Tropical polyhedra are equivalent to mean payoff games

Marianne Akian

(INRIA Saclay - Île-de-France and CMAP, École Polytechnique)

joint work with Stéphane Gaubert (INRIA Saclay and CMAP) and Alexander Guterman (Moscow State Univ.), see arXiv:0912.2462

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Max-plus or tropical algebra (semiring)

\[ \mathbb{R}_{\text{max}} := ( \mathbb{R} \cup \{ -\infty \} , \max , + , -\infty , 0 ) \]

\[ \oplus \quad \otimes \quad 0 \quad 1 \]

\[ \lor \quad + \]

“+” concatenation

- \( 2 \oplus 3 = 3, 2 \otimes 3 = 5. \)
- \( a \oplus b = a \lor b = "a + b"; \)
- \( a \otimes b = a + b = "ab". \)
- \( \mathbb{R}_{\text{max}} \) is idempotent: \( a \oplus a = a. \)
- Hence there are no opposites,
- The natural order \( (a \leq b \text{ if } a \oplus b = b) \) is the usual order and all numbers are \( \geq 0. \)
A max-plus linear operator $A : \mathbb{R}^n_{\text{max}} \to \mathbb{R}^m_{\text{max}}$ can be represented by a matrix $A \in \mathbb{R}^{m \times n}$:

$$(Ax)_i = \max_{j \in [n]} (A_{ij} + x_j), \quad i \in [n] := \{1, \ldots, n\}.$$ 

Several ways to define a hypersurface:

- with two-sided equations:

  $$S = \{x \in \mathbb{R}^n_{\text{max}} \mid \max_{j \in [n]} (a_j + x_j) = \max_{j \in [n]} (b_j + x_j)\}$$

- with “one-sided” equations, as in tropical geometry:

  $$S = \{x \in \mathbb{R}^n_{\text{max}} \mid \text{the max in } \max_{j \in [n]} (a_j + x_j) \text{ is attained at least twice}\}$$

  denoted “$\sum_j a_jx_j = 0$” or “$\max_j(a_j + x_j) = 0$”.


Example

- The tropical line “\(x + y + 1 = 0\)” is the set of points where \(\max(x, y, 0)\) is attained at least twice:

- this is the limit of the amoeba:
  \[
  \lim_{t \to 0^+} \left\{ -\frac{1}{\log t} (\log(|x|, \log|y|); \ ax + by + c = 0 \right\} \text{ where } a, b, c \in \mathbb{C}.
  \]

See Gelfand, Kapranov & Zelevinsky, Passare . . .
Tropical segments:

$$[f, g] := \{ (\lambda + f) \lor (\mu + g) \mid \lambda, \mu \in \mathbb{R}_{\text{max}}, \lambda \lor \mu = 0 \}.$$
$C \subset \mathbb{R}_{\text{max}}^n$ is a tropical convex set if $f, g \in C \implies [f, g] \in C$

Tropical convex cones $\iff$ subsemimodules over $\mathbb{R}_{\text{max}}^n$. 
A **tropical half-space** is a set of the form

$$H = \{ x \in \mathbb{R}^n_{\max} \mid \max_j (a_j + x_j) \leq \max_j (b_j + x_j) \}$$

It is also the union of sectors (usual convex sets) delimited by the tropical hyperplane: “$$\max_j (c_j + x_j) = 0$$” with $$c_j = a_j \lor b_j$$.

From the **separation theorem**, we have:

**Theorem**

*Every closed tropical convex cone of $$\mathbb{R}^n_{\max}$$ is the intersection of tropical half-spaces:*

$$C = \{ x \in \mathbb{R}^n_{\max} \mid Ax \leq Bx \}$$

*with $$A, B \in \mathbb{R}^{I \times [n]}_{\max}$$, and $$I$$ possibly infinite.*

See for instance [Zimmermann 77], [Cohen, Gaubert, Quadrat 01 and LAA04].
Tropical polyhedral cones are defined as the intersection of finitely many tropical half-spaces \((I = [m])\), or equivalently, the convex hull of finitely many rays. See the works of [Gaubert, Katz, Butkovič, Sergeev, Schneider,...]. See also the tropical geometry point of view [Sturmfels, Develin, Joswig, Yu,...].
Tropical convex cones and games

- $Ax \leq Bx \iff x \leq f(x)$ with $f(x) = A^\#Bx$:
  $$(f(x))_j = \inf_{i \in I} (-A_{ij} + \max_{k \in [n]} (B_{ik} + x_k)) .$$

- $f$ is the dynamic programming operator of a zero-sum two player deterministic game: the states and actions are in $I$ and $[n]$, Min plays in states $j \in [n]$, choose a state $i \in I$ and receive $A_{ij}$ from Max, Max plays in states $i \in I$, choose a state $j \in [n]$ and receive $B_{ij}$ from Min. 
  The vector of values $v_j^N$ of the game after $N$ turns (Min + Max) starting in state $j$ satisfies:
  $$v^N = f(v^{N-1}), \ v^0 = 0 .$$

- $f$ is a min-max function [Olsder 91] when $I$ is finite, and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ when the columns of $A$ and the rows of $B$ are not $\equiv -\infty$.
- $f$ is order preserving ($x \leq y \Rightarrow f(x) \leq f(y)$) and additively homogeneous ($f(\lambda + x) = \lambda + f(x)$).
Every order preserving and additively homogeneous map $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ can be written as the dynamic programming operator of a zero-sum two player deterministic game (with infinite action space $I$):

$$[g(x)]_j = \inf_{i \in I} \max_{k \in [n]} (r_{ijk} + x_k)$$

(take $I = \mathbb{R}^n$ and $r_{jyk} = g(y)_j - y_k$) [Kolokoltsov; Gunawardena, Sparrow; Rubinov, Singer].

