Multiscale problems in differential games

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Plan

1. Dynamic Programming and the Isaacs PDE for differential games: a brief historical overview
2. DGs with random parameters
   - A two-scale model
   - Examples from finance and marketing
   - Averaging via Bellman-Isaacs equations
3. Singular Perturbations of differential games

Zero-sum differential games

We are given a (nonlinear) system with two controls

$$\dot{x}_s = f(x_s, \alpha_s, \beta_s) \qquad x_s \in \mathbf{R}^n,\ \alpha_s \in A,\ \beta_s \in B,$$
$$x_0 = x$$

with A, B compact sets, and a cost functional

$$J(t, x, \alpha, \beta) := \int_0^t l(x_s, \alpha_s, \beta_s)\, ds + h(x_t)$$

Player 1 governing α_s wants to MINIMIZE J,
Player 2 governing β_s wants to MAXIMIZE J.

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Part I: Dynamic Programming and the Isaacs PDE

Classical idea: the value function of the game

$$"V(t,x) := J(t,x,\alpha^*,\beta^*)$$

$(\alpha^*,\beta^*) =$ a saddle point of the game within feedback controls"

should be a solution of the Isaacs Partial Differential Equation

$$\frac{\partial V}{\partial t} + \min_{b\in B}\max_{a\in A}\{-D_x V\cdot f(x,a,b) - l(x,a,b)\} = 0 \quad \text{in } \mathbf{R}_+ \times \mathbf{R}^n$$

with the initial condition

$$V(0,x) = h(x) \qquad \text{in } \mathbf{R}^n.$$

Moreover, from the Hamiltonian computed on V one can (in principle!) synthesize the saddle feedbacks.

To make it rigorous must answer some questions:

1. Does the definition of $V(t,x)$ make sense ?
2. What happens at point where V is not differentiable ?
3. Can the Cauchy problem for the Isaacs PDE be solved ?
4. Does it determine the value function V ?
5. How can we synthesize the saddle if $D_x V$ does not exist, or $\text{argmin}_b \ \text{argmax}_a$ are discontinuous?

Mathematical tools: definitions of value

Question 1: rigorous definition of value.

- by discretization: Fleming '60s, Friedman 71
- by nonanticipating strategies (causal maps from the open loop controls of one player to those of the other player):
  Varaiya, Roxin, Elliott - Kalton 67 - 74; the lower value is

$$V(t,x) := \inf_{\alpha\in\Gamma(t)} \sup_{\beta\in\mathcal{B}(t)} J(t,x,\alpha[\beta],\beta),$$

  and the upper value is

$$\tilde{V}(t,x) := \sup_{\beta\in\Delta(t)} \inf_{\alpha\in\mathcal{A}(t)} J(t,x,\alpha,\beta[\alpha]),$$

  if they coincide the game has a value.
- by generalized motions of the system: Krassovski - Subbotin '70s, Berkovitz '80s,...

Mathematical tools: the viscosity method

Questions 2-3-4: v solves Isaacs PDE in a generalized sense

- 1980 Subbotin: minimax solutions and Krassovski-Subbotin value,
- 1981 Crandall - P.-L. Lions: viscosity solutions, existence and uniqueness for the Cauchy problem,
- 1984 L.C. Evans - Souganidis: V-R-E-K value is the viscosity solution to the Cauchy problem,
- 1989 M.B. - Soravia: viscosity solutions for pursuit-evasion games.

Surveys:

- Subbotin's book 1995
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Set $\quad H(x,p) := \min_{b\in B}\max_{a\in A}\{-p\cdot f(x,a,b) - l(x,a,b)\}$
and assume data at least continuous, f Lipschitz in x....
Consider the Cauchy problem

$$\text{(CP)} \qquad \frac{\partial u}{\partial t} + H(x, D_x u) = 0 \text{ in } R_+ \times \mathbf{R}^n, \quad u(0,x) = h(x).$$

Main results

i) Comparison Principle: a viscosity subsolution u and supersolution v of (CP) satisfy $u \le v \quad \forall t, x$; so (CP) has at most one visco. solution;

ii) the lower value V is the continous visco. solution of (CP);

iii) the VREK upper value $\tilde{V}$ is the continous visco. solution of (CP) with $\quad \tilde{H} := \max_{a\in A}\min_{b\in B}\{...\}$; so $V \le \tilde{V}$;

iv) if $H = \tilde{H}$ (Isaacs condition) then $V = \tilde{V}$ and the game has a value.

