\xi).$$

Due to the assumptions of the theorem, there exists $\alpha>d-2$ such that (4.3) holds. Let us decompose this integral into the sum J_1+J_2 of the integrals over the domains D_1 and D_2 with

$$D_1=\{\xi:|x-\xi|\le(t-s)^{-\delta+1/2}\}$$

and D_2 its complement. Choosing $\delta>0$ in such a way that $\alpha(-\delta+1/2)-d/2>-1$ (which is possible due to the assumption on α) we get from (4.3) that

$$J_1\le KC\int_0^t(2\pi\sqrt{1+\epsilon^2})^{-d/2}(t-s)^{\alpha(-\delta+1/2)-d/2}\,ds=$$

$$=KC(1+\alpha(-\delta+1/2)-d/2)^{-1}(2\pi\sqrt{(1+\epsilon^2)})^{-d/2}t^{1+\alpha(-\delta+1/2)-d/2}.$$

In D_2 the integrand is uniformly exponentially small, and therefore using the boundedness of the measure $|V|$ we obtain for J_2 an even better estimate than for J_1. Higher order terms are again estimated by induction giving the required result.

Proposition 4.3. *Assume the assumptions of Proposition 4.2 holds. If the operator $-\Delta/2+V$ is selfadjoin and bounded from below, then one can take the limit as $\epsilon \to 0$ in (4.4) to obtain the solutions to equation (1.10).*

It was proved in [AFHK] that under the assumptions of Proposition 4.2 there exists a constant ω_0 such that the operator $-\Delta/2+\omega|V|$ is selfadjoin and bounded below for all $\omega<\omega_0$. Therefore, for these operators the statement of Proposition 4.3 holds. A concrete example of interest is given by measures on $\mathcal{R}^3$ concentrated on a Brownian path, because their dimensionality equals 2 almost surely. As shown e.g. in [AFHK], potentials being the finite sums of the Dirac measures of closed hypersurfaces in $\mathcal{R}^d$ satisfy the assumptions of the above corollary (without an assumption of a small coupling constant), see also [Koch2]. For the particular case of mesaures on spheres of codimension 1, see [AntGS], [DaSh], where one can find the references on physical papers dealing with these models.

Less exotic examples of potentials satisfying the asumptions of Proposition 4.3 are given by measures with a density $V(x)$ having a bounded support and such that $V\in L^p(\mathcal{R}^d)$ with $p>d/2$ (which one checks by the Hölder inequality). Moreover, it is not difficult to check that one can combine the potentials from Proposition 4.3 and Theorem 3.1, for example, one can take $V\in L^\infty(\mathcal{R}^d)+L^p(\mathcal{R}^d)$ with $p>d/2$, which includes, in particular, the Coulomb potential in $d=3$.

Notice for conclusion that solutions to the Schrödinger equation with a certain class of singular potentials can be obtained in terms of the Feynman integral defined as a generalised functional in Hida's white noise space [HKPS].

5. Semiclassical asymptotics

In this section we answer the following question: how to define a measure on a path space in such a way that the solutions to the Schrödinger or heat equation (deterministic or stochastic) can be expressed as integrals with respect to this measure of the exponential of the classical action. We start with the heat equation, where no regularisation is needed.

Consider the equation (3.1.1). We are going to interprete the formula for its Green function given in Proposition 3.4.1 of Chapter 3, under the assumptions of regularity of the corresponding diffusion (see Chapter 3), in terms of a path integral of type (1.15). Namely, as in Section 3, let us introduce the set CPC of continuous piecewise classical paths, i.e. continuous paths that are smooth and satisfy the classical equations of motion

