Statistical Mechanics of Strategic Substitutes on Networks

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Strategic Games on Networks

1) agents are the nodes of a graph G

2) set X of possible pure strategies/actions (e.g. X = {0,1})

3) local utility/payoff function $u_i\left[x_i, f\left(\left\{x_j \middle| j \in \partial i\right\}\right)\right]$

→ At least one Nash Equilibrium always exists in pure strategies (or in mixed strategies if X is discrete)

Strategic Games on Networks

Two main cases:

- Strategic Complements (actions mutually reinforce one another)

→ Usually the Nash equilibrium is unique

- Strategic Substitutes (actions mutually offset one another)

→ there could be *many Nash equilibria* (exponentially in N) with very different properties

Best-Shot Game

- provision of local public goods, information
  (A. Galeotti, S. Goyal, M. O. Jackson, F. Vega-Redondo, and L. Yariv,
  "Network Games", forthcoming in *The Review of Economic Studies*
  also Y. Bramoullé and R. Kranton, J. Econ. Theory **135, 478** 2007.)

- binary actions X = {0,1}

- Utility $u_i\left(x_i, \hat{x}_{\partial i}\right)$ with $\hat{x}_{\partial i} = \sum_{j \in \partial i} x_j$

  1) $u_i(1,0) > u_i(0,0)$

  2) $u_i(1,\hat{x}) < u_i(0,\hat{x})$ for any $\hat{x} > 0$

Best-Shot Game

$\vec{x} = (x_1, \ldots, x_N)$ is a Nash Equilibrium

$$\Updownarrow$$

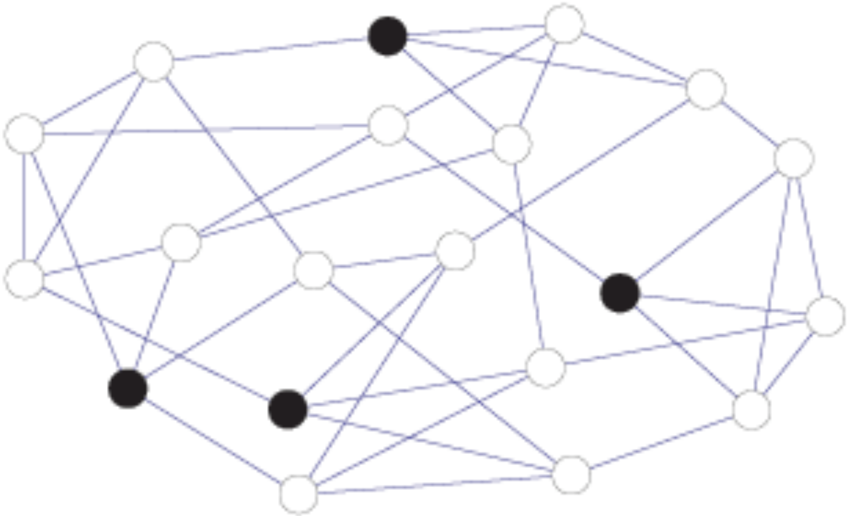
$$\forall i \quad \left(x_i = 1 \;\wedge\; \sum_{j \in \partial i} x_j = 0 \right) \vee \left(x_i = 0 \;\wedge\; \sum_{j \in \partial i} x_j > 0 \right)$$

