Problems for 'Mathematical methods in finance'.

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1 One period models: utility and risk, portfolios and CAPM, factor models

1. **St. Petersburg paradox**. In the 18th century D. Bernoulli suggested to St. Petersburg Academy the following game. A fair coin is tossed one time after another. When the first Head appears, on a certain step k, you receive the payoff 2^k. Expectation of your gain equals

$$\frac{1}{2} \times 2 + \frac{1}{2^2} \times 2^2 + ... = 1 + 1 + ... = \infty.$$

However, nobody would agree that a fair price for a privilege to play this game should be infinite, or even very large? Assume your utility function is logarithmic: $u(x) = \log_2(x)$. Calculate the expected utility of the St. Petersburg game.

Answer: 2.

2. **Dominance and utility**. Let A, B be r.v. with values in $[a,b]$ and F_A, F_B their distribution functions. (a) Show that

$$F_A(x) \leq F_B(x) \quad \forall x \in [a,b]$$

(non strict first order dominance) iff $\mathbf{E}u(A) \geq \mathbf{E}u(B)$ for all utility functions u with a non-negative u'.

Hint: use integration by parts.

(b) Show that

$$\int_a^x F_A(y)\,dy \leq \int_a^x F_B(y)\,dy \quad \forall x \in [a,b] \tag{1}$$

(non strict second order dominance) iff $\mathbf{E}u(A) \geq \mathbf{E}u(B)$ for all utility functions u on $[a,b]$ such that $u' \geq 0$ and $u'' \leq 0$.

3. **Second order dominance and stop loss**. Show that (1) holds (i) iff $\mathbf{E}(B-d)^- \geq \mathbf{E}(A-d)^-$ for all $d \in [a,b]$ or (ii) iff $\mathbf{E}\min(d,B) \leq \mathbf{E}\min(d,A)$ for all $d \in [a,b]$.

4. **Utility and ARA**. (a) Show that there exists a unique utility function u on $\mathbf{R}$ with a given ARA $\rho(x)$ and such that $u(0) = 0, u'(0) = 1$. Express u in terms of ρ.

(b) Let u_i, $i = 1,2$, be two utility functions with ARA $\rho_i(x)$ and such that $u_i(0) = 0, u_i'(0) = 1$. Show that if $\rho_1(x) \geq \rho_2(x)$ for $|x| \leq \delta$, then $u_1(x) \leq u_2(x)$ for all these x. Similarly, if $\rho_1(x) > \rho_2(x)$ for $|x| \leq \delta$, then $u_1(x) < u_2(x)$ for $x \in (-\delta,0) \cup (0,\delta)$.

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(c) Recall that an agent with utility function u accepts a gamble (or a simple lottery) at wealth x iff $\mathbf{E}u(x+g) > u(x)$. Show that if $\rho_1(x) > \rho_2(x)$ and utilities u_1, u_2 are concave ($u_i'' < 0$) and monotone ($u_i' > 0$), then there exists a gamble g that agent 2 accepts at wealth x, and agent 1 rejects at x.

Solution: It is enough to consider the case $x = 0$. So assume $u_i(0) = 0, u_i'(0) = 1$, $i = 1, 2$, and $\rho_1(0) > \rho_2(0)$. Hence $\rho_1(x) > \rho_2(x)$ for all $x \in (-3\delta, 3\delta)$ with some $\delta > 0$, and consequently (by (b)) $u_1(x) < u_2(x)$ for $x \in (-3\delta, 0) \cup (0, 3\delta)$. Therefore, the functions

$$f_i(x) = \frac{1}{2}(u_i(x-\delta) + u_i(x+\delta)), \quad i = 1, 2,$$

satisfy

$$f_1(x) < f_2(x), \quad x \in [0, \delta]. \tag{2}$$

By concavity of u, $f_1(0) < f_2(0) \leq u_2(0) = 0$, and by monotonicity, $f_1(\delta) = u_1(\delta)/2 > u_1(0)/2 = 0$. Hence $f_1(y) = 0$ at some $y \in (0, \delta)$. By (2), $f_2(y) > 0$, implying that $f_2(y - \eta) > 0 > f_1(y - \eta)$ for small enough η. Consequently, if g is a gamble yielding $-\delta + y - \eta$ or $\delta + y - \eta$ with equal chances, then

$$\mathbf{E}u_2(g) = f_2(y - \eta) > 0 > f_1(y - \eta) = \mathbf{E}u_1(g),$$

and agent 2 accepts g while agent 1 rejects it.

(d) An agent 1 is called *no less risk-averse* than 2 if, at any given wealth, 2 accepts any gamble that 1 accepts. Assume both agents have concave strictly increasing utilities. Deduce from (b) and (c) that 1 is no less risk-averse than 2 if and only if $\rho_1(x) \geq \rho_2(x)$ for all x.

5. **Certainty equivalent and insurance premium in terms of ARA.** An insurance premium π_i that an agent with utility u and ARA ρ is ready to pay for avoiding a gamble g with the expectation $\mathbf{E}g$ and variance σ^2 can be defined from the equation $\mathbf{E}u(g) = u(\mathbf{E}g - \pi_i)$. Prove the approximate formula

$$\pi_i = -\frac{1}{2}\sigma^2 \rho(\mathbf{E}g),$$

which is valid for small π_i and small higher moments of $g - \mathbf{E}g$.

Hint: Use Taylor's expansion for u around the value $\mathbf{E}g$.

6. **Basic utilities**.

(a) For a general quadratic utility function $u(x) = a + bx + cx^2$ calculate ARA, RRA. Find the domains of non-satiation and risk-aversion. Show that if a utility u is quadratic, then $\mathbf{E}u(g)$ is a quadratic function of the expectation and the standard deviation of g.

(b) A utility is called CARA (constant ARA) if its ARA is a constant. Find all CARA utility functions. Check that for a CARA utility, the acceptance of a gamble g does not depend on an initial capital, that is the validity of the inequality $\mathbf{E}u(x+g) > u(x)$ does not depend on a constant x.

(c) A utility is called CRRA (constant RRA) if its RRA is a constant. Find all CRRA utility functions on $\mathbf{R}_+$. Which of them are increasing, and which of them are risk averse?

(d) A utility is called HARA (hyperbolic ARA), if its ARA is inverse to linear, that is of the form $1/(ax+b)$ with constants a, b. Find all HARA utilities. Which of them are increasing, and which of them are risk averse?

Hint. Condition HARA yields $\log(u') = \alpha \log(ax + b) + c$.

7. **Finding utility**. Suppose $X = \{a, b, c\}$ and $\leq$ is a preference order on simple lotteries (probability measures) on X. Suppose that

$$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \sim \left(\frac{1}{2}, 0, \frac{1}{2}\right)$$

and

$$\left(\frac{1}{2}, \frac{1}{2}, 0\right) \leq \left(0, \frac{1}{2}, \frac{1}{2}\right).$$

(i) Find a utility function U corresponding to $\leq$.
(ii) Find out whether

$$\left(\frac{2}{3}, \frac{1}{3}, 0\right) \leq \left(0, \frac{1}{3}, \frac{2}{s}\right)?$$

Answer. (i) $(u(a), u(b), u(c)) = (\lambda, \mu, \lambda + 3\mu)$ with $\mu > 0$ (u is defined up to two constants λ, μ). (ii) Yes.