$$\dot{x}=p, \quad \dot{p}=\frac{\partial V}{\partial x}, \tag{5.1}$$

where H is the Hamiltonian corresponding to equation (3.1.1) (see Chapter 3), except at a finite number of points, where their derivatives may have discontinuities of the first kind. Let $CPC^{x,y}(0,t)$ denote the class of paths $q:[0,t]\mapsto\mathcal{R}^d$ from CPC joining y and x in time t, i.e. such that $q(0)=y$, $q(t)=x$. We denote by $CPC_n^{x,y}(0,t)$ the subclass consisting of all paths from $CPC^{x,y}(0,t)$ that have exactly n jumps in their derivative. Obviously, to any σ-finite measure M on $\mathcal{R}^d$ there corresponds a unique σ-finite measure M^{CPC} on $CPC^{x,y}(0,t)$, which is the sum of the measures M_n^{CPC} on $CPC_n^{x,y}(0,t)$, where M_0^{CPC} is just the unit measure on the one-point set $CPC_0^{x,y}(0,t)$ and for $n>0$ M_n^{CPC}

is the direct product of the Lebesgue measure on the simplex (1.14) of the jump times $0 < s_1 < ... < s_n < t$ of the derivatives of the paths $q(.)$ and of n copies of the measure M on the values $q(s_j)$ of the paths at these times. In other words, if

$$q^s_\eta = q^{s_1 \dots s_n}_{\eta_1 \dots \eta_n}(s) \in CPC^{x,y}_n(0,t) \tag{5.2}$$

denotes the path that takes values η_j at the times s_j, is smooth and classical between these time-points, and Φ is a functional on $CPL^{x,y}(0,t)$, then

$$\int_{CPC^{x,y}(0,t)} \Phi(q(.)) M^{CPC}(dq(.)) = \sum_{n=0}^\infty \int_{CPC^{x,y}_n(0,t)} \Phi(q(.)) M^{CPC}_n(dq(.))$$

$$= \sum_{n=0}^\infty \int_{Sim^n_t} ds_1 ... ds_n \int_{\mathcal{R}^d} ... \int_{\mathcal{R}^d} M(d\eta_1)...M(d\eta_n)\Phi(q(.)). \tag{5.3}$$

We will use this construction only for the case when the measure M is the standard Lebesgue measure M_{Leb}. Clearly, from Proposition 3.4.3 one obtains directly the following result.

Proposition 5.1. *Under the assumptions of Proposition 3.4.3*

$$u_G(t,x,y) = \int_{CPC^{x,y}(0,t)} \exp\{-\frac{1}{h} I(q(.))\} \Phi_{sem}(q(.)) M^{CPC}_{Leb}(dq(.)), \tag{5.4}$$

where $I(y(.)) = \int_0^t L(y(s), \dot{y}(s)\, ds$ *denote the classical action on a path* $y(.)$. *The explicit formula for the "semiclassical density"* Φ_{sem} *follows from Proposition 3.4.3. For example, in the simplest case of equation*

$$h \frac{\partial u}{\partial t} = \frac{h^2}{2} \Delta u - V(x) u, \quad x \in \mathcal{R}^d, \quad t > 0, \tag{5.5}$$

one has

$$\Phi_{sem}(q^s_\eta) = (2\pi h)^{-dn/2} J(t - s_n, x, \eta_n)^{-1/2} \prod_{k=1}^n \Delta(J(s_k - s_{k-1}, \eta_k, \eta_{k-1})^{-1/2}) h^n \tag{5.6}$$

on a typical path (5.2), where $J(t,x,x_0)$ *is the Jacobian, given by*

$$J(t,x,\xi) = \det \frac{\partial X}{\partial p}(t, \xi, p_0(t,x,\xi)),$$

on the classical path joining x_0 *and* x *in time* t *and where* $s_0 = 0$, $\eta_0 = y$.

Hence the Green function is expressed as the integral of the exponential of the classical action functional $-\frac{1}{h} I(q(.))\}$ over the measure $\Phi_{sem} M^{CPC}_{Leb}$, which is actually a measure on the Cameron-Martin space of paths that are absolutely continuous and have their derivatives in L_2.

A similar result can be obtained for the stochastic heat equation and for the stochastic Schrödinger equations of Theorems 7.2.2, 7.2.3.

