Evolution and market behavior with endogenous investment rules

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Research questions

Consider a market for a risky asset and an ecology of investment strategies competing to gain superior returns. The open questions are:

- $\Rightarrow$ which are the strategies surviving in the long run?
- $\Rightarrow$ is it possible to establish an order relationship among them?
- $\Rightarrow$ is a strategy dominating all the others?

Answers to these questions help to clarify specific issues (think of financial markets) as well as general issues ("as if" point).

Where do we stand?

On this issue

- Behavioral Finance (a survey is Barberis and Thaler, 2003)

  Pros Ecology of strategies behaviorally grounded
  Cons No wealth-driven strategy selection
  Focus Market biases

- HAM Finance (a survey is Hommes, 2006)

  Pros Focus on price feedbacks
  Cons No wealth-driven strategy selection (mostly CARA), deterministic
  Focus Stylized facts

- Evolutionary Finance (Kelly, 1956; Blume and Easley, 1992; a survey is Evstigneev, Hens, and Schenk-Hoppe, 2009)

  Pros Multi-asset stochastic general equilibrium framework
  Cons Absence of price feedbacks (no endogenous investment rules)
  Focus Market selection

$\Rightarrow$ Our approach: evolutionary finance with endogenous (price dependent) investment rules.

Framework

- Trading is repeated and occurs in discrete time
- Many assets in constant supply with uncertain dividends
- Market is complete
- Agents care about consumption, thus wealth
- A strategy is a portfolio of wealth fractions (CRRA)
- Walrasian market clearing
- Intertemporal budget constraint
- Market dynamics is formalized as a random dynamical system

A toy market

- Two states of the world, $s = 1, 2$, which occur with probability π and $1 - \pi$. Bernoulli process $\omega = (\dots, \omega_t, \dots, \omega_0) \in \Omega$.
- Two (short-lived) Arrow's securities, $k = 1, 2$, paying $D_{k,s} = \delta_{k,s}$.
- Fraction of consumption is constant and uniform, $\alpha_0 = c$. All the rest is invested.
- Define normalized prices $p_{s,t} = \frac{P_{s,t}}{W_t}$ so that $p_{1,t} + p_{2,t} = 1 - \alpha_0$, $\forall t$.
- Two agents, $i = 1, 2$, with wealth fractions ϕ_t and $1 - \phi_t$.
- Endogenous strategies with one memory lag, $L = 1$,
  - $\alpha^1_{1,t} = \alpha^1_1(p_{1,t-1})$ describes the portfolio choice of the first agent,
  - $\alpha^2_{1,t} = \alpha^2_1(p_{1,t-1})$ describes the portfolio choice of the second agent.

A toy market

⇒ Evolutionary finance literature shows that, among constant investment rules, $\alpha_s^* = \pi_s$ dominates and

$$I_\pi(\alpha) = \sum_{s=1}^{S} \pi_s \log\left(\frac{\pi_s}{\alpha_s}\right)$$

can be used to establish an ordering relationship.

A toy market

Strategy i dominates strategy j, $i > j$, if

$$\forall \epsilon > 0 \ , \quad \exists T \quad \text{s.t.} \quad \text{Prob} \left\{ \frac{\phi_t^j}{\phi_t^i} < \epsilon, \quad \forall t > T \right\} = 1 \ .$$

Two agents: the random dynamical system

Given $x_t = (\phi_t, p_t, q_t = p_{t-1})$, the state of our market at time t, the random dynamical system is the composition of the following maps

$$
\begin{cases}
\phi_{t+1} = \begin{cases} \frac{\alpha_1^1(q_t)\phi_t}{p_t} & \text{with probability} \quad \pi \\ \frac{(1-\alpha_0-\alpha_1^1(q_t))\phi_t}{1-\alpha_0-p_t} & \text{with probability} \quad 1-\pi \end{cases}, \\
p_{t+1} = \alpha_1^1(p_t)\phi_{t+1} + \alpha_1^2(p_t)(1-\phi_{t+1}), \\
q_{t+1} = p_t .
\end{cases}
$$

That is, $x_{t+1} = f_\pi(x_t)$ with probability π and $x_{t+1} = f_{1-\pi}(x_t)$ with probability $1-\pi$, depending on the realization of ω_t.