8. **Investment wheel.** An investment wheel has m sectors with probabilities p_i and returns r_i, $i = 1, ..., m$, so that at each turn only one sector wins and the capital put on this sector is multiplied by the corresponding return (the capital put on other sectors being lost). A strategy is a collection of non-negative numbers $\alpha = (\alpha_1, ..., \alpha_m)$ with $\sum_i \alpha_i = 1$ so that for any initial capital V, $\alpha_i V$ specifies the amount put on the sector i. Let V_1 denote a random result of this investment and let $R(\alpha)$ denote the corresponding return: $R = V_1/V$. (i) Prove that if utility is logarithmic $u(R) = \log R$, the optimal strategy is $\alpha_i = p_i$. (ii) Use the law of large numbers to describe the behavior of the return capital V_n/V obtained after n turns, if this strategy is applied for a large number of turns? (iii) What is the maximum of $u(R)$ if all p_i are equal and $r_i = 2^i$?

Hint: (i)

$$\mathbf{E} \log R(\alpha) = \sum_{i=1}^{m} p_i \log(\alpha_i r_i).$$

Hence, we are looking for stationary points of the Lagrange function

$$L = \sum_{i=1}^{m} p_i \log(\alpha_i r_i) - \lambda \sum_{i=1}^{m} \alpha_i,$$

yielding the equations $p_i/\alpha_i = \lambda$. Summing up yields $\lambda = 1$ and hence $p_i = \alpha_i$.

(ii) For this λ

$$\mathbf{E} \log R = M = \sum_{i=1}^{m} p_i \log(p_i r_i)$$

so that

$$\lim_{n\to\infty} (V_n/V)^{1/n} = e^M$$

(justify this statement taking log of both sides and using the law of large numbers!).

$$(iii) \quad M = \frac{1}{m} \sum_{i=1}^{m} (\log 2^i - \log m) = \frac{1}{2}(m + 1) \log 2 - \log m.$$

9. **Measures of risk.**

An investor is contemplating an investment with a return of $\$R$ where $R = 30 - 50U$, where U is $[0,1]$-uniform.

Calculate each of the following four measures of risk:

(i) variance σ^2, (ii) downside semi-variance; (iii) shortfall probability with respect to shortfall level \$10; (iv) VaR at the level 5%.

Answers: (i) $\sigma^2(R) = 625/3$; (ii) 625/6; (iii) 3/5; (iv) 17.5.

10. **VaR for normals.** $\mathbf{P}(X \le -VaR(X)) = p$. If X is $N(\mu, \sigma^2)$, express VaR in terms of the standard normal distribution function Φ. Hint: $X = \mu + \sigma Z$ with Z being $N(0,1)$. Hence $-(\mu + Var(X))/\sigma = \Phi^{-1}(p)$ or $Var(X) = -\mu - \sigma\Phi^{-1}(p)$.

11. **Investing in a 'bad' security is optimal**! Average return from two securities are 1 and 0.9 resp. and covariance matrix is

$$\begin{pmatrix} 0.1 & -0.1 \\ -0.1 & 0.15 \end{pmatrix}$$

Find the min-variance portfolio for the expected return not less than 0.96 when borrowing (short selling) is not allowed. Show, in particular, that though the second security is by far worse than the first one, it is still optimal to invest in it (the effect is due to negative correlations).

Answer: optimal weight of the first security is $x^* = 0.6$.

12. **Capital market line.** Let the expected return and standard deviation of the efficient market portfolio are $\bar{r}_M = 23\%$, $\sigma_M = 32\%$ and the risk free rate is $r_f = 7\%$.

(i) Write down the equation for the capital market line. Sketch the graph of the line in the (expected) rate of return - standard deviation plane. Show the regions, where the investors lend the money and where they borrow them.

(ii) If an expected return 39% is desired, which is the standard deviation for the corresponding position on the capital market line? How to allocate \$200 in order to achieve this position?

(iii) What is the expectation of the amount of money you would get when investing \$30 in the risk-free asset and \$70 in the market portfolio?

(iv) If an investor has the exponential utility $u_\alpha(x) = -e^{-\alpha x}$ and if r_M is normally distributed, which return $\bar{r}$ will be chosen? Describe the levels α, for which the corresponding investors lend money at the risk-free rates.

Answer: (i) $\bar{r} = 0.07 + 0.5\sigma$. (ii) $\sigma = 64\%$. Borrow \$200 and invest \$400 in the market. (iii) 118.2. (iv) $r^* = 0.07 + 0.25\alpha^{-1}$. Lend those with $\alpha^{-1} \le 0.64$.

13. **Small world.** $\exists$ two risky assets A, B and a risk free asset F. Two risky assets are in equal supply in the market $M = (A+B)/2$. It is known that $r_f = 10\%$, $\sigma_A^2 = 0.04$, $\sigma_B^2 = 0.02$, $\sigma_{AB} = 0.01$, and the expectation of the market rate of return $\bar{r}_M = 0.18$. (i) Find the general expression for $\sigma_M^2, \beta_A, \beta_B$, and, assuming CAPM, also $\bar{r}_A, \bar{r}_B$. (ii) Calculate $\bar{r}_A, \bar{r}_B$ and the market price of risk.

Partial answer: $\bar{r}_A = 0.15$, $\bar{r}_B = 0.70$. The market price of risk is $(\bar{r}_M - r_f)/\sigma_M$.

14. **Simple land.** $\exists$ two risky assets A, B with the number of shares outstanding 100 and 150, price per share \$1.50 and \$2.00, expected rate of return $r_A = 15\%$, $r_B = 12\%$ and standard deviations of return $\sigma_A = 15\%$, $\sigma_B = 9\%$. Moreover, the correlation is $\rho_{AB} = 1/3$, there exists a risk-free asset and CAPM is satisfied.

(i) What the expected return $\bar{r}_M$ and the standard deviation σ_M of the market portfolio? (ii) Calculate β_A, β_B. (iii) Calculate the risk-free rate r_f.

Partial answers:

$$\beta_A = \frac{35}{27}, \quad \beta_B = \frac{23}{27}, \quad r_f = 0.0625.$$

15. **Arbitrage pricing for linear single factor models**. Suppose return on some class of assets is given by

$$r_i = a_i + b_i f, \quad i = 1, ..., m,$$

where a_i, b_i are constants and f a random variable (single factor). Assume all b_i are different. Show by the arbitrage argument that a_i should be given by certain fixed (independent of i) linear functions of b_i.

Solution: A portfolio mixing two assets i and j yields a return

$$r = wa_i + (1-w)a_j + [wb_i + (1-w)b_j]f.$$

Choosing $w = b_j/(b_j - b_i)$, makes the coefficient at f zero and the corresponding asset risk free. Hence its return should be the same number r_f for all pairs (and equal to the underlying basic rate of return of a risk free asset, if this is available), i.e.

$$\frac{a_i b_j}{b_j - b_i} + \frac{a_j b_i}{b_i - b_j} = r_f.$$

Rearranging yields

$$\frac{a_j - r_f}{b_j} = \frac{a_i - r_f}{b_i}.$$

As this ratio should be the same or all i, it is a constant, say c. Consequently,

$$a_i = r_f + b_i c, \quad i = 1, ..., m.$$

16. **CAPM and index models.** A market consists of three securities A, B and C with capitalizations of \$22 bn, \$33 bn, \$22 bn. Their standard deviations of annual return are 40%, 20%, 10% respectively, and the correlation between each pair is 0.5. The expected rate of return of the market portfolio is known to be $\bar{r}_M = 22.86\%$ p.a., and the risk-free rate of return is $r_f = 3.077\% p.a.$

(a) Assuming CAPM, find the expected return rates for A, B, C.