Remark. One sees directly from the path integral (5.4) that the main contribution to the asymptotics as $h \to 0$ of (5.4) comes from a small neighborhood of the classical path joining x and x_0 in time t, since a stationary point of the classical action defined on piecewise classical paths is clearly given by a strictly smooth classical path.

As usual, the case of the Schrödinger equation requires some regularisation. We shall use here the regularisation given by the Schrödinger equation in complex time (1.25), exploiting the semiclassical asymptotics for this equation constructed in Chapter 7. This will give us the required result more or less straightforwardly. However, to use this regularisation one needs rather strong assumption on the potential, namely local analyticity.

Notice that equation (1.25) can be considered as a particular case of equation (7.1.1.) with $\alpha = 0$. Therefore, Theorem 7.2.2 implies straightforwardly the following result.

Proposition 5.2. *Let V be analytic in the strip $St_b = \{x = y + iz : |z| < b\}$ and suppose that all its second and higher order derivatives are uniformly bounded in this strip. Let $S_\epsilon(t, x, x_0)$ be the two-point function for the complex Hamiltoian equation*

$$\frac{\partial S}{\partial t} + (1 - i\epsilon)\left[\frac{1}{2}(\frac{\partial S}{\partial x})^2 + V(x)\right] = 0 \tag{5.6}$$

with the Hamiltonian $H_\epsilon = (1 - \epsilon i)(p^2/2 + V(x))$. Let χ_R be a smooth bounded function on the positive halfline such that $\chi(s)$ vanishes for $s > R$ and $\chi_R(s) = 1$ for $s \leq R - \delta$ for some $\delta > 0$. Then there exists t_0 such that for any R there exists ϵ_0 and $c > 0$ such that for all $t \leq t_0$, $\epsilon \leq \epsilon_0$ and $x, x_0 \in St_{b/2}$ with $|x - x_0| \leq R$, there exists a unique trajectory of the Hamiltonian system with Hamiltonian H_ϵ which joins x_0 and x in time t, lies completely in St_b, and has the initial momentum $p_0 : |p_0| \leq c/t$. Moreover, the Green function for equation (1.25) exists, and for $t \leq t_0$, $\epsilon \leq \epsilon_0$ it can be represented as the absolutely convergent series

$$u_G^\epsilon = u_{as}^\epsilon + h u_{as}^\epsilon \otimes F + h^2 u_{as}^\epsilon \otimes F \otimes F + ..., \tag{5.7}$$

with

$$u_{as}^\epsilon = (2\pi h i)^{-m/2} \chi_R(|x - x_0|) J_\epsilon^{-1/2}(t, x, x_0) \exp\{\frac{i}{h} S_\epsilon(t, x, x_0)\}. \tag{5.8}$$

and $F = \tilde{F} \exp\{\frac{i}{h} S_\epsilon(t, x, x_0)\}$, where $\tilde{F}(t, x, x_0)$ equals

$$[O(t^2) + O((th)^{-1})\Theta_{R-\epsilon,R}(|x - x_0|)](2\pi h i)^{-d/2} \chi_R(|x - x_0|) J_\epsilon^{-1/2}(t, x, x_0)$$

(the exact form of F can be found in Chapter 7).

The series (5.7) can again be interpreted as a path integral. Namely, as in the case of a real diffusion considered above we obtain the following.

Proposition 5.3. *The Green function (5.7) can be written in the form*

$$u_G^\epsilon(t, x, y) = \int_{CPC^{x,y}(0,t)} \exp\{\frac{i}{h} I_\epsilon(q(.))\} \Phi_{sem}^\epsilon(q(.)) M_{Leb}^{CPC}(dq(.)) \tag{5.9}$$

where the path space CPC is defined by the real Hamiltonian system with the Hamiltonian H_0, but the action and the Jacobian are defined with respect to the complex trajectoties (solutions to the Hamiltonian system with the Hamiltonian H_ϵ) with the same end points as the corresponding real trajectories, and where

$$\Phi^\epsilon_{sem}(q^s_\eta) = (2\pi hi)^{-dn/2} J_\epsilon^{-1/2}(t - s_n, x, \eta_n)$$

$$\times \chi_R(|x - \eta_n|) h^n \prod_{k=1}^n \tilde{F}_\epsilon(s_k - s_{k-1}, \eta_k, \eta_{k-1}). \tag{5.10}$$