Every dynamic programming operator $g$ as above can be written as $g(x) = A^\# Bx$ for some (infinite) matrices $A, B \in \mathbb{R}^{I' \times [n]}$ (take $I' = I \times [n]$, $A_{(i,\ell),j} = \delta_{\ell,j}$, $B_{(i,\ell),j} = r_{\ell,i,j}$).

Thus $C := \{ x \in (\mathbb{R} \cup \{-\infty\})^n \mid x \leq g(x) \}$ is a tropical convex cone.
Corollary

Every dynamic programming operator of a deterministic game (resp. every order preserving additively homogeneous map) yields an external representation of a closed tropical convex cone, and vice versa. In this correspondence, games with finite action spaces, or equivalently min-max functions, are mapped to polyhedral cones.
Perron-Frobenius tools for order preserving homogeneous maps

exp : \( x \mapsto (\exp(x_j))_{j \in [n]} \) maps \( \mathbb{R}^n_{\max} \) to the positive cone \( \mathbb{R}^n_+ \) of \( \mathbb{R}^n \), and send order preserving additively homogeneous maps of \( (\mathbb{R} \cup \{-\infty\})^n \) into order preserving positively homogenous maps of \( \mathbb{R}^n_+ \).

Spectral radius, Collatz-Wielandt number, and dual CW number:

\[
\rho(f) := \max\{\lambda \in \mathbb{R}_{\max} | \exists u \in \mathbb{R}^n_{\max \setminus \{-\infty\}}, f(u) = \lambda + u\},
\]

\[
cw(f) := \inf\{\mu \in \mathbb{R} | \exists w \in \mathbb{R}^n, f(w) \leq \mu + w\},
\]

\[
cw'(f) := \sup\{\lambda \in \mathbb{R}_{\max} | \exists u \in \mathbb{R}^n_{\max \setminus \{-\infty\}}, f(u) \geq \lambda + u\}.
\]

Theorem (see [Nussbaum, LAA 86] for general cones of \( \mathbb{R}^n \))

Let \( f \) be a continuous, order preserving and additively homogeneous self-map of \( (\mathbb{R} \cup \{-\infty\})^n \), then

\[
\rho(f) = cw(f).
\]
Proposition

The following limit exists and is independent of the choice of $x$:

$$
\bar{\chi}(f) := \lim_{N \to \infty} \max_{j \in [n]} f_j^N(x)/N,
$$

and we have:

$$
cw'(f) = \rho(f) = cw(f) = \bar{\chi}(f).
$$

Moreover, there is at least one coordinate $j \in [n]$ such that

$$
\chi_j(f) := \lim_{N \to \infty} f_j^N(x)/N 
$$

exists and is equal to $\bar{\chi}(f)$.

See [Vincent 97, Gunawardena, Keane 95, Gaubert, Gunawardena 04] for the existence of $\bar{\chi}$ when $f$ preserves $\mathbb{R}^n$.

$\chi_j(f)$ is the mean payoff of the game starting in state $j$.

When $f$ is a min-max function which preserves $\mathbb{R}^n$, this can be shown also using Kohlberg's theorem (80) on the existence of invariant half-lines $f(u + t\eta) = u + (t + 1)\eta$ for $t$ large. Then $\chi_j(f)$ exists for all $j$ and $\bar{\chi}(f) = \max_{j \in [n]} \chi_j(f)$.
\[ C := \{ x \mid \max_{j \in [n]} (A_{ij} + x_j) \leq \max_{j \in [n]} (B_{ij} + x_j), \quad i \in I \} \]

**Theorem**

\[ \exists x \in C \setminus \{0\} \text{ iff Max has at least one winning position in the mean payoff game with dynamic programming operator} \]

\[ f_j(x) = (A^\# B x)_j = \inf_{i \in I} (-A_{ij} + \max_{k \in [n]} (B_{ik} + x_k)) , \]

i.e., \( \exists j \in [n], \chi_j(f) \geq 0. \)
$A = \begin{pmatrix} 2 & -\infty \\ 8 & -\infty \\ -\infty & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -\infty \\ -3 & -12 \\ -9 & 5 \end{pmatrix}$

players receive the weight of the arc
\[2 + x_1 \leq 1 + x_1\]
\[8 + x_1 \leq \max(-3 + x_1, -12 + x_2)\]
\[x_2 \leq \max(-9 + x_1, 5 + x_2)\]
\[ 2 + x_1 \leq 1 + x_1 \]
\[ 8 + x_1 \leq \max(-3 + x_1, -12 + x_2) \]
\[ x_2 \leq \max(-9 + x_1, 5 + x_2) \]

\[ \chi(g) = (-1, 5), \ x = (-\infty, 0) \ \text{solution} \]
Theorem
When $C$ is a polyhedron, the set of winning initial positions
\[ \{ j \in [n] \mid \chi_j(f) \geq 0 \} \]
is exactly the union of supports (indices of finite entries) of the elements of $C$.
The proof relies on Kohlberg’s theorem of existence of invariant half-lines.