(b) Derive a single index model (with index equal to r_M, the random return on the market portfolio) with the same expected return and variances as in the CAPM. Calculate the variances of all random variables in this model. Emphasize the difference between CAPM and the index model.

Answer:

$$\bar{r}_A = 0.4, \quad \bar{r}_B = 0.2, \quad \bar{r}_C = 0.1.$$

(b) The single index model is

$$r_i = r_f + \beta_i(r_M - r_f) + \epsilon_i,$$

where ϵ_i are uncorrelated with each other and with r_M. Hence

$$\sigma_i^2 = Var(\epsilon_i) = Var(r_i) - \beta_i^2 \sigma_M^2$$

(with $Var(r_i)$ and β_i the same as in CAPM), so that

$$\sigma_A^2 = 0.0320, \quad \sigma_B^2 = 0.0131, \quad \sigma_C^2 = 0.0055.$$

The single index model differs from CAPM by the values of covariances between the assets.

17. **Wilkie model.** Wilkie model (in the simplest form) describes the evolution of the force of inflation $I(t)$ as an AR(1) model (auto-regression of length one). I is expressed via the *Consumer price index* Q (called also *PRI index*, or *inflation index*) as

$$I(t) = \log \frac{Q(t)}{Q(t-1)}.$$

The *inflation* over year t equals $Q(t)/Q(t-1)$.

Assume AR(1) model is given as

$$I(t) = 0.03 + 0.55[I(t-1) - 0.03] + 0.45Z(t),$$

where $Z(t)$ is a standard normal r.v.

(a) Calculate the long term average rate of inflation.

(b) Calculate the 95% confidence interval of the force of inflation over the following year given inflation over the past year was 2.75%.

Solution: (a) The long term average rate is the average which is stable under the given transformation, that is

$$I_\infty = 0.03 + 0.55[I_\infty - 0.03].$$

Hence $I_\infty = 0.33$.

(b) From

$$\frac{Q(t-1)}{Q(t-2)} = e^{I(t-1)} = 1.0275,$$

it follows $I(t-1) = \log(1.0275)$. The 95% confidence interval for $I(t)$ has the upper and lower levels respectively

$$0.03 + 0.55(\log(1.0275) - 0.03) + 0.45 \times 1.96 = 0.910,$$

$$0.03 + 0.55(\log(1.0275) - 0.03) - 0.45 \times 1.96 = -0.854,$$

since $|Z| \leq 1.96$ with probability 0.95, that is $2\Phi(-1.96) = 0.05$.

Recommended reading.

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2 Derivative securities in discrete time: arbitrage pricing and hedging

1. **Lookback option.** Suppose $S_0 = 4, u = 2, d = 1/2$, interest rate $r = 1/4$. Then $\tilde{p} = \tilde{q} = 1/2$ (check this). A lookback option in three periods is defined via payoff

$$V_3 = \max_{0 \le n \le 3} S_n - S_3.$$

Find the values V_3 of the option at times 3 (8 values), the values V_2 at time 2 (4 values), the values V_1 and the value V_0. How many shares an agent has to buy in order to hedge her short position in the option?

Partial answer: $V_0 = 1.376$, $\Delta_0 = 0.1733$.

2. **Asian option.** Let $Y_n = \sum_{k=0}^{n} S_k$. An Asian option expiring at time n and with the strike price K is defined via the payoff

$$V_n = (\frac{1}{n} Y_n - K)^+.$$

(a) Let $v_k(s, y)$ denote the price of this option at time k and $\delta_n(s, y)$ the number of shares of stock that should be held by the replicating portfolio at time k, if $S_k = s, Y_k = y$. In particular $v_n(s, y) = (-K + y/n)^+$. Develop an algorithm for computing v_k, δ_k recursively, that is, write v_k, δ_k in terms of v_{k+1}.

Answer:

$$v_k(y, s) = \frac{1}{r+1} [\tilde{p} v_{k+1}(y + us, us) + \tilde{q} v_{k+1}(y + ds, ds)],$$

$$\delta_k(y, s) = \frac{v_{k+1}(y + us, us) - v_{k+1}(y + ds, ds)}{s(u - d)}.$$

(b) Let $S_0 = 4, u = 2, d = 1/2$, $K = 4$, interest rate $r = 1/4$, $n = 3$. Compute $v_0(4, 4)$.

3. **Chooser or "As you like it" option.** Times $1 \le m \le N - 1$ given, Chooser option confers the right to receive either call or put at time m, both having maturity N and strike price K. Show that its time-zero price is the sum of the time-zero put with expiration N and strike K and a call with expiration m and strike $K(1 + r)^{-(N-m)}$.

Hint: Use put-call parity to deduce that (risk-neutral expectations are assumed)

$$\mathbf{E}_m \frac{(S_N - K)^+}{(1+r)^{N-m}} = \mathbf{E}_m \frac{(K - S_N)^+}{(1+r)^{N-m}} + S_m - \frac{K}{(1+r)^{N-m}},$$

and hence

$$\max\left(\mathbf{E}_m \frac{(S_N - K)^+}{(1+r)^{N-m}}, \mathbf{E}_m \frac{(K - S_N)^+}{(1+r)^{N-m}}\right) = \mathbf{E}_m \frac{(K - S_N)^+}{(1+r)^{N-m}} + \left(S_m - \frac{K}{(1+r)^{N-m}}\right)^+.$$

4. **Option on a dividend-paying stock.** In a dividend-paying model, adapted random variables A_n with values in $(0, 1)$ are given, specifying the payment $A_n Y_n S_{n-1}$ at time n, where Y_n equals u or d that occurred at time n. After the dividend is payed, the stock price at time n becomes $S_n = (1 - A_n) Y_n S_{n-1}$. Wealth equations becomes

$$X_{n+1} = \Delta_n S_{n+1} + (1 + r)(X_n - \Delta_n S_n) + \Delta_n A_{n+1} Y_{n+1} S_n$$

$$= \Delta_n Y_{n+1} S_n + (1+r)(X_n - \Delta_n S_n).$$

Show that the discounted wealth process is a martingale under the risk-neutral measure (which are the same as in the model without dividends), that is

$$\mathbf{E}_m \frac{X_n}{(1+r)^n} = \frac{X_m}{(1+r)^m}, \quad n \geq m,$$

implying that the risk-neutral pricing formula

$$V_n = \mathbf{E}_n \frac{V_N}{(1+r)^{N-n}}$$

for option pricing still applies. Show that if $A_n = a$ is a constant, then

$$\frac{S_n}{(1-a)^n(1+r)^n}$$

is also a martingale.

Hint: It is sufficient to show the first formula for $n = m+1$. Then

$$\mathbf{E}_m \frac{X_{m+1}}{(1+r)^{m+1}} = \frac{\Delta_m S_m}{(1+r)^{m+1}} \mathbf{E} Y_{m+1} + \frac{X_m - \Delta m S_m}{(1+r)^m}$$

$$= \frac{\Delta_m S_m}{(1+r)^m} + \frac{X_m - \Delta m S_m}{(1+r)^m} = \frac{X_m}{(1+r)^m}.$$

5. **Real options: A gold mine**. The current market price for gold is S_0 per ounce and is supposed to change per year according to a binomial model with parameters u, d. Interest rate $r = 10\%$. Gold can be extracted from the mine at a rate A ounces per year at a cost c per ounce. Assume that the price obtained for gold mines in a given year is the price that held at the beginning of the year, but cash flows occur at the end of the year. In order to value a 10-years lease of this mine, let us consider the price of a lease as a derivative instrument whose underlying security is gold. Develop an algorithm for computing the price of a lease v_k.