Remark. The path space in (5.9) can also be defined as the set $CPC^{x,y}_\epsilon(0,t)$ of paths that are piecewise classical for the Hamiltonian H_ϵ. This seems to be more appropriate when considering equation (1.25) on its own, but we have defined this measure differently in order to have the same measure for all $\epsilon > 0$.

For $\epsilon = 0$ the integral in (5.9) is no longer absolutely convergent. However, under the assumptions of Proposition 5.2 the operator $-\Delta + V$ is selfadjoint (if V is real for real x). Therefore from (1.24) we obtain the following result.

Proposition 5.4. *Under the assumptions of Proposition 5.2, if V is real for real x, one has the following path integral representation for the solutions $\psi(t,x)$ of the Cauchy problem for equation (1.10) with the initial data $\psi_0 \in L^2(\mathcal{R}^d)$:*

$$\psi(t,x) = \lim_{\epsilon \to 0} \int_{CPC^{x,y}(0,t)} \exp\{\frac{i}{h} I_\epsilon(q(.))\} \Phi^\epsilon_{sem}(q(.)) \psi_0(y) M^{CPC}_{Leb}(dq(.)) dy.$$

6. Fock space representation

The paths of the spaces CRC and CPL used above are parametrised by finite sequences $(s_1, x_1), ..., (s_n, x_n)$ with $s_1 < ... < s_n$ and $x_j \in \mathcal{R}^d$, $j = 1, ..., d$. Denote by $\mathcal{P}^d$ the set of all these sequences and by $\mathcal{P}^d_n$ its subset consisting of sequences of the length n. Thus, functionals on the path spaces CPC or CPL can be considered as functions on $\mathcal{P}^d$. To each measure ν on $\mathcal{R}^d$ there corresponds a measure $\nu_\mathcal{P}$ on $\mathcal{P}^d$ which is the sum of the measures ν^n on $\mathcal{P}^d_n$, where ν^n are the product measures $ds_1...ds_n d\nu(x_1)...d\nu(x_n)$. The Hilbert space $L^2(\mathcal{P}^d, \nu_\mathcal{P})$ is known to be isomorphic to the Fock space Γ^d_ν over the Hilbert space $L^2(\mathcal{R}_+ \times \mathcal{R}^d, dx \times \nu)$ (which is isomorphic to the space of square integrable functions on $\mathcal{R}_+$ with values in $L^2(\mathcal{R}^d, \nu)$). Therefore, square integrable functionals on CPL can be considered as vectors in the Fock space $\Gamma^d_{V(dx)}$. It is well known that the Wiener, Poisson, general Lévy and many other interesting processes can be naturally realised in a Fock space: the corresponding probability space is defined as the spectrum of a commutative von Neumann algebra of bounded linear operators in this space. For example, the isomorphism between $\Gamma^0 = \Gamma(L^2(\mathcal{R}_+))$ and $L^2(W)$, where W is the Wiener space of continuous real functions on halfline is given by the Wiener chaos decomposition, and a construction of a Lévy process with the Lévy measure ν in the Fock space Γ_ν can be found in [Par] or [Mey]. Therefore, using Fock space representation, one can give different stochastic representations for path integrals over CPL or CPC rewriting them as expectations with respect to different stochastic processes.

For example, let us express the solution to the Cauchy problem of equation (1.26) in terms of an expectation with respect to a compound Poisson process. The following statement is a direct consequence of Proposition 4.2 and the standard properties of Poisson processes.