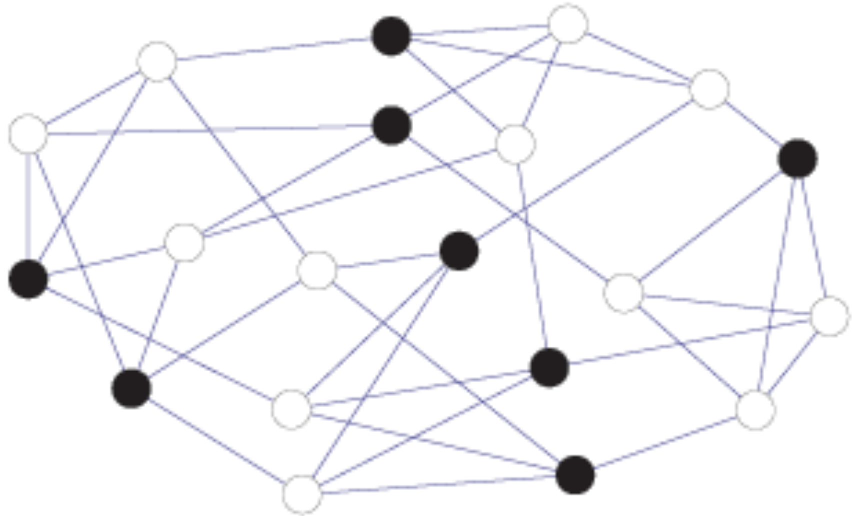
$$\Updownarrow$$

$\vec{x}$ is a maximal independent set of graph G

Best-Shot Game

Examples for a regular random graph of degree K = 4





Multiple Nash Equilibria

1) Complete network knowledge

→ Graphical Games (belief propagation algorithms)

(e.g. Kearns, chapter 7 in *Algorithmic Game Theory* by Nisan et al., 2007)

2) Incomplete information by incomplete network knowledge

→ Bayes-Nash Equilibrium (mean-field)

(e.g. Galeotti et al. 2009, Lopez-Pintado, 2008)

Incomplete Information

1) agents know their own degree k
2) have a belief P(k,k') on neighbors' degree k'

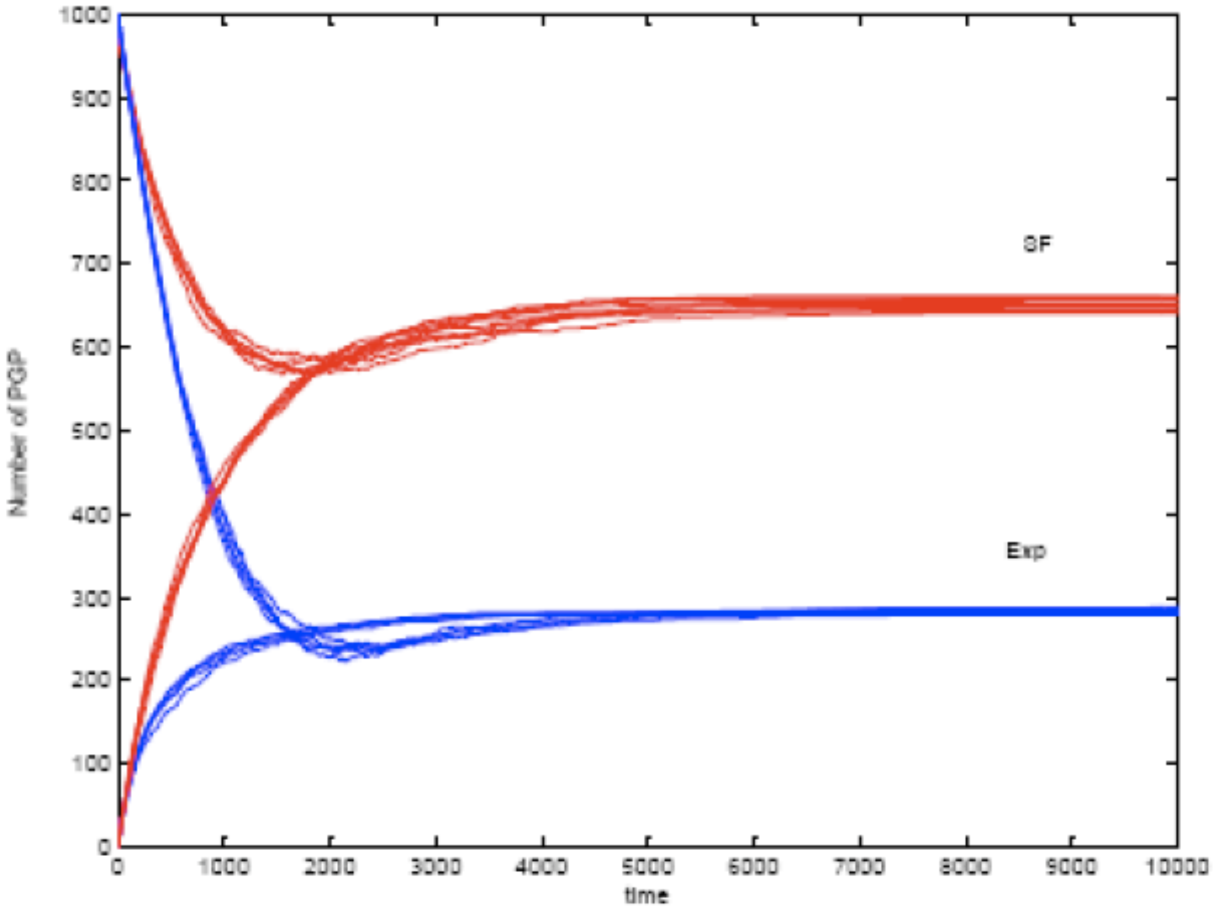
F(k, ρ) = prob. that a random agent of degree k chooses 1 when anticipating that each neighbor will choose 1 with independent prob ρ