Hint: argue inductively starting with $v_9(s) = A(s-c)^+$.

Answer:

$$v_k(s) = \frac{1}{1+r}[\tilde{p} v_{k+1}(su) + \tilde{q} v_{k+1}(sd) + A(s-c)^+].$$

6. **Bull spread**. Buy a call with strike price K_1 and sell a call with strike price $K_2 > K_1$ (same expiration). Draw a payoff curve for such a spread. When it reaches its maximum? Is the initial cost of the spread positive?

7. **Call strikes.** Let $C(K)$ denote the price of a European call with strike K (and some fixed expiration). Sketch the graph of this function (for any given maturity and stock price S). Show that

(a) $C(K)$ is a decreasing function.

(b) $K_2 > K_1 \implies C(K_1) - C(K_2) \leq K_2 - K_1$.

(c) $K_3 > K_2 > K_1$ implies

$$C(K_2) \leq \left(\frac{K_3 - K_2}{K_3 - K_1}\right) C(K_1) + \left(\frac{K_2 - K_1}{K_3 - K_1}\right) C(K_3).$$

Hint: By risk neutral pricing one should only prove it at the time of expiration. Thus (b) follows from

$$(S - K_1)^+ - (S - K_2)^+ \leq K_2 - K_1, \quad K_2 > K_1.$$

Why is it true? Statement (c) follows from the convexity of the function $C(K)$.

8. **One period Δ**. Let Δ denote the amount of shares needed to hedge a position of an agent that is selling a standard European put, in a one period binomial model with strike price K. Sketch the graph of Δ against the initial value of stock S. Describe the domains where this function is linear, concave or convex.

9. **Setting binomial models and deflator calculations.** We are interested in pricing a derivative security, written on a stock with volatility $\sigma = 0.2$ p.a., that matures in one year. Continuously compounded risk-free rate is 4% p.a.

(a) Set up a four period binomial model and compute the risk-neutral probabilities.

(b) Suppose additionally that the expected return on the stock is 2% per month. Calculate the real world probabilities and the state price deflators (or density).

(c) Price a European powered call option with expiry one year and strike K, specified by the payoff $V(S) = ((S-K)^+)^p$. Construct a hedge portfolio.

Hint: (a)

$$\Delta t = 1/4, \quad 1 + r = e^{0.04/4} = e^{0.01}, \quad u = e^{0.2/\sqrt{4}} = e^{0.1}, \quad d = e^{-0.1}.$$

$$\tilde{p} = (e^{0.01} - e^{-0.1})/(e^{0.1} - e^{-0.1}).$$

(b) Since $\mathbf{E}(S_1/S_0) = (1.02)^3 = pu + (1-p)d$ (where p is real world prob.), it follows

$$p = \frac{(1.02)^3 - d}{u - d}.$$

Deflator is a random variable taking value

$$e^{-0.04}\left(\frac{\tilde{p}}{p}\right)^l \left(\frac{1-\tilde{p}}{1-p}\right)^{4-l}$$

on the event when l among 4 moves of the stock were upwards.

10. **Backward calculations for European and American derivatives**. Let a current stock price is \$62 and the stock volatility is $\sigma = 0.20$ Interest rates are 10% compounded monthly. Using binomial approach with period length 1 month, determine the price of a European call and American put, both with expiration 5 months and strike \$60. Display the prices of stocks and options for all months in tables. On the table for American put stress the values, where the earlier exercise is optimal. Write down the recurrent equation for determining American put $v_n(s)$ (at time n, when the stock price is S).

Partial answer: Table of stock prices:

62.00	65.68	69.59	73.72	78.11	82.75
	58.52	62.00	65.68	69.59	73.72
		55.24	58.52	62.00	65.68
			52.14	55.24	58.52
				49.21	52.14
					46.45

Table of the American put prices:

1.56	0.61	0.12	0	0	0
	2.79	1.23	0.28	0	0
		4.80	2.45	0.65	0
			7.86	**4.76**	**1.48**
				10.79	**7.86**
					13.55

11. **Multi-factor model**. Assume there are two risky assets. In one period, the initial values (S_0^1, S_0^2) can move to $(u_1 S_0^1, u_2 S_0^2)$ with probability p_{uu}, to $(u_1 S_0^1, d_2 S_0^2)$ with probability p_{ud}, to $(d_1 S_0^1, u_2 S_0^2)$ with probability p_{du}, or to $(d_1 S_0^1, d_2 S_0^2)$ with probability p_{dd}. Write down the equations for (risk neutral) probabilities $\tilde{p}_{uu}$, $\tilde{p}_{ud}$, $\tilde{p}_{du}$, $\tilde{p}_{dd}$ that ensure that the discounted stock price processes $\tilde{S}_m^1/(1+r)^m$, $\tilde{S}_m^2/(1+r)^m$ are martingales? Is there a unique solution for these equations?

Answer:

$$(\tilde{p}_{uu} + \tilde{p}_{ud})u_1 + (\tilde{p}_{du} + \tilde{p}_{dd})d_1 = 1 + r,$$

$$(\tilde{p}_{uu} + \tilde{p}_{du})u_2 + (\tilde{p}_{ud} + \tilde{p}_{dd})d_2 = 1 + r,$$

$$\tilde{p}_{uu} + \tilde{p}_{ud} + \tilde{p}_{du} + \tilde{p}_{dd} = 1.$$

Not unique.

Recommended reading.

S. E. Shreve. Stochastic Calculus for Finance I. The Binomial Asset Pricing Model. Springer Finance. Springer, 2005.

N. H. Bingham, R. Kiesel. Risk-neutral valuation. Pricing and hedging of financial derivatives. 2nd Ed. Springer Finance. Springer, 2004.

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3 Stochastic calculus and derivatives in continuous time

W denotes the standard Wiener process or Brownian motion, and Φ the distribution function of standard normal r.v.

1. **Simplest product rule.** Let $dX(t) = \alpha(t)dt + \beta(t)dW(t)$ and $dY(t) = \omega(t)dt$, where $\alpha(t), \beta(t), \omega(t)$ are adapted continuous processes. Show that

$$d(Y(t)X(t)) = X(t)dY(t) + Y(t)dX(t) = \omega(t)X(t)dt + Y(t)dX(t), \tag{3}$$

or

$$d(Y(t)X(t)) = [\omega(t)X(t) + Y(t)\alpha(t)]dt + Y(t)\beta(t)dW(t).$$

2. **Vasicek interest rate model**. This is given by the equation

$$dR(t) = (\alpha - \beta R(t))\, dt + \sigma\, dW(t), \tag{4}$$

where $\alpha, \beta, \sigma > 0$.

(a) Recall (and check!) that the solution to the ODE

$$\dot{x} = -\beta x + g(t), \quad x(0) = x_0,$$

is given by the formula (Duhamel principle)

$$x = e^{-\beta t} x_0 + \int_0^t e^{-\beta(t-s)} g(s)\, ds.$$

Applying this result formally to the Vasicek equation, deduce its solution:

$$R(t) = e^{-\beta t} R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t} \int_0^t e^{\beta s}\, dW(s). \tag{5}$$

(b) Check that (5) solves (19) applying Ito's formula to the function

$$f(t,x) = e^{-\beta t} R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t} x$$

and $X(t) = \int_0^t e^{\beta s}\, dW(s)$, or, alternatively, applying the Ito's product rule (stochastic integration by parts).