Proposition 6.1 *Suppose a measure V satisfies the assumptions of Proposition 4.2. Let $\lambda_V = V(\mathcal{R}^d)$. Let paths of CPL are parametrised by (3.1) and let E denote the expectation with respect to the process of jumps η_j which are identicaly independently distributed according to the probability measure V/λ_V and which occur at times s_j from $[0,t]$ that are distributed according to Poisson process of intensity λ_V. Then the function (4.4) can be written in the form*

$$\psi_\epsilon(t,x) = e^{t\lambda_V} \int_{\mathcal{R}^d} \psi_0(y) E(\Phi_\epsilon(q(.)))\, dy. \tag{6.1}$$

As an example of the repersentation of path integral in terms of the Wiener measure, let us consider the Green function (3.5). Let us first rewrite it as the integral of an element of the Fock space $\Gamma^0 = L^2(Sim_t)$ with $Sim_t = \cup_{n=0}^\infty Sim_t^n$ (which was denoted $\mathcal{P}^0$ above), where Sim_t^n is as usual the simplex (1.14). Let $g_0^V = G_\epsilon^{free}$ (see (3.9)) and let

$$g_n^V(s_1,...,s_n) = \int_{\mathcal{R}^{nd}} \Phi_\epsilon(q_{\eta_1...\eta_n}^{s_1...s_n})\, d\eta_1...d\eta_n$$

for $n = 1,2,...$, where Φ_ϵ and $q_{\eta_1...\eta_n}^{s_1...s_n}$ are given by (3.3) and (3.1). Considering the series of functions $\{g_n^V\}$ as a single function g^V on Sim_t we shall rewrite the r.h.s. of (3.5) in the following concise notation:

$$\int_{Sim_t} g^V(s)\, ds = \sum_{n=0}^\infty \int_{Sim_t^n} g_n^V(s_1,...,s_n)\, ds_1...ds_n. \tag{6.2}$$

Now, the Wiener chaos decomposition theorem states (see e.g. [Mey]) that, if $dW_{s_1}...dW_{s_n}$ denotes the n-dimensional stochastic Wiener differential, then to each $f = \{f_n\} \in L^2(Sim_t)$ there corresponds an element $\phi_f \in L^2(\Omega_t)$, where Ω_t is the Wiener space of continuous real functions on $[0,t]$, given by the formula

$$\phi_f(W) = \sum_{n=0}^\infty \int_{Sim_t^n} f_n(s_1,...,s_n)\, dW_{s_1}...dW_{s_n}, \tag{6.3}$$

or in concise notations

$$\phi_f(W) = \int_{Sim_t} f(s)\, dW_s.$$

Moreover the mapping $f \mapsto \phi_f$ is an isometric isomorphism, i.e.

$$E_W(\phi_f(W)\bar{\phi}_g(W)) = \int_{Sim_t} f(s)\bar{g}(s)\, ds, \tag{6.4}$$

where E_W denotes the expectation with respect to the standard Wiener process. One easily sees that under the assumptions of Theorem 3.1 the function g^V belongs not only to $L^1(Sim_t)$ (as shown in the proof of Theorem 3.1) but also to $L^2(Sim_t)$. Therefore, the function ϕ_{g^V} is well defined. Since (see e.g. again [Mey])

$$\int_{Sim_t} dW_s = e^{W(t)-t/2},$$

formula (6.4) implies the following result.

Proposition 6.2. *Under the assumptions of Theorem 3.1 the Green function (3.5) can be written in the form*

$$G_\epsilon(t,x,x_0) = E_W(\phi_{g^V} \exp\{W(t)-t/2\}), \tag{6.5}$$

where E_W denotes the expectation with respect to the standard Wiener process.