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v) All other notions of (lower) value coincide with V;

vi) if u^ε solves

$$\frac{\partial u^\varepsilon}{\partial t} + H(x, D_x u^\varepsilon) = \varepsilon \Delta u^\varepsilon \quad \text{in } R_+ \times \mathbf{R}^n, \quad u^\varepsilon(0, x) = h(x).$$

then $\quad u^\varepsilon \to V \quad$ locally uniformly;

vii) any monotone and consistent approximation scheme for (CP) converges to V.

Remark: vi) is the vanishing viscosity approximation of (CP).

In game terms $u^\varepsilon(t, x) = \inf_\alpha \sup_\beta E[J]$ for the stochastic system

$$dx_s = f(x_s, \alpha_s, \beta_s)ds + \varepsilon\sqrt{2}dW_s$$

where W_s is a Brownian motion, i.e.,
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A more general context: stochastic differential games

The theory of viscosity solutions works also for STOCHASTIC control and d.g.'s, i.e.,

$$dx_s = f(x_s, \alpha_s, \beta_s)\, ds + \sigma(x_s, \alpha_s, \beta_s)\, dW_s, \quad x_0 = x,$$

$$J(t, x, \alpha, \beta) := E_x \left[\int_0^t l(x_s, \alpha_s, \beta_s)\, ds + h(x_t) \right].$$

The value function is the unique solution of the Cauchy problem for the (degenerate) parabolic PDE

$$\frac{\partial u}{\partial t} + \min_{b \in B} \max_{a \in A} L^{a,b} u = 0$$

where $L^{a,b}$ is the generator of the diffusion process with constant controls $\alpha_s = a, \beta_s = b$:

$$L^{a,b} u := -\frac{1}{2} \text{trace}(\sigma \sigma^T D^2 u) - f \cdot Du$$

1 player: P.-L. Lions 1983; 2 players: Fleming - Souganidis 1989

Costructive and computational methods

Questions 3 and 5: - solve explicitly or numerically the Isaacs PDE,

- compute the optimal (saddle) strategies.

- Study of singular surfaces in low dimensions:
  Isaacs, Breakwell, Bernhard,...

  Surveys:
  - Lewin's book 1994
  - Melikyan's book 1998
- Semi-discrete schemes: discretize time

$$x_{n+1} = x_n + \Delta t\, f(x_n, a_n, b_n),$$

  find the value function $V_{\Delta t}(n, x)$ and the feedback saddle form the D.P. finite difference equation, then let $\Delta t \to 0$ and show convergence to $V(t, x)$:
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- Fully discrete schemes: discretize time and space, solve the game on a finite graph, then prove (by viscosity methods)

$$V_{\Delta t, \Delta x} \to v \quad \text{as } \Delta t, \Delta x \to 0$$

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- Methods from the theory of positional differential games and minimax solutions:
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Part II: Games with random parameters

Usually the system and costs depend on a vector of parameters y:

$$f = f(x, y, a, b), \quad l = l(x, y, a, b),$$

summarizing all the un-modeled variables.
In practical applications, for short time, one often models the parameters as CONSTANTS: one gets some historical values $y_1, ..., y_N$ and then estimates $\phi = f, l$ by

$$\phi \approx \frac{1}{N} \sum_{i=1}^{N} \phi_i, \quad \phi_i := \phi(x, y_i, a, b),$$

the arithmetic mean of the observed data.

QUESTION: is this correct? and why?

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Rmk: the data $y_1, ..., y_N$ often look like samples of a stochastic process. How can we model them?