(c) Find the variance of $R(t)$.

Answer: $\sigma^2(1 - e^{-2\beta t})/2\beta$.

(d) Find the expectation of $R(t)$ and show that it converges to a constant, α/β (mean-reverting property).

(e) **Hull-White model.** Suppose α, β, σ in (19) are not constants, but rather continuous deterministic functions of time. Deduce the corresponding generalization of the solution formula (5).

3. **Cox-Ingersoll-Ross (CIR) interest rate model**. This is given by

$$dR(t) = (\alpha - \beta R(t))\, dt + \sigma \sqrt{R(t)}\, dW(t), \tag{6}$$

where $\alpha, \beta, \sigma > 0$.

(a) Using (3), show that $X(t) = e^{\beta t} R(t)$ satisfies the equation

$$dX(t) = \alpha e^{\beta t}\, dt + \sigma e^{\beta t/2} \sqrt{X(t)}\, dW(t). \tag{7}$$

Use this formula to show that the expectation of $R(t)$ coincides with the expectation of the Vasicek model.

(b) From the product rule deduce that

$$d(X^2(t)) = (2\alpha + \sigma^2) e^{\beta t} X(t)\, dt + 2\sigma e^{\beta t/2} X^{3/2}(t)\, dW(t). \tag{8}$$

Then calculate $\mathbf{E}(R^2(t))$ and Var $(R(t))$.

Partial answer: Var $R(t)$ equals

$$\frac{\sigma^2}{\beta} R(0)(e^{-\beta t} - e^{-2\beta t}) + \frac{\alpha\sigma^2}{2\beta^2}(1 - 2e^{-\beta t} + e^{-2\beta t}).$$

4. **CIR and OU processes**. Let $W_1, ..., W_d$ be independent standard BM and X_j be the *Ornstein-Uhlenbeck processes* (OU) solving the equations

$$dX_j(t)) = -\frac{b}{2} X_j(t) dt + \frac{1}{2}\sigma dW_j(t), \tag{9}$$

$j = 1, ..., d$.

(a) Show that the process

$$R(t) = \sum_{j=1}^{d} X_j^2(t)$$

solves the CIR equation (6) with $\alpha = d\sigma^2/4$, where

$$W(t) = \sum_{j=1}^{d} \int_0^t \frac{X_j(s)}{\sqrt{R(s)}} dW_j(s)$$

is a BM (check it by Lévy's theorem!).

(b) Noting that the equations for X_j are particular cases of the Vasicek equation (19), deduce that $X_j(t)$ are iid $N(\mu(t), v(t))$ r.v. with

$$\mu(t) = e^{-bt/2}\sqrt{R(0)/d}, \quad v(t) = \frac{\sigma^2}{4b}[1 - e^{-bt}].$$

(c) Recall that the Laplace transform and the MGF of a r.v. Z are defined respectively as the functions

$$L_Z(u) = \mathbf{E} e^{-uZ}, \quad M_Z(u) = \mathbf{E} e^{uZ} = L_Z(-u).$$

Find the Laplace transform $L_{X_j^2}(u)$ of X_j^2. Answer:

$$\frac{1}{\sqrt{1 + 2uv(t)}} \exp\left\{ \frac{-u\mu^2(t)}{1 + 2uv(t)} \right\}.$$

(d) Find the Laplace transform and the MGM of the solution $R(t)$ to the CIR model.

5. **Linear equations and generalized geometric BM.** Consider the equation

$$dS(t) = S(t)(\alpha(t)dt + \sigma(t)dW(t)). \tag{10}$$

Let $S(t) = S_0 e^{Y(t)}$.

(a) Show that

$$dY(t) = (\alpha(t) - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dW(t),$$

and hence deduce the formula for the solution of (10).

(b) Deduce the SDE for the discounted stock price $D(t)S(t)$, where $D(t) = \exp\{-\int_0^t R(s)ds\}$ and $R(t)$ denotes the interest rate.

(c) Show, in particular, that if α, σ, R are constants, then $\log(S(T)/S_0)$ is normal $N((\alpha - \sigma^2/2)t, \sigma^2 t)$ and $\log(D(T)S(T)/S_0)$ is normal $N((\alpha - R - \sigma^2/2)t, \sigma^2 t)$.

(d) Assuming α, σ, R to be constants, write down the expression for $S(t)$ in terms of the BM $\tilde{W}(t) = \theta t + W(t)$, where $\Theta = (\alpha - R)/\sigma$ is the market price of risk. Conclude that under $\tilde{W}$, $\log(S(T)/S_0)$ is normal $N((R - \sigma^2/2)t, \sigma^2 t)$ and $\log(D(T)S(T)/S_0)$ is normal $N(-t\sigma^2/2, t\sigma^2)$.

Answer:

$$S(t) = S_0 \exp\{\sigma\tilde{W}(t) + t(R - \sigma^2/2)\}.$$

6. **Moving normal means by a change of measure.** Let X be $N(0,1)$ r.v. defined on a probability space with the law $\mathbf{P}$. Let $\theta > 0$,

$$Z = \exp\{-\theta X - \frac{1}{2}\theta^2\},$$

and probability $\tilde{\mathbf{P}}$ is defined via $\mathbf{E}Y = \mathbf{E}(ZY)$, or equivalently

$$\tilde{\mathbf{P}}(A) = \int_A Z(\omega)d\mathbf{P}(\omega). \tag{11}$$

Show that X is $N(-\theta, 1)$ under $\tilde{\mathbf{P}}$. Hint:

$$\tilde{E}X = \frac{1}{\sqrt{2\pi}} \int xe^{-\theta x - \theta^2/2} e^{-x^2/2}\, dx.$$

7. **Change of measure under filtration.** Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ be a filtered probability space, $t \in [0,T]$, and Z an a.s. positive r.v. satisfying $\mathbf{E}Z = 1$.

(a) Show that the *Radon-Nikodym derivative process* $Z_t = \mathbf{E}(Z|\mathcal{F}_t)$ is a martingale.

(b) Let $\tilde{\mathbf{P}}$ be defined via (11). Show that

$$\tilde{\mathbf{E}}Y = \mathbf{E}(YZ_t) \tag{12}$$

for any (integrable) $\mathcal{F}_t$-measurable r.v. $Y, \quad t \in [0,T]$.

(c) Let $s \le t \le T$ and Y be again an $\mathcal{F}_t$-measurable r.v. Show that

$$\tilde{\mathbf{E}}(Y|\mathcal{F}_s) = \frac{1}{Z_s}\mathbf{E}(YZ_t|\mathcal{F}_s). \tag{13}$$

Hint: Observe that (13) is equivalent to having

$$\tilde{\mathbf{E}}[\xi \frac{1}{Z_s}\mathbf{E}(YZ_t|\mathcal{F}_s)] = \tilde{\mathbf{E}}[\xi Y]$$

for any $\mathcal{F}_s$-measurable ξ and use (12) to prove it.

8. **Calculating BS from risk-neutral evaluation formula**.