Definition

The state $x^* = (\phi^*, p^*, q^* = p^*)$ is a deterministic fixed point of the random dynamical system generated by the maps f_π and $f_{1-\pi}$, that is, $\varphi(t, \omega, x) = \dots f_\pi \circ \dots \circ f_{1-\pi} \dots$ if it holds

$$\varphi(t, \omega, x^*) = x^* \quad \forall \omega \in \Omega \tag{1}$$

or, in terms of the maps, if it holds both

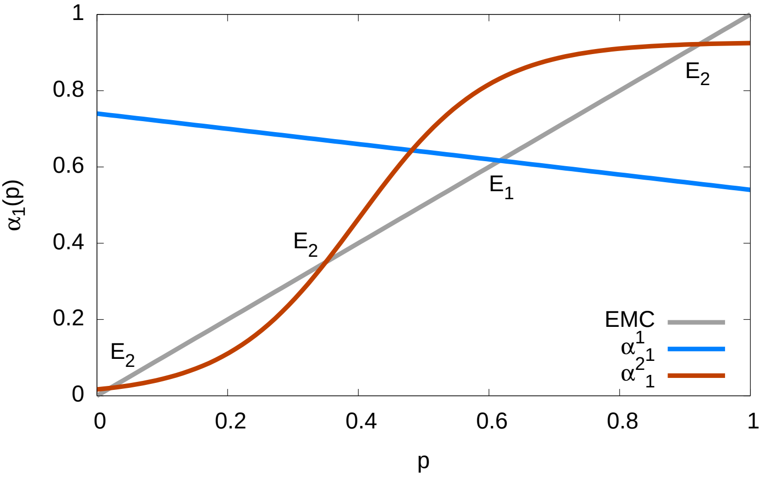
$$f_\pi(x^*) = x^* \quad \textit{and} \quad f_{1-\pi}(x^*) = x^* \,. \tag{2}$$

Theorem

Fixed points of the random dynamical system that represents the toy market dynamics are given by

$$
\begin{aligned}
x_1^* &= (\phi^* = 1, p^* = \alpha_1^1(p^*), q^* = p^*) \\
x_2^* &= (\phi^* = 0, p^* = \alpha_1^2(p^*), q^* = p^*) \\
x_{1/2}^* &= (\phi^*, p^* = \alpha_1^1(p^*) = \alpha_1^2(p^*), q^* = p^*)
\end{aligned}
$$

Fixed points on a plot: the Equilibrium Market Curve



Definition

A fixed point x^* of the random dynamical system $\varphi(t, \omega, x)$ is called locally stable if $\lim_{t\to\infty} ||\varphi(t, \omega, x) - x^*|| \to 0$ for all x in a neighborhood $U(\omega)$ of x and for all $\omega \in \Omega$.

Theorem

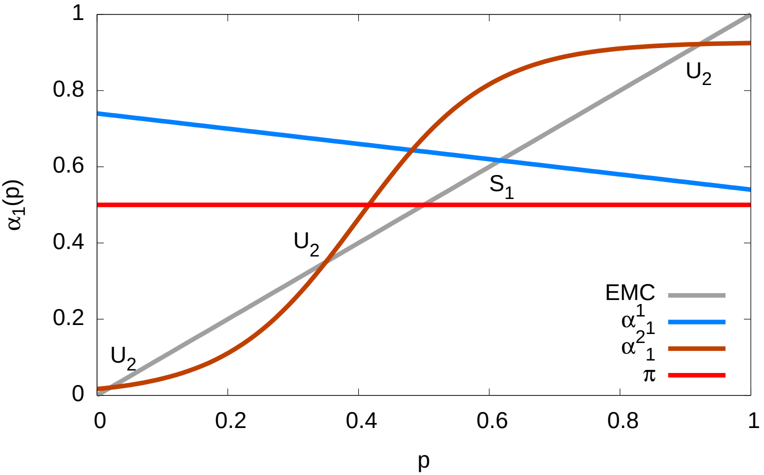
Provided that the eigenvalues of the iterated map are inside the unit circle the deterministic fixed point is locally stable (use Multiplicative Ergodic Theorem and Local Hartman-Grobman Theorem). For fixed points of the type $(1, \alpha_1^1(p^*), p^*)$ *eigenvalues are*

$$\mu = \exp\left(I_\pi(\alpha^1) - I_\pi(\alpha^2)\right) \quad \textit{and} \quad \lambda = \left.\frac{\partial \alpha_1^1(p)}{\partial p}\right|_{p^*} \tag{3}$$

and for fixed points of the type $(\phi^*, \alpha_1^1(p^*) = \alpha_1^2(p^*), p^*)$