Corollary
Whether an (affine) tropical polyhedron
\[ \{ x \mid \max_{j \in [n]} \max (A_{ij} + x_j), c_i) \leq \max_{j \in [n]} \max (B_{ij} + x_j), d_i), i \in [m] \} \]
is non-empty reduces to whether a specific state of a mean payoff game is winning.
Corollary

Each of the following problems:

1. Is an (affine) tropical polyhedron empty?
2. Is a prescribed initial state in a mean payoff game winning?

Can be transformed in linear time to the other one.

One can then compute $\chi(f)$ by dichotomy solving the emptiness problem for convex polyhedra.
It follows that all these problems

- belong to $\text{NP} \cap \text{co-NP}$ ([Condon 92], [Zwick and Paterson 96])
- can be solved in pseudo-polynomial time (value iteration).
- other algorithms with experimentally fast average execution time:
  - pumping algorithm [Gurvich, Karzanov, and Khachiyan 88],
  - policy iteration ([Cochet, Gaubert, Gunawardena 98],....), parity game algorithm of [Jurdziński and Vöge 00], but the number of iterations may be exponential, see [Fridman, 2009].

- the existence of a polynomial algorithm is an open problem.
Mean payoff games associated to linear independence

Let $A \in M_{m,n}(\mathbb{R}_{\text{max}})$. The columns of $A$ are *tropically linearly dependent* if we can find scalars $x_1, \ldots, x_n \in \mathbb{R}_{\text{max}}$, not all equal to $-\infty$, such that “$Ax = 0$”, that is for all $i \in [m]$, when evaluating the expression

$$\max_{j \in [n]} (A_{ij} + x_j)$$

the maximum is attained (at least) twice.

Equivalently, the rows of $A$ belongs to the tropical hyperplane

$$\{ z \mid \max_{j \in [n]} z_j + x_j \text{ attained twice} \}.$$
We define the game with dynamic programming operator

\[ g_j(x) = \min_{i \in [m], (i,j) \in E} \left( -A_{ij} + \max_{k \in [n], k \neq j} (A_{ik} + x_k) \right), \]

where \( E = \{(i,j) | A_{ij} \neq -\infty \} \). 
\( k \neq j \): the backspace move is forbidden for Max. So \( \chi(g) \leq 0 \).

**Theorem**

*The columns of A are linearly dependent if and only if Max has at least one winning position in the game with operator g.*
We define the game with dynamic programming operator

\[
g_j(x) = \min_{i \in [m], (i,j) \in E} \left( -A_{ij} + \max_{k \in [n], k \neq j} (A_{ik} + x_k) \right),
\]

where \( E = \{ (i,j) \mid A_{ij} \neq -\infty \} \).

\( k \neq j \): the backspace move is forbidden for Max. So \( \chi(g) \leq 0 \).

**Theorem**

The columns of \( A \) are linearly dependent if and only if Max has at least one winning position in the game with operator \( g \).

**Idea of the proof.** If in \((Au)_i\) the max is attained only once, then, there is an index \( j \) such that \( A_{ij} + u_j > \max_{k \neq j} A_{ik} + u_k \). We deduce that \( u_j > g_j(u) \).
\[ a = (0 \ 2 \ 0) \ b = (0 \ 3 \ 2) \ c = (0 \ 1 \ 1) \ d = (1 \ 3 \ 0) \]
\( a = (0, 2, 0) \quad b = (0, 3, 2) \quad c = (0, 1, 1) \quad e = (1, 1, 0) \)
If one replaces \( d \) by \( e \), we leave it to the reader to check that Max looses at all states.
A $n \times n$ matrix $B$ is tropically nonsingular iff the optimal assignment problem

$$\max_{\sigma} \sum_{i \in [n]} B_{i\sigma(i)}$$

has a unique optimal solution. We get a game proof of what follows:

**Corollary**

If $m \geq n$, the columns of $A$ are linearly independent if and only if there is a $n \times n$ tropically nonsingular submatrix (unique optimal assignment).

[Develin, Santos, Sturmels 05]: mixed subdivision proof (special case finite entries), see also [Izhakian, Rowen 09]. Can we find this matrix in polynomial time?
Concluding remarks

- Tropical convexity yields a geometrical point of view on mean payoff games.
- Several tropical problems reduce to mean payoff games. See also current works of Gaubert and co-authors.
- Mean payoff deterministic games with finite action spaces $\iff$ tropical linear programming.
- Can we find new algorithms for mean payoff games using the correspondance with tropical polyhedra?
- Can we find polynomial algorithms for all these problems?