(a) Suppose X is log-normal, so that $X = e^Y$ with a normal $N(\mu, \sigma^2)$ r.v. Y, and let K be a positive constant. Show that

$$\mathbf{E}\max(X-K,0) = \tilde{\mu}\Phi\left(\frac{\ln(\tilde{\mu}/K)+\sigma^2/2}{\sigma}\right) - K\Phi\left(\frac{\ln(\tilde{\mu}/K)-\sigma^2/2}{\sigma}\right), \qquad (14)$$

where

$$\tilde{\mu} = \mathbf{E}(X) = \exp\{\mu+\sigma^2/2\}.$$

Hint: Firstly

$$\mathbf{E}\max(X-K,0) = \mathbf{E}\max(\exp\{\mu+\sigma Z\}-K,0)$$

with a standard $N(0,1)$ r.v. Z. Consequently

$$\mathbf{E}\max(X-K,0) = \int_{(\log K-\mu)/\sigma}^{\infty}(e^{\mu+\sigma x}-K)\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}dx$$

$$= \tilde{\mu}\int_{(\ln K-\mu)/\sigma}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(x-\sigma)^2}{2}\right\}dx - K\int_{(\ln K-\mu)/\sigma}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}dx,$$

which rewrites as (14).

(b) The risk-neutral evaluation principle yields the value

$$c(t,x) = e^{-r(T-t)}\tilde{\mathbf{E}}[\max(S_T-K,0)|S(t)=x] = xe^{-r(T-t)}\tilde{\mathbf{E}}\left[\max\left(\frac{S_T}{x}-\frac{K}{x},0\right)|S(t)=x\right],$$

for the price of a standard European option with maturity T and strike price K, where $\tilde{\mathbf{E}}$ means the expectation with respect to the risk-neutral distribution, which is log-normal with $\mathbf{E}(S_T) = e^{r(T-t)}x$, that is with $\log(S(T)/x))$ being $N((r-\sigma^2/2)(T-t), \sigma^2(T-t))$. Prove the BS formula

$$c(t,x) = x\Phi(d_1(T-t,x)) - Ke^{-r(T-t)}\Phi(d_2(T-t,x))], \qquad (15)$$

where

$$d_1(s,x) = \frac{\log(x/K)+(r+\sigma^2/2)s}{\sigma\sqrt{s}},$$

$$d_2(s,x) = \frac{\log(x/K)+(r-\sigma^2/2)s}{\sigma\sqrt{s}}.$$

Hint: Use (a) with $\tilde{\mu} = e^{r(T-t)}$ and $\sigma\sqrt{s}$ instead of σ.

9. **Basic Greeks and PDE for BS**. Assume (15).

(a) Find the *delta* of the option $\frac{\partial c}{\partial x}$. Answer: $\Phi(d_1)$.

(b) Find the *theta* of the option $\frac{\partial c}{\partial t}$. Answer:

$$-rKe^{-r(T-t)}\Phi(d_2) - \frac{\sigma x}{2\sqrt{T-t}}\Phi'(d_1).$$

(c) Find the *gamma* of the option $\frac{\partial^2 c}{\partial x^2}$. Answer:

$$\frac{1}{\sigma x\sqrt{T-t}}\Phi'(d_1(T-t,x)).$$

(d) Use the above formulas for Greeks to check that the function (15) satisfies the BS equation

$$\frac{\partial c}{\partial t}+rx\frac{\partial c}{\partial x}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 c}{\partial x^2}=rc, \quad 0\le t<T, \quad x>0. \tag{16}$$

(e) Show that

$$\lim_{t\to T} d_{1,2}=\begin{cases} +\infty, & x>K,\\ -\infty, & x<K,\\ 0, & x=K\end{cases}$$

and hence the terminal condition

$$\lim_{t\to T} c(t,x)=(x-K)^+, \quad x>0.$$

(f) Show that

$$\lim_{x\to 0} d_{1,2}=-\infty,$$

and hence the boundary condition

$$\lim_{x\to 0} c(t,x)=0.$$

(g) Show that

$$\lim_{x\to\infty} d_{1,2}=\infty.$$

Use this fact to deduce the boundary condition

$$\lim_{x\to\infty}[c(t,x)-(x-e^{-r(T-t)}K)]=0, \quad 0\le t<T.$$

10. **Dividends**. Evolution of a stock price with a continuous dividend paying is given by

$$dS(t)=S(t)[\alpha(t)dt+\sigma(t)dW(t)-A(t)dt], \tag{17}$$

where $\alpha(t),\sigma(t),A(t)$ are given continuous adapted processes, and of the portfolio value

$$dX(t)=\Delta(t)dS(t)+\Delta(t)A(t)S(t)dt+R(t)[X(t)-\Delta(t)S(t)]dt,$$

where $R(t)$ and $\Delta(t)$ are adapted processes denoting respectively the (given) interest rate and the number of shares held at time t.

(a) Show that

$$dX(t)=R(t)X(t)dt+\Delta(t)\sigma(t)S(t)[\Theta(t)dt+dW(t)],$$

where $\Theta(t)=(\alpha(t)-R(t))/\sigma(t)$ denotes the *market price of risk.*

(b) Introducing the new BM via

$$\tilde{W}(t)=W(t)+\int_0^t\Theta(u)du,$$

derive for the discount portfolio the equation

$$d[D(t)X(t)] = \Delta(t)D(t)S(t)\sigma(t)d\tilde{W}(t),$$

where $D(t) = \exp\{-\int_0^t R(u)\,du\}$ is the discount process.

(c) Deduce the risk-neutral pricing formula for a derivative security based on S.

11. **An example of a call.** Suppose an option pays \$1 at time T, if the share price $S(T)$ lies in the interval $[a, b]$ and nothing otherwise. Find its price at time zero.

Hint:

$$c = e^{-rT}\tilde{\mathbf{E}}\mathbf{1}_{S(T)\in[a,b]}$$

where $\log(S(T)/S_0))$ is $N((r - \sigma^2/2)(T - t), \sigma^2(T - t))$. Hence

$$c = e^{-rT}\left(\Phi\left[\frac{\log(a/S_0) - (r - \sigma^2/2)T}{\sqrt{T}\sigma}\right] - \Phi\left[\frac{\log(b/S_0) - (r - \sigma^2/2)T}{\sqrt{T}\sigma}\right]\right).$$

12. **Comparing Greeks.** (a) A portfolio consists of 1 thousand shares of a stock and a number, n, of European put options, P, on this stock. The delta and the gamma of the put are -0.2 and 0.4 respectively. Find n, if the delta of the whole portfolio is zero.

Hint: (a) $-0.2n + 1000 = 0$, so $n = 5000$.

(b) Two other derivatives are available on the market: a European call, C, with the same maturity and strike as P, and a security D, with a delta of 0.2 and gamma of 0.2. Calculate the number c of derivatives C and the number d of D that have to be purchased and added to the portfolio so that the delta and gamma of the expanded portfolio vanish.

Hint: Use the put-call parity to deduce $\Delta(C) = \Delta(P) + 1$ and $\Gamma(C) = \Gamma(P)$. Next, the Gamma of the original portfolio is $0.4 \times 5000 = 2000$. Hence we need

$$c(1 - 0.2) + 0.2d = 0, \quad 5000 \times 0.4 + 0.4c + 0.2d = 0.$$

13. **Greeks and implied volatility calculations**. A stock is priced at \$8.20. A writer of 100 units of a one year European call with an exercise price of \$8 has hedged the option with a portfolio of 75 shares and a loan. The annual risk-free interest rate is 7%.

(a) Find the Delta of the option.

(b) Calculate the implied volatility of the stock to within 0.1% p.a., assuming that it is below 100%.

(c) Find the value of the loan and the price of the option contract.