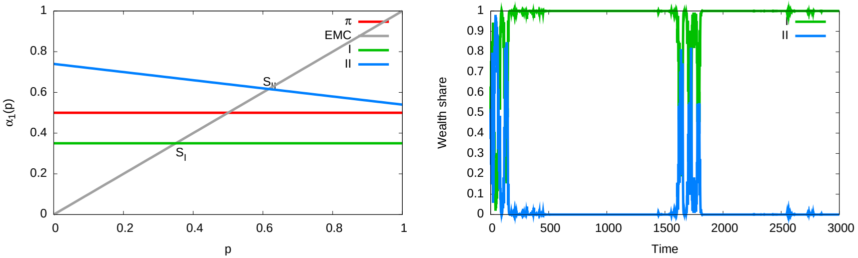
$$\mu = 1 \quad \textit{and} \quad \lambda = \phi^* \left.\frac{\partial \alpha_1^1(p)}{\partial p}\right|_{p^*} + (1-\phi^*) \left.\frac{\partial \alpha_1^2(p)}{\partial p}\right|_{p^*} \tag{4}$$

Local stability on the EMC plot



Ordering is not complete

Coexistence of stable equilibria



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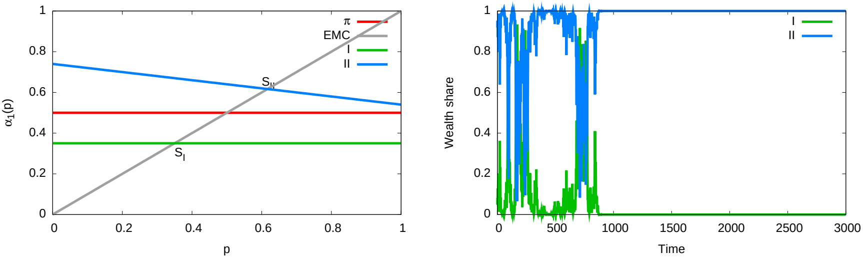

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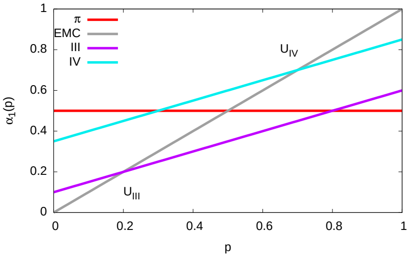
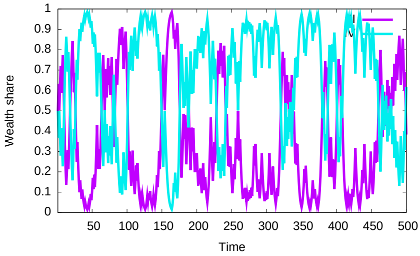
Multiple unstable equilibria





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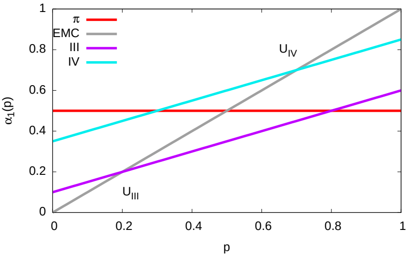
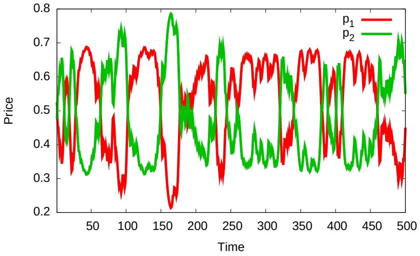
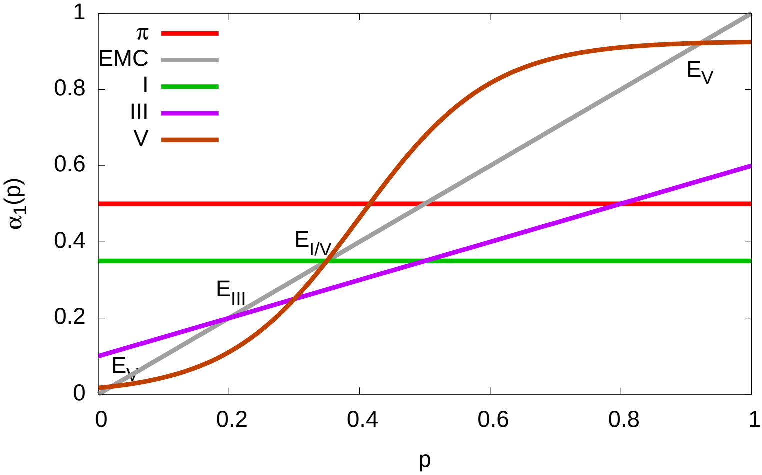