A symmetric Bayes-Nash equilibrium exists in mixed strategies (Kakutani's fixed point th.)

$$\Rightarrow \quad \rho_k = \sum_{k'} P(k,k')F(k,\rho_{k'})$$

Simulations on Random Graphs

The B.-N.E. is the fixed point solution of mean-field equations for best-response dynamics



N = 1000

D. Lopez-Pintado, *Eastern Econ. J.*, 34 (2008)

CSP Representation

If N.E. can be expressed as a set of local conditions (on G), then

Nash Equilibria are solutions of a constraint satisfaction problem

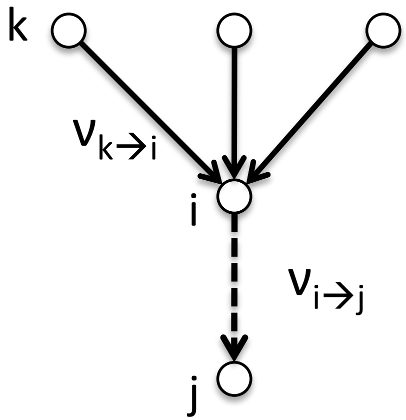
Partition function:

$$Z(\mu) = \sum_{\vec{x}} e^{-\mu \sum_{i=1}^{N} x_i} \prod_i I_i(x_i, \{x_j\}_{j \in \partial i})$$

Standard methods of statistical mechanics of disordered systems
(see e.g. M. Mezard and A. Montanari "Information, Physics, and Computation",2009)

Cavity Approach

Probability marginal ν of having configuration $\{x_i, \{x_k\}_{k \in v(i) \setminus j}\}$ on node *i* and its neighbors k on the cavity graph



$$\nu_{i\to j}(x_i, x_{i\to j}) = \frac{1}{Z_{i\to j}} \sum_{x_{k\to i}} \prod_{k\in\partial i\setminus j} I_k I_{ik} \nu_{k\to i}(x_k, x_{k\to i}) e^{-\mu x_i}$$