A process $\tilde{y}_\tau$ is ergodic with invariant measure μ if for all measurable ϕ

$$\lim_{T\to+\infty} E\left[\frac{1}{T}\int_0^T \phi(\tilde{y}_\tau)\, d\tau\right] = \int \phi(y)\, d\mu(y) =: E[\phi].$$

Define $y_t^\varepsilon := \tilde{y}_{t/\varepsilon}$. Suppose you observe y_t^ε at the times $t = i/N$, $i = 1, ..., N$. Want to estimate the system and cost $\phi = f, l$ by

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For N large and ε small, setting $\tau = t/\varepsilon$ we get

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Conclusion:
The arithmetic mean of data is a good approximation of a function of the random parameters if

- there are many data, and
- the parameters are an ergodic process evolving on a time scale much faster than the state variables.

QUESTION 1:
What are the right quantities to average?

The system dynamics f and costs l themselves or something else?

QUESTION 2:
Is this model fit to real observed data in applications ?

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Two-scale model of DGs with random parameters

If $\tilde{y}_\tau$ solves

(FS) $$d\tilde{y}_\tau = g(\tilde{y}_\tau)\, d\tau + \nu(\tilde{y}_\tau)\, dW_\tau,$$

and $y_t = \tilde{y}_{t/\varepsilon}$, we get the two-scale system

(2SS) $$\begin{aligned} \dot{x}_s &= f(x_s, y_s, \alpha_s, \beta_s) \qquad x_s \in \mathbf{R}^n, \\ dy_s &= \frac{1}{\varepsilon} g(y_s)\, ds + \frac{1}{\sqrt{\varepsilon}} \nu(y_s)\, dW_s, \qquad y_s \in \mathbf{R}^m, \end{aligned}$$

Want to understand the limit as $\varepsilon \to 0$:

a Singular Perturbation problem.

Main assumption: the fast subsystem (FS) is ergodic, i.e., it has a unique invariant measure μ.

Example 1. Any non-degenerate diffusion

$$d\tilde{y}_\tau = g(\tilde{y}_\tau)\, d\tau + \nu(\tilde{y}_\tau)\, dW_\tau, \quad \det \nu \neq 0$$

on a compact manifold, e.g., the torus $\mathbb{T}^m$, is ergodic.

Example 2. The Ornstein-Uhlenbeck process

$$d\tilde{y}_t = (m - \tilde{y}_t)\, dt + \sqrt{2}\, \nu\, dW_t$$

(m, ν constant) is ergodic with Gaussian invariant measure

$$\mu \sim \mathcal{N}(m, \nu^2).$$

It is also mean-reverting, i.e., the drift pulls the process back to its mean value m.

Fouque, Papanicolaou, Sircar give empirical data showing that a good model for the volatility in financial markets is

$$\sigma = \sigma(y_t^\varepsilon), \quad y_t^\varepsilon := \tilde{y}_{t/\varepsilon}$$

for some $\sigma(\cdot) > 0$.

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A natural candidate limit system is

$$\dot{x}_s = \langle f \rangle (x_s, \alpha_s, \beta_s), \qquad \langle f \rangle (x, a, b) = \int_{\mathbf{R}^m} f(x, y, a, b)\, d\mu(y).$$

More generally, we can consider a stochastic control system with random parameters, so the 1st equation in (2SS) becomes

$$\text{(Sx)} \qquad dx_s = f(x_s, y_s, \alpha_s, \beta_s)\, ds + \sigma(x_s, y_s, \alpha_s, \beta_s)\, dW_s,$$

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$$J^\varepsilon(t,x,y,\alpha_\cdot,\beta_\cdot) := E\left[\int_0^t l(x_s,y_s,\alpha_s,\beta_s)\,ds + h(x_t)\right].$$

The value function is $V^\varepsilon(t,x,y) := \inf_{\alpha\in\Gamma(t)} \sup_{\beta\in\mathcal{B}(t)} J^\varepsilon(t,x,y,\alpha[\beta],\beta)$.

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Convergence for split system with a single controller

An answer is known by probabilistic methods if B is a singleton, so the problem is a minimization for the single player a.