Hints: (a) $100\Delta = 75$, so that $\Delta = 0.75$.

(b) It is known that $\Delta = \Phi(d_1)$. Hence $d_1 = 0.6745$. It follows that

$$0.02469 + 0.07 + 0.5\sigma^2 = 0.6745\sigma.$$

Solving the equation (choosing the root less than one): $\sigma = 0.159 = 15.9\%$.

(c) The loan is $100Ke^{-rT}\Phi(d_2) = 519.7$ and the option contract $100 \times 8.2 \times 0.75 - 519.7 = 95.2$.

14. **Term-structure equation for bond prices (and interest rates).** If interests rate evolve according to an SDE of form

$$dR(t) = \beta(t, R(t))dt + \gamma(t, R(t))d\tilde{W}, \tag{18}$$

where $\tilde{W}$ is BM under a risk-neutral measure, the *discount process* is $D(t) = \exp\{-\int_0^t R(s)ds\}$ and the *zero-coupon bond pricing formula* is

$$B(t,T) = \tilde{\mathbf{E}}\left(e^{-\int_0^t R(s)ds}|\mathcal{F}_t\right).$$

Since $R(t)$ is Markov, $B(t,T) = f(t, R(t))$ for some function $f(t,r)$. Find the PDE (called *term-structure equation*) for f and state the terminal condition for $t = T$.

Hint 1: $D(t)B(t,T) = D(t)f(t,R(t))$ should be a martingale. Hence the dt term of its differential should vanish. Terminal condition $f(T,r) = 1$ for all r.

Hint 2:

$$d(D(t)f(t,R(t)) = D(t)[-Rf + \frac{\partial f}{\partial t} + \beta\frac{\partial f}{\partial r}\frac{1}{2}\gamma^2 + \frac{\partial^2 f}{\partial r^2}] + D(t)\gamma\frac{\partial f}{\partial r}d\tilde{W}.$$

15. **Bond prices under the Hull-White interest rate model.** In the Hull-White model equation (18) is of the form

$$dR(t) = (a(t) - b(t)R(t))\,dt + \sigma(t)\,dW(t), \tag{19}$$

where α, β, σ are positive (deterministic) functions. The equation for $f(t,r)$ describing the bond price via $B(t,T) = f(t,R(t))$, see previous Exer., takes the form

$$\frac{\partial f}{\partial t} + (a(t) - b(t)r)\frac{\partial f}{\partial r} + \frac{1}{2}\sigma^2(t)\frac{\partial^2 f}{\partial r^2} = rf.$$

Find the functions $C(t,T)$ and $A(t,T)$ such that this equation is satisfied by f of the form

$$f(t,r) = e^{-rC(t,T)-A(t,T)}.$$

Show that adding the terminal condition $f(T,r) = 1$ for all r specify these C and A uniquely.

Partial answer: $C(T,T) = A(T,T) = 0$ and

$$\dot{C}(t,T) = b(t)C(t,T) - 1,$$

$$\dot{A}(t,T) = -a(t)C(t,T) + \frac{1}{2}\sigma^2(t)C^2(t,T).$$

16. **Equivalent constant rate of interest**.

(a) Show, by the arbitrage argument, that if $R = R(t,T_1,T_2)$ is the equivalent constant rate of interest over the period $[T_1,T_2]$ (pay 1 at time T_1 and receive $e^{R(T_2-T_1)}$ at T_2), as seen from time $t \le T_1$, then

$$e^{R(T_2-T_1)} = \frac{B(t,T_1)}{B(t,T_2)},$$

where $B(t,T)$ denotes the zero-coupon bond with maturity T (or T-bond), or equivalently

$$R = R(t,T_1,T_2) = -\frac{\log B(t,T_2) - \log B(t,T_1)}{T_2 - T_1}.$$

Hint: Sell T_1 bond and buy $\frac{B(t,T_1)}{B(t,T_2)}$ of T_2 bonds – investment zero. Then you will pay 1 at T_1 and receive $\frac{B(t,T_1)}{B(t,T_2)}$ at T_2.

17. **Interest rate evolutions (various representations)**. Define: the instantaneous forward rate with maturity T, at time t

$$f(t,T) = -\frac{\partial \log B(t,T)}{\partial T}, \tag{20}$$

or equivalently

$$B(t,T) = \exp\{-\int_t^T f(t,s)\,ds\} \tag{21}$$

and the instantaneous short rate $r(t) = f(t,t)$ at time t.

Define Short-rate Dynamics:

$$dr(t) = a(t)\,dt + b(t)\,dW(t), \tag{22}$$

Bond-price Dynamics

$$dB(t,T) = B(t,T)[m(t,T)\,dt + v(t,T)\,dW(t)], \tag{23}$$

Forward-rate Dynamics

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dW(t) \tag{24}$$

(starting analysis from an exogenously given SDE for forward rates is often referred to as *Heath-Jarrow-Morton methodology*).

Show: (i) If $B(t,T)$ satisfies (23), then f satisfies (24) with

$$\alpha(t,T) = \frac{\partial v}{\partial T}(t,T)v(t,T) - \frac{\partial m}{\partial T}(t,T), \quad \sigma(t,T) = -\frac{\partial v}{\partial T}(t,T)$$

(hint: use the Ito formula);

(ii) If $f(t,T)$ satisfies (24), then short-rate satisfies (22) with

$$a(t) = \frac{\partial f}{\partial T}(t,t) + \alpha(t,t), \quad b(t) = \sigma(t,t); \tag{25}$$

and $B(t,T)$ satisfies (23) with

$$m(t,T) = r(t) + A(t,T) + \frac{1}{2}\|S(t,T)\|^2, \quad v(t,T) = -S(t,T), \tag{26}$$

where

$$A(t,T) = -\int_t^T \alpha(t,s)\,ds, \quad S(t,T) = -\int_t^T \sigma(t,s)\,ds.$$

Solution: By (24),

$$f(t,t) = r(t) = f(0,t) = \int_0^t \alpha(s,t)\,ds + \int_0^t \sigma(s,t)\,dW(s), \tag{27}$$

$$\frac{\partial f}{\partial T}(t,t) = \frac{\partial f}{\partial T}(0,t) = \int_0^t \frac{\partial \alpha}{\partial T}(s,t)\,ds + \int_0^t \frac{\partial \sigma}{\partial T}(s,t)\,dW(s). \tag{28}$$

Inserting there

$$\alpha(s,t) = \alpha(s,s) + \int_s^t \frac{\partial \alpha}{\partial T}(s,u)\, du,$$

$$\sigma(s,t) = \sigma(s,s) + \int_s^t \frac{\partial \sigma}{\partial T}(s,u)\, du,$$

yields

$$r(t) = f(0,t) + \int_0^t \alpha(s,s)\, ds + \int_0^t \int_s^t \frac{\partial \alpha}{\partial T}(s,u)\, duds$$
$$+ \int_0^t \sigma(s,s)\, dW(s) + \int_0^t \int_s^t \frac{\partial \sigma}{\partial T}(s,u)\, dudW(s),$$

which by changing order (stochastic Fubbini's theorem) rewrites as

$$r(t) = f(0,t) + \int_0^t \alpha(s,s)\, ds + \int_0^t \left(\int_0^u \frac{\partial \alpha}{\partial T}(s,u)\, ds \right) du$$

$$+ \int_0^t \sigma(s,s)\, dW(s) + \int_0^t \left(\int_0^u \frac{\partial \sigma}{\partial T}(s,u)\, dW(s) \right) du,$$

yielding (25).