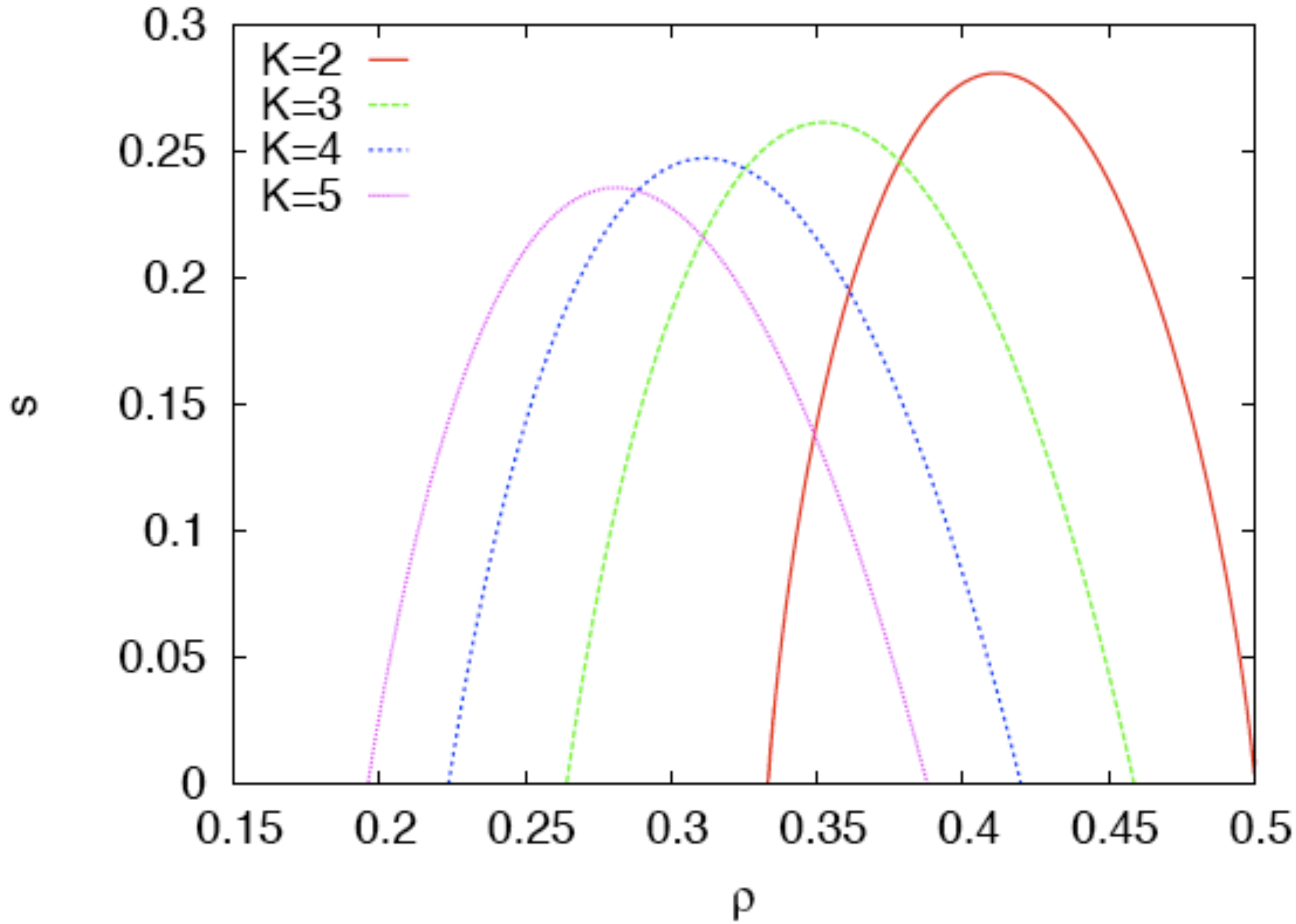
On random graphs the resulting self-consistent equations can be solved analytically (and numerically on any graph)