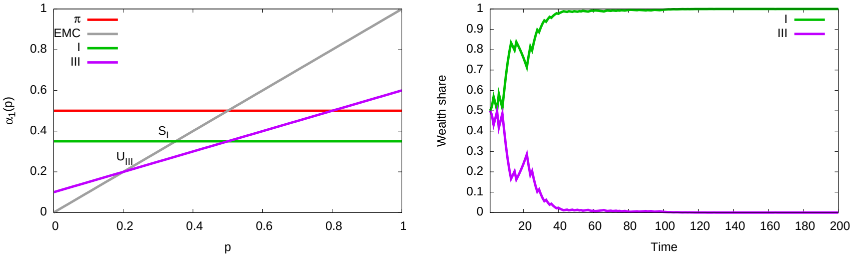
Ordering is not transitive

I $>$ *III* $>$ *V* $\sim$ *I*



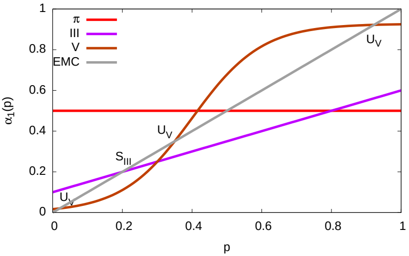
Ordering is not transitive

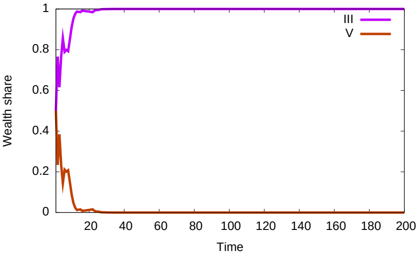
$I > III$



Ordering is not transitive

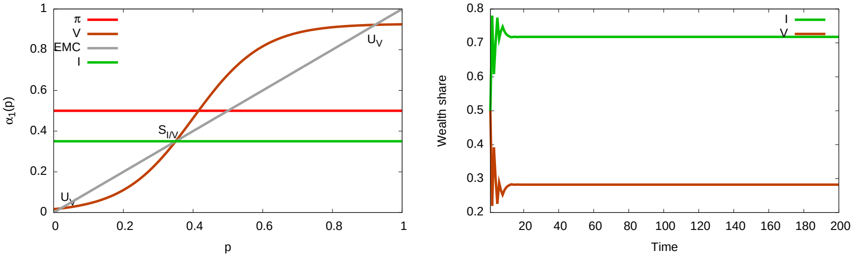
III > V





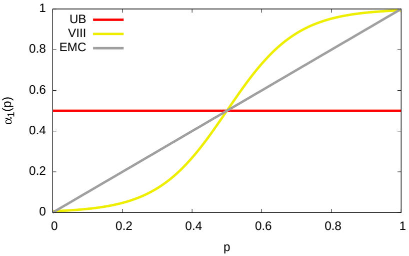
Ordering is not transitive

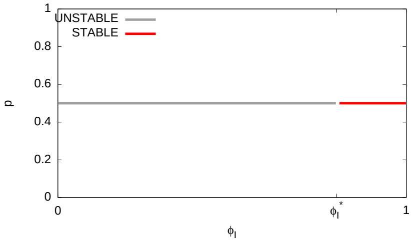
$V \sim I$



Does it exist a dominant strategy?

Yes, but not strictly





Same type of results holds with I agents, L memory lag, $S = K$ assets.
For x^* with $\phi^I = 1$ and $p^* = \alpha^I(p^*)$, eigenvalues are
$\Lambda = (\mu_1, ..., \mu_{I-1}, \lambda_{1,1}, ..., \lambda_{k,l}, \ldots, \lambda_{K-1,L})$, with

$$\mu_i = \prod_{k=1}^{K} \left(\frac{\alpha_k^i(p^*)}{\alpha_k^I(p^*)} \right)^{\pi_k} , \tag{5}$$

and, for a any given k, $\lambda_{k,l}$ one of the L solutions of the following equation

$$\lambda^L + \sum_{l=0}^{L-1} \lambda^l (\alpha_k^I)^{(L-1-l,k)} = 0\,, \tag{6}$$

where

$$(\alpha_k^I)^{(0,k)} = \left.\frac{\partial \alpha_k^I}{\partial p_k}\right|_{p^*} , \quad (\alpha_k^I)^{(l,k)} = \left.\frac{\partial \alpha_k^I}{\partial p_k^l}\right|_{p^*} \qquad l = 1, \ldots, L\,,\ k = 1, \ldots, K-1\,.$$

Conclusion

- Many fixed points, located on the Equilibrium Market Curve, whose local stability depends both on
  - Entropy w.r.t. dividend payment process
  - Price feedbacks being not too strong

$\Rightarrow$ No ordering relation based on market dominance can be established

$\Rightarrow$ Constant investment rule that minimize entropy $I_\pi(\alpha)$ is (locally) dominating all others.