Turning to the second part. By (21) and Ito's formula, to prove (26) it is sufficient to show that $Z(t) = -\int_t^T f(t,s)\, ds$ satisfies the equation

$$dZ(t,T) = (r(t) + A(t,T))dt + S(t,T)dW(t). \tag{29}$$

But from (24),

$$Z(t,T) = -\int_t^T f(0,s)ds - \int_0^t \int_t^T \alpha(u,s)dsdu - \int_0^t \int_t^T \sigma(u,s)dsdW(u).$$

Splitting and changing the order yields

$$Z(t,T) = Z(0,T) + \int_0^t r(s)ds - \int_0^t \left(\int_u^T \alpha(u,s)ds \right) du - \int_0^t \left(\int_u^T \sigma(u,s)ds \right) dW(u),$$

implying (29).

18. **SDEs: uniqueness in law and path-wise.** Show that the SDE (B is a standard BM)

$$dX_t = \mathrm{sgn(X_t)dB_t}$$

(a) enjoys uniqueness in law, (b) but not a path-wise uniqueness.

Hint: (a) Use the Lévy theorem to show that any solution is a BM. (b) Show that, for a given B, if X is a solution then so is $-X$.

19. **Integral representation for functionals of BM B_t.** Let $T > 0$ is given and $f \in C(\mathbf{R}^d)$. Define the martingale $M_t = \mathbf{E}(f(B_T)|\mathcal{F}_t)$. Show that it has a representation

$$M_t = \mathbf{E}(f(B_T)) + \int_0^t \omega_s(B_s)dB_s,$$

where

$$\omega_t(x) = \int \frac{\partial}{\partial x} p_t(x-y) f(y)\, dy,$$

with p_t being the heat kernel

$$p_t(x) = (2\pi t)^{-d/2} \exp\{-\frac{x^2}{2t}\}.$$

Hint. By the Markov property, $M_t = \phi_t(B_t)$, where $\phi(t,x) = \mathbf{E}(f(B_T)|B_t = x)$. In particular, $M_0 = \phi_0(0)$. The function

$$\phi_t(x) = \int p_{T-t}(x-y) f(y) dy$$

satisfies the (inverse time) heat equation. Hence it follows from Ito's formula that

$$M_t = \phi_0(0) + \int_0^t \frac{\partial}{\partial x} \phi_s(B_s)\, dB_s.$$

Integration by parts yields $\frac{\partial}{\partial x}\phi_t(x) = \omega_t(x)$.

20. **Credit risk: setting up a Markov chain model.** (a) Suppose τ is a positive r.v. with a differentiable distribution function. Show that there exists a non-negative continuous function $\lambda(t)$ s.t.

$$\mathbf{P}(\tau > t|\tau > s) = e^{-\int_s^t \lambda(u)\, du}, \quad s \le t.$$

When τ denotes a default time, $\lambda(u)$ is called *default intensity* or *hazard rate.*

Hint: Write down

$$\mathbf{P}(\tau > t|\tau > s) = \mathbf{P}(\tau > t)/\mathbf{P}(\tau > s),$$

take logarithms and denote $\lambda(u) = -\frac{d}{dt} \log \mathbf{P}(\tau > t)$.

(b) For a τ as above let us define a Markov chain X_t on the space of two points $\{D, N\}$ (default, no default) in such a way that D is an absorbing state and if $X_t = N$, then it stays there a random time τ_t: s.t. $\mathbf{P}(\tau_t > s) = \mathbf{P}(\tau > t|\tau > s)$ and then jumps to D. Show that this Markov chain has the following transitions:

$$p_{s,t}(D,D) = 1, \quad p_{s,t}(D,N) = 0, \quad p_{s,t}(N,N) = e^{-\int_s^t \lambda(u)\, du}, \quad p_{s,t}(N,D) = (1-e^{-\int_s^t \lambda(u)\, du}),$$

and the following (time-nonhomogeneous) Q-matrix

$$Q(t) = \begin{pmatrix} -\lambda(t) & \lambda(t) \\ 0 & 0 \end{pmatrix}$$

so that

$$\frac{d}{dt}|_{t=s} \mathbf{E} f(X_t)|X_s = D) = 0, \quad \frac{d}{dt}|_{t=s} \mathbf{E} f(X_t)|X_s = N) = \lambda(s)(f(D) - f(N)),$$

for a function f on the state space $\{D, N\}$.

(c) Let $B(t,T)$ denote the price at time t of a zero-coupon bond that matures at time T with payment π_T that equals 1 or δ in cases N (no default till time T) or D respectively. If $\lambda(t)$ above are given with respect to the risk-neutral measure Q, and if at time t there is no default, then

$$B(t,T) = e^{-r(T-t)}\mathbf{E}_Q(\pi_T).$$

Show that

$$B(t,T) = e^{-r(T-t)}\left(\delta + (1-\delta)e^{-\int_t^T \lambda(u)\,du}\right),$$

and hence deduce the formula

$$\lambda_T = -\frac{\partial}{\partial T}\log[e^{r(T-t)}B(t,T) - \delta].$$

(d) Extend this model to the case of n states (credit ranking), where the last state, n, is absorbing (default) and other transitions are specified by certain rates $\lambda_{ij}(t)$ (The Jarrow-Lando-Turnbull model).

21. **Credit risk: hedging.**

In a two state model let r be constant risk free rate (continuously compounded) and the default intensity is a constant, λ, under a measure P. Consider a zero-coupon defaultable bond that is due to redeem at par in two years time. Suppose its price is given by

$$B_t = \begin{cases} \delta e^{-r(2-t)}, \text{ default occurred prior to time } t \\ e^{-r(2-t)}[\delta + (1-\delta)e^{-\lambda(2-t)}] \text{ otherwise} \end{cases}$$

(a) Show that P is EMM for this model.

Hint: One has to show that

$$\mathbf{E}(e^{-rt}B_t|\mathcal{F}_s) = e^{-rs}B_s.$$

Analyze separately the cases when default occurred prior to s and otherwise. Say, in the first case, $e^{-rt}B_t = e^{-rs}B_s = \delta e^{-2r}$.

(b) Suppose a derivative contract pays \$100 after two years iff the bond has defaulted. Determine a constant portfolio in the defaultable bond and cash which replicates the derivative. Calculate the price of the derivative.

Hint: For portfolio of the form $xB_t + y$ we should have $x\delta + y = 100$ and $x + y = 0$ at the maturity, so that $y = -x = 100/(1-\delta)$. The fair price is the cost of this portfolio at time t:

$$ye^{-2r} + xB_0.$$

22. **Credit risk: Merton's model.** A company has issued a zero-coupon bond of nominal value \$6 mln with maturity of one year. The value of the assets of the company is \$10 mln and this value is expected to grow at 10% p. a. compound with an annual volatility of 30%. Shares are currently traded at a capitalization of \$4 mln. The company is expected to be wound up after one year when the assets will be used to pay off the bond holders with the remainder being distributed to the equity holder. Write down the equation for the risk free rate of interest r p.a. Which one of the above conditions is superfluous?

Hint: $S_0 = 10$, $\sigma = 0.3$, $T = 1$, $K = 6$. The value of the call is 4.

Answer:

$$4 = 10\Phi\left[\frac{\log(10/6) + r + (0.3)^2/2}{0.3}\right] - 8e^{-r}\Phi\left[\frac{\log(10/6) + r - (0.3)^2/2}{0.3}\right].$$