They provide an exact heuristic on locally tree-like graphs

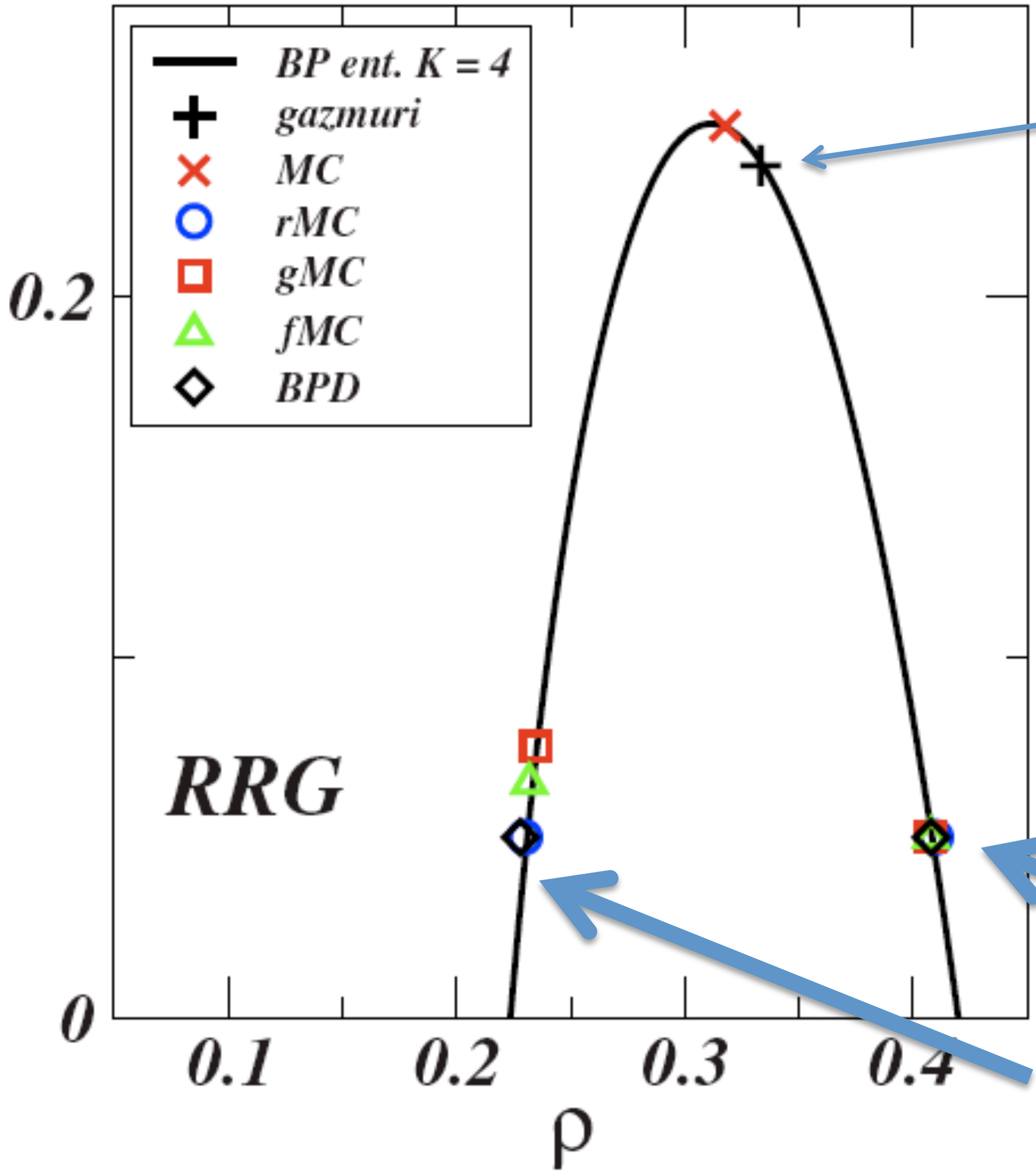
Cavity Approach

Density of contributors ρ,

Entropy of N.E. $s(\rho)$, that means $N_{NE}(\rho) \approx exp(N\ s(\rho))$



Equilibrium Selection (?)



Best-response dynamics

Finding optimal N.E. is difficult!!!

Space of N.E.