Theorem [Kushner, book 1990]

If the system (Sx) for the slow variables x_s has $\sigma = \sigma(x, y)$ possibly degenerate but independent of the control and

$$f(x, y, a) = f_0(x, y) + f_1(x, a), \qquad l(x, y, a) = l_0(x, y) + l_1(x, a),$$

then

$$\lim_{\varepsilon \to 0} V^\varepsilon(t, x, y) = V(t, x) := \inf_{\alpha_\cdot} \overline{J}(t, x, \alpha_\cdot).$$

Convergence for games with split system

Theorem [M.B. et al. 2009]

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It is proved by PDE instead of probabilistic methods.
It is a special case of the general result we show later.
N.B.: split system and uncontrolled diffusion σ are restrictive assumptions: see the next examples.

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Financial models: option pricing

The evolution of stock S with stochastic volatility σ is

$$d\log S_s = \gamma\, ds + \sigma(y_s)\, dW_s$$
$$dy_s = \frac{1}{\varepsilon}(m - y_s) + \frac{\nu}{\sqrt{\varepsilon}} d\tilde{W}_s$$

There is NO control, $W_.$ and $\tilde{W}_.$ can be correlated, $l \equiv 0$, and, e.g., the terminal cost at time t is $h(S_t) = (S_t - K)^+$ for European call options.

Then, as $\varepsilon \to 0$,

$$V^\varepsilon(t,x,y) := E[h(S_t) \,|\, S_0 = x, y_0 = y] \to V(t,x) =$$

the Black-Scholes formula of the model with (constant) mean historical volatility

$$d\log S_s = \gamma\, ds + \langle\sigma\rangle\, dW_s, \quad \langle\sigma\rangle^2 = \int_{\mathbf{R}} \sigma^2(y) \frac{1}{\sqrt{2\pi\nu^2}} e^{-(y-m)^2/2\nu^2}\, dy.$$

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Merton portfolio optimization problem

Invest β_s in the stock S_s, $1 - \beta_s$ in a bond with interest rate r.
Then the wealth x_s evolves as

$$d\,x_s = (r + (\gamma - r)\beta_s)x_s\,ds + x_s\beta_s\,\sigma(y_s)\,dW_s$$
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and want to maximize the expected utility at time t, $E[h(x_t)]$ for some h increasing and concave.
N.B.: the diffusion term depends on the control and is not in split form, the previous theory does not apply.

QUESTIONS:
Is the limit as $\varepsilon \to 0$ a Merton problem with constant volatility?
If so, is the previous averaged system still correct ?

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An advertising model

Consider a duopoly: in a market with total sales M the sales of firm 1 are S_s, those of firm 2 are $M - S_s$, and $\alpha_s, \beta_s \geq 0$ are the advertising efforts. Take Lanchester dynamics

$$\dot{S}_s = (M - S_s)\alpha_s - \beta_s S_s$$

and objective functionals ($r_i, \theta_i > 0$)

$$J_1 = \int_0^t \left(r_1 S_s - \theta_1 \alpha_s^2\right) ds, \quad J_2 = \int_0^t \left(r_2(M - S_s) - \theta_2 \beta_s^2\right) ds.$$

This can be written as a 0-sum game with cost functional ($\theta > 0$)

$$J = \int_0^t \left(r S_s + \theta {\alpha_s}^2 - {\beta_s}^2\right) ds,$$

see Jorgensen and Zaccour, book 2004.

If the three parameters M, r, θ are random, assume they are functions of a fast ergodic process y_s^ε.

The system is not split because there is a term $M(y_s)\alpha_s$.

QUESTIONS:
Is the limit as $\varepsilon \to 0$ a Lanchester system with objective functional linear in the state and quadratic in the control ?
If so, what are the effective parameters ?

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Convergence via Bellman-Isaacs equations

1. Write the Hamilton-Jacobi-Bellman-Isaacs equation for the value function V^ε;
2. find a limit (effective) PDE such that V^ε converges to its solution V;
3. identify the limit PDE as a Bellman-Isaacs for a new system $\bar{f}$ and cost functional $\bar{J}$, so V is the value function of this new game.