It is not difficult to see that in order to similarly rewrite formula (4.4) in terms of the Wiener integral one needs a stronger assumption on the measure V than in Proposition 4.2: namely, one needs to assume that $dim(V) > d-1$. For general V from Propositions 4.1 or 4.2, the corresponding function g^V may well belong to $L^1(Sim_t)$, but not to $L^2(Sim_t)$. In that case, formula (6.5) should be modified. Consider, for example the case of one-dimensional heat or Schrödinger equation with point interactions. Namely, consider the (formal) complex diffusion equation

$$\frac{\partial \psi}{\partial t} = G(\Delta/2 - \sum_{j=1}^m a_j \delta_{x_j}(x))\psi, \quad x \in \mathcal{R}, \tag{6.6}$$

where $a_1,...,a_m$ are positive real numbers, $x_1,...,x_m$ are some points on the real line, and G is a complex number with a non-negative real part. This is an equation of the type considered in Section 9.4. The path integral representation for its heat kernel from Proposition 4.1 can be written in the form

$$\psi_G^\delta(t,x,x_0) = \sum_{n=0}^\infty \int_{Sim_t^n} g_n^\delta(s_1,...,s_n)\, ds_1...ds_n = \int_{Sim_t} g^\delta(s)\, ds, \tag{6.7}$$

where

$$g_0^\delta = \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{(x-x_0)^2}{2t}\} \tag{6.8}$$

and

$$g_n^\delta(s_1,...,s_n) = \prod_{k=1}^{n+1} \frac{1}{\sqrt{2\pi(s_k - s_{k-1})}} \sum_{j_1=1}^m \cdots \sum_{j_n=1}^m$$

$$\exp\{-\frac{(x-x_{j_n})^2}{2G(t-s_n)} - \frac{(x_{j_n}-x_{j_{n-1}})^2}{2G(s_n-s_{n-1})} - ... - \frac{(x_{j_1}-x_0)^2}{2Gs_1}\} \prod_{k=1}^n a_{j_k} \tag{6.9}$$

for $n \geq 1$, where it is assumed that $s_0 = 0$ and $s_{n+1} = t$.

Formula (6.7) has a clear probabilistic interpretation in the spirit of Proposition 6.1: it is an expectation with respect to the measure of the standard Poisson process of jump-times $s_1, ..., s_n$, of the sum of the exponentials of the classical actions $(2G)^{-1} \int \dot{q}^2(s)\, ds$ of all (essential) paths joining x_0 and x in time t each taken with a weight which corresponds to the weights of singularities x_j of the potential. On the other hand, since the function g^δ from (6.7) is not an element of $L^2(Sim_t)$, but only of $L^1(Sim_t)$, formula (6.5) does not hold.

One way to rewrite (6.7) in terms of an integral over the Wiener measure is by factorising g^δ in a product of two functions from $L^2(Sim_t)$. For example, in the case of the heat equation, i.e. when $G = 1$ in (6.6), the function g^δ is positive, which implies the following statement.

Proposition 6.3. *The Green function of equation (6.6) with $G = 1$ can be written in the form*

$$\psi_G^\delta(t, x, x_0) = E_W \left(\int_{Sim_t} \sqrt{g^\delta(s)}\, dW_s \right)^2 .$$

It is also possible to write a regularised version of (6.5). Namely, let $g^{\delta,\alpha}$ with $\alpha < 2$ is defined by (6.8),(6.9) where instead of the multipliers $\sqrt{t}$ and $\sqrt{s_j - s_{j-1}}$ one plugs in the multipliers t^α and $(s_j - s_{j-1})^\alpha$ respectively. If $\alpha \in (0, 1/2)$, the corresponding $g^{\delta,\alpha}$ belongs to $L^2(Sim_t)$ and therefore $\phi_{g^{\delta,\alpha}}$ is well defined and belongs to $L^2(W)$. This implies the following result.

Proposition 6.4. *The Green function ψ^δ of equation (6.6) has the form*:

$$\psi_G^\delta(t, x, x_0) = \lim_{\alpha \to 1/2} E_W \left(\exp\{W(t) - t/2\} \int_{Sim_t} g^{\delta,\alpha}(s)\, dW_s \right) .$$

Similar representations in terms of Poisson or Wiener processes can be given for the path integral over the path space CPL from Section 9.5. Let us notice also that Theorem 3.1 and Propositions 6.1, 6.2 can be easily modified to include the case of the Schrödinger equation for a quantum particle in a magnetic field with a bounded vector-potential.