- take a N.E.
- flip a node's action from 0 to 1 (or viceversa)
- let the other nodes rearrange their actions by best-response

1) The space of N.E. is connected under this operation (distance between equilibria is o(N))

2) Rearrangements up to the second neighborhood

3) Typical N.E. are "stable"

Space of N.E.

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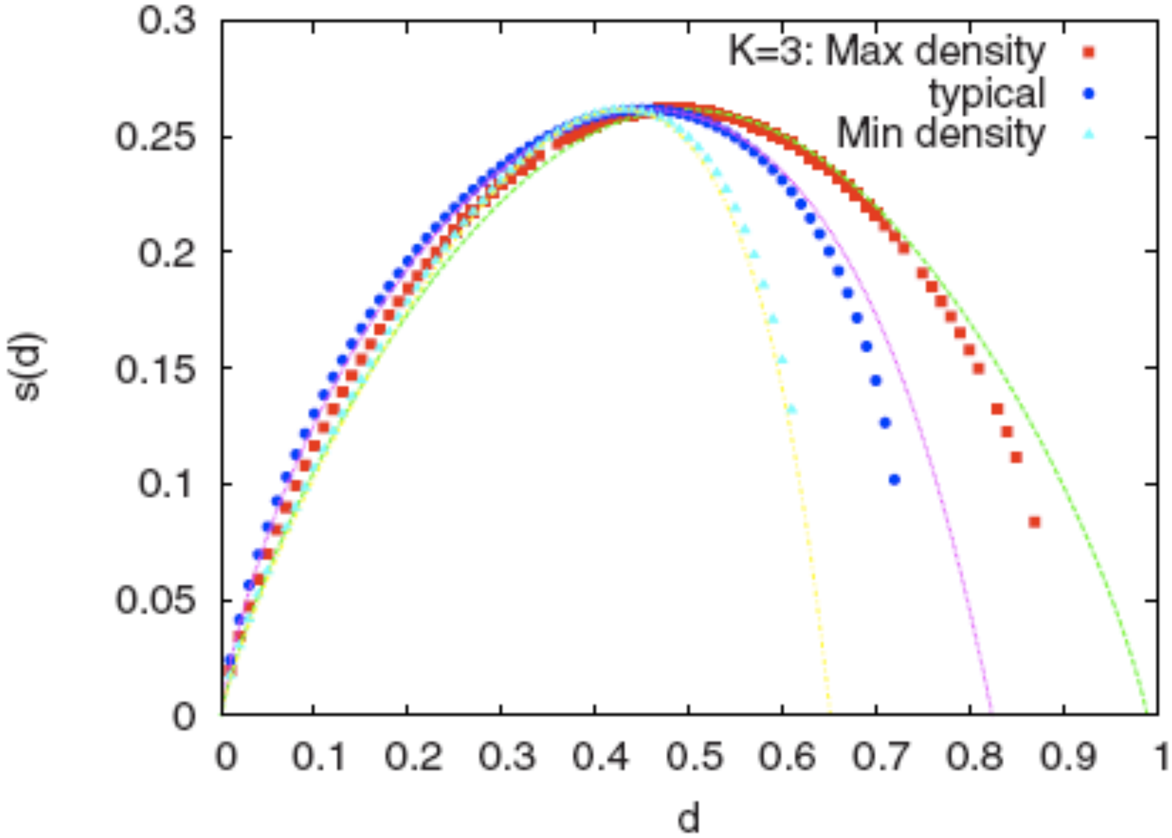
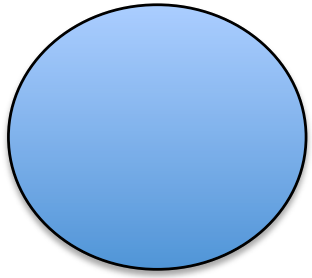
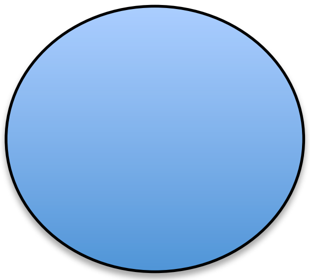


Space of N.E.