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Step 1: V^ε solves

$$\begin{cases} \frac{\partial V^\varepsilon}{\partial t} + \mathcal{H}\left(x, y, D_x V^\varepsilon, D^2_{xx} V^\varepsilon\right) - \frac{1}{\varepsilon}\mathcal{L} V^\varepsilon = 0 \quad \text{in } \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}^m, \\ \\ V^\varepsilon(0, x, y) = h(x) \qquad \text{in } \mathbf{R}^n \times \mathbf{R}^m, \end{cases}$$

$$\mathcal{H}\left(x, y, D_x\,, D^2_{xx}\,\right) := \min_{b \in B} \max_{a \in A} \left\{ -\text{trace}(\sigma\sigma^T D^2_{xx}\,) - f \cdot D_x - l \right\}$$

$$\mathcal{L} := \text{trace}(\nu\nu^T D^2_{yy}\,) + g \cdot D_y$$

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Theorem [M.B., Cesaroni, Manca 2009]

$$V^\varepsilon(t,x,y) \to V(t,x) \quad \text{as } \varepsilon \to 0, \quad \text{locally uniformly}$$

and V solves (in viscosity sense)

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where μ is the invariant measure of the fast subsystem (FS).

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Corollary

For split systems, i.e.,

$$\sigma = \sigma(x, y), \quad f = f_0(x, y) + f_1(x, a, b), \quad l = l_0(x, y) + l_1(x, a, b),$$

the limit (effective) system and cost are obtained by averaging w.r.t. $\mu(y)$:

$$\bar{f} = \langle f \rangle = \int f_0(x, y)\, d\mu(y) + f_1(x, a, b),$$

$$\overline{\sigma\sigma^T} = \langle \sigma \rangle \langle \sigma \rangle^T = \int \sigma\sigma^T\, d\mu(y), \quad \bar{l} = \langle l \rangle = \int l_0(x, y)\, d\mu(y) + l_1(x, a, b)$$

Proof: under these assumptions $\int d\mu$ and $\min_{b \in B} \max_{a \in A}$ commute

$$\overline{H} = \int \min_{b \in B} \max_{a \in A} \{...\}\, d\mu(y) = \min_{b \in B} \max_{a \in A} \int \{...\}\, d\mu(y).$$

Merton problem with stochastic volatility

Maximize $E[h(x_t)]$ for the system in $\mathbf{R}^2$

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with $\gamma > r$, $\sigma > 0$, $\beta_s \in [0, \infty)$,
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Assume the utility h has $h' > 0$ and $h'' < 0$.
Then expect a value function strictly increasing and concave in x, i.e., $V^\varepsilon_x > 0$, $V^\varepsilon_{xx} < 0$. The HJB equation becomes

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By the Theorem, $V^\varepsilon(t,x,y) \to V(t,x)$ as $\varepsilon \to 0$ and V solves

$$\frac{\partial V}{\partial t} - rxV_x + \frac{(\gamma - r)^2 V_x^2}{2V_{xx}} \int \frac{1}{\sigma^2(y)} d\mu(y) = 0 \quad \text{in } \mathbf{R}_+ \times \mathbf{R}$$

So the limit problem is a Merton problem with constant volatility

$$\overline{\sigma} := \left(\int \frac{1}{\sigma^2(y)} d\mu(y) \right)^{-1/2}$$

a harmonic average of σ

So if I have N empirical data $\sigma_1, ..., \sigma_N$ of the volatility, in the Black-Scholes formula for option pricing I use the arithmetic mean

$$\sigma_a^2 = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^2$$

whereas in the Merton problem I use the harmonic mean

$$\sigma_h^2 = \left(\frac{1}{N} \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \right)^{-1} \leq \sigma_a^2 .$$

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The advertising model with random parameters

$$\dot{S}_s = (M(y_s) - S_s)\alpha_s - \beta_s S_s, \quad S_0 = x$$
$$dy_s = \frac{1}{\varepsilon}(m - y_s) + \frac{\nu}{\sqrt{\varepsilon}} dW_s, \quad y_0 = y$$

The objective functional of the 0-sum duopoly game is

$$J^\varepsilon = E\left[\int_0^t \left(r(y_s)S_s + \theta(y_s)\alpha_s^2 - \beta_s^2\right) ds\right]$$

with $\theta > 0$. By the Theorem, $V^\varepsilon(t,x,y) \to V(t,x)$ as $\varepsilon \to 0$ and V solves

$$\frac{\partial V}{\partial t} - \int \left(r(y)x + (M(y) - x)^2 \frac{V_x^2}{4} - \frac{x^2 V_x^2}{4\theta(y)}\right) d\mu(y) = 0 \quad \text{in } \mathbf{R}_+ \times \mathbf{R}$$

Denote with $\langle \phi \rangle := \int \phi d\mu$.