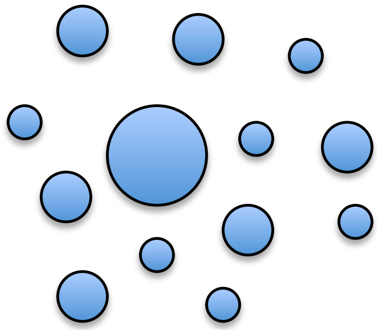
N.E. are well connected in a single cluster



Is it still true at *fixed density* ρ of contributors? NO



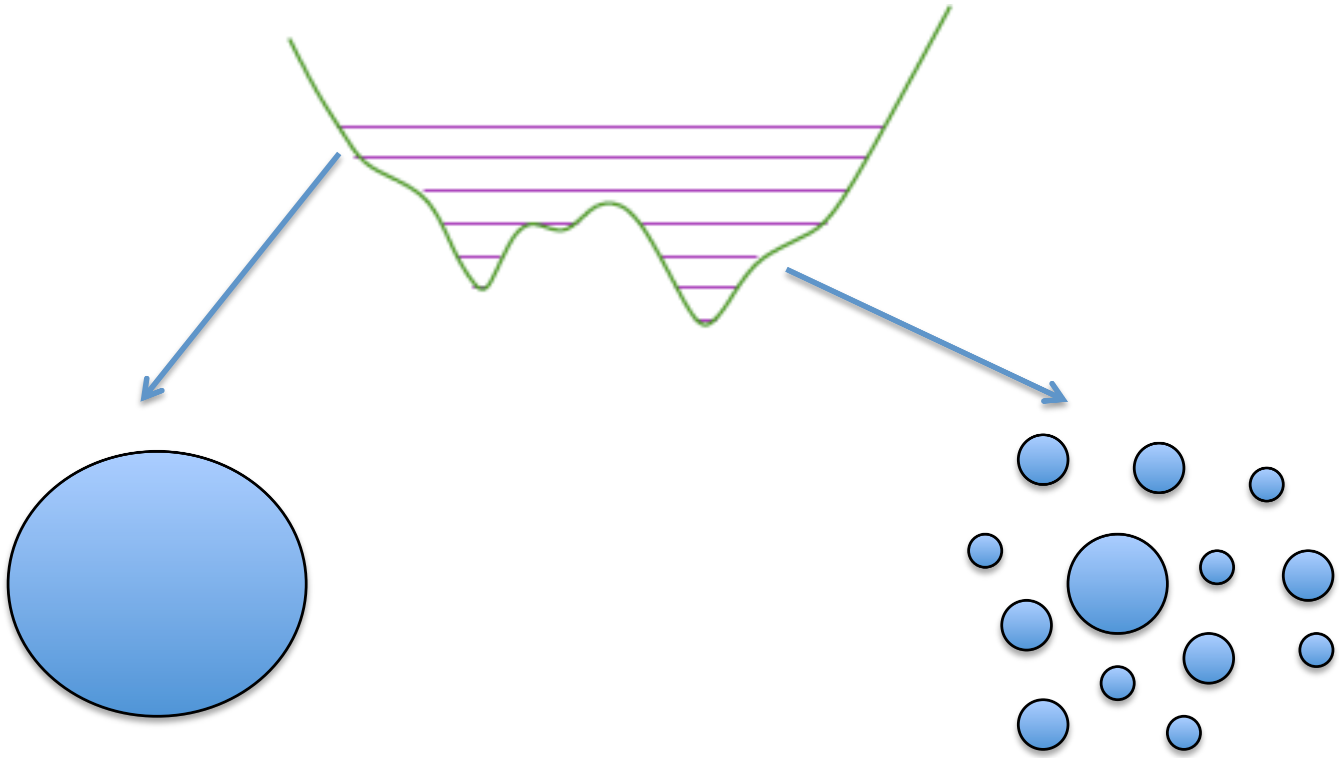
TIPICAL DENSITY ρ



LOW DENSITY ρ

Space of N.E.

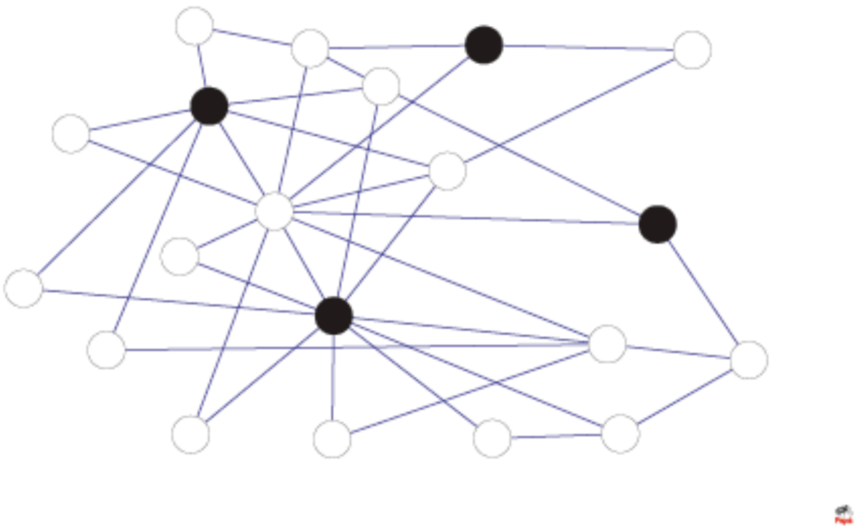
Is it still true at *fixed density* ρ of contributors? NO

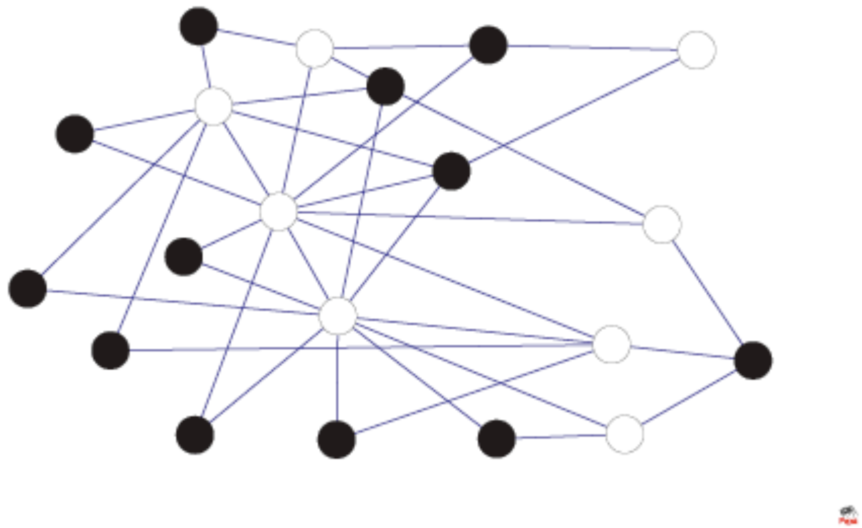


TIPICAL DENSITY ρ LOW DENSITY ρ

Effects of Degree Heterogeneity

Low density N.E. become less accessible and less stable for larger <k^2> (e.g. mean preserving spread)





Conclusions

- We investigated the space of N.E. of a network game (of strategic substitutes)
- These statistical properties can be used to extract info to design incentives or to define proper refinements of N.E.
- Interesting problems in Algorithmic GT.

References:

L. D., P. Pin, A. Ramezanpour, *Phys. Rev. E* **80**, 061136 (2009)
L. D., P. Pin and A. Ramezanpour, *Optimal Equilibria of the Best Shot Game, under revision in JPET.*