The PDE for V is the Isaacs equation for the game with system

$$\dot{S}_s = \sqrt{\langle M^2 \rangle - 2\langle M \rangle S_s + S_s^2}\, \alpha_s - \beta_s S_s$$

that is NOT a Lanchester dynamics, and objective functional

$$J = \int_0^t \left(\langle r \rangle S_s + \langle \frac{1}{\theta} \rangle^{-1} \alpha_s^2 - \beta_s^2 \right) ds$$

that is still linear in state and quadratic in the controls but with different averages of the parameters.

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Conclusions

In control and game problems with random parameters driven by a fast ergodic process the limit problem can be

1. of the same form and with parameters the historical mean of the random ones (as in uncontrolled problems!)
2. of the same form, but the parameters are obtained by a different averaging of the random ones (as in Merton)
3. of a form different from the original problem (as in the advertising game).

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Part III: Singular Perturbations of differential games

Singularly perturbed deterministic (for simplicity) control system

$$\dot{x}_s = f(x_s, y_s, \alpha_s, \beta_s) \qquad x_s \in \mathbf{R}^n,\ \alpha_s \in A,\ \beta_s \in B,$$

$$\dot{y}_s = \frac{1}{\varepsilon} g(x_s, y_s, \alpha_s, \beta_s) \qquad y_s \in \mathbf{R}^m,$$

$$x_0 = x, \quad y_0 = y.$$

Here y_s are fast state variables depending on x_s and the controls α_s, β_s. The value function

$$V^\varepsilon(t, x, y) := \inf_{\alpha \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} \int_0^t l(x_s, y_s, \alpha[\beta]_s, \beta_s)\, ds + h(x_t)$$

solves the Isaacs equation

$$\frac{\partial V^\varepsilon}{\partial t} + H\left(x, y, D_x V^\varepsilon, \frac{D_y V^\varepsilon}{\varepsilon}\right) = 0 \quad \text{in } \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}^m,$$

$$H(x, y, p, q) :=$$

$$\min_{b \in B} \max_{a \in A} \{-p \cdot f(x, y, a, b) - q \cdot g(x, y, a, b) - l(x, y, a, b)\}$$

Ergodicity of a game for the fast subsystem

Freeze x, p and take the (lower) value function for the game in $\mathbf{R}^m$

$$w(t, y; x, p) := \inf_{\alpha \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} \int_0^t L(y_\tau, \alpha[\beta]_\tau, \beta_\tau; x, p)\, d\tau,$$

$$L(y, a, b; x, p) := p \cdot f(x, y, a, b) + l(x, y, a, b),$$

where y_τ is the trajectory of

(FS) $$\dot{y}_\tau = g(x, y_\tau, \alpha[\beta]_\tau, \beta_\tau), \qquad y_0 = y$$

Definition:

(FS) is ERGODIC for the cost L if, for all x, p,

$$\lim_{t \to +\infty} \frac{w(t, y; x, p)}{t} = \text{constant (in } y\text{), uniformly in } y \;=: \overline{H}(x, p)$$

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Convergence theorem (M.B. - Alvarez, ARMA 2003)

Assume the fast variables y live on the torus $\mathbb{T}^m$
(i.e., all data are $\mathbb{Z}^m$- periodic in y).
Fast subsystem (FS) ergodic for the cost L $\implies$

$$V^\varepsilon(t,x,y) \to V(t,x) \quad \text{as } \varepsilon \to 0,$$

(in the sense of weak viscosity limits), and V solves

$$(\overline{CP}) \qquad \frac{\partial V}{\partial t} + \overline{H}(x, D_x V) = 0, \quad V(0,x) = h(x).$$

If, moreover,

$$(1) \qquad |\overline{H}(x,p) - \overline{H}(z,p)| \le C|x-z|(1+|p|),$$

then $(\overline{CP})$ has a unique viscosity solution and

$$V^\varepsilon \to V \qquad \text{locally uniformly.}$$

Conclusion: dimension reduction

The initial $n + m$-dimensional game is split into

- an m-dimensional ergodic-type game (that determines $\overline{H}$)
- if we have a representation

$$\overline{H}(x,p) = \min_{b'\in B'} \max_{a'\in A'} \left\{ -\bar{f}(x,a',b') \cdot p - \bar{l}(x,a',b') \right\}$$

  for some control sets A', B' and effective system and cost $\bar{f}, \bar{l}$, the PDE in $(\overline{\text{CP}})$ is the Isaacs equation of a n-dimensional "effective" game

$\Longrightarrow$ we got a SEPARATION OF SCALES and a reduction to two lower dimensional games.

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$\Longrightarrow$ we got a SEPARATION OF SCALES and a reduction to two lower dimensional games.

Example 1 (M.B. - Alvarez, Mem. A.M.S. 2010).
The 1st player controls the slow variables x_s, the 2nd player the fast ones y_s

$$\dot{x}_s = f(x_s, y_s, \alpha_s)$$
$$\dot{y}_s = \frac{1}{\varepsilon} g(x_s, y_s, \beta_s)$$

and the fast subsystem (FS) is Bounded-Time Controllable by the 2nd player , i.e. $\forall x \, \exists S > 0$ such that $\forall y, \tilde{y}$ the 2nd player can drive (FS) from y to $\tilde{y}$ within a time $\leq S$.

Then (FS) is ergodic (for all costs L).

If also $g(x, y, B) \ni 0$ and l is independent of b

$$\overline{H}(x,p) := \min_{y \in \mathbb{T}^m} \max_{a \in A} \{-p \cdot f(x, y, a) - l(x, y, a)\}$$

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Example 2. Assume $y = (y^A, y^B) \in \mathbb{T}^{m^A} \times \mathbb{T}^{m^B}$ such that

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Singular Perturbations of Stochastic DGs

$$dx_s = f(x_s, y_s, \alpha_s, \beta_s)\, ds + \sigma(x_s, y_s, \alpha_s, \beta_s)\, dW_s$$

$$dy_s = \frac{1}{\varepsilon} g(x_s, y_s, \alpha_s, \beta_s)\, ds + \frac{1}{\sqrt{\varepsilon}} \nu(x_s, y_s, \alpha_s, \beta_s)\, dW_s$$

$$x_0 = x, \quad y_0 = y$$

$$J(t, x, y, \alpha, \beta) := E_{(x,y)} \left[\int_0^t l(x_s, y_s, \alpha_s, \beta_s)\, ds + h(x_t) \right]$$

General principle still holds: ergodic fast subsystem $\implies$ convergence of the value function of the S.P. problem.

We saw before examples modeling random parameters:
y_s uncontrolled ergodic process.

Theorem

(M.B. - Alvarez, Mem. A.M.S. to 2010)

Assume $\exists \eta > 0$ such that

$$\nu\nu^T(x, y, a, b) \geq \eta I_m \quad \forall x, y, a, b,$$

i.e., the noise is uniformly nondegenerate in the Fast Subsystem

(FS) $$dy_\tau = g(x, y_\tau, \alpha_\tau, \beta_\tau)\, d\tau + \nu(x, y_\tau, \alpha_\tau, \beta_\tau)\, dW_\tau.$$

Then (FS) is ergodic for all costs and so the value function of the SP problem converge as $\varepsilon \to 0$.

The proof is done by working on the Isaacs parabolic equation and uses some deep results in the theory of elliptic PDEs (Krylov - Safonov estimates).

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References on Singular Perturbations

For control systems with only 1 player

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- Bensoussan's book 1988
- Kushner's book 1990
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- Gaitsgory, Artstein, Leizarowitz,... 1992 - 2004.
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Thanks for your attention!