Evolution and market behavior with endogenous investment rules

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Research questions

Consider a market for a risky asset and an ecology of investment strategies competing to gain superior returns. The open questions are:

⇒ which are the strategies surviving in the long run?
⇒ is it possible to establish an order relationship among them?
⇒ is a strategy dominating all the others?

Answers to these questions help to clarify specific issues (think of financial markets) as well as general issues ("as if" point).
Where do we stand?

On this issue

- Behavioral Finance (a survey is Barberis and Thaler, 2003)
  - Pros: Ecology of strategies behaviorally grounded
  - Cons: No wealth-driven strategy selection
  - Focus: Market biases

- HAM Finance (a survey is Hommes, 2006)
  - Pros: Focus on price feedbacks
  - Cons: No wealth-driven strategy selection (mostly CARA), deterministic
  - Focus: Stylized facts

- Evolutionary Finance (Kelly, 1956; Blume and Easley, 1992; a survey is Evstigneev, Hens, and Schenk-Hoppe, 2009)
  - Pros: Multi-asset stochastic general equilibrium framework
  - Cons: Absence of price feedbacks (no endogenous investment rules)
  - Focus: Market selection

⇒ Our approach: evolutionary finance with endogenous (price dependent) investment rules.
Framework

- Trading is repeated and occurs in discrete time
- Many assets in constant supply with uncertain dividends
- Market is complete
- Agents care about consumption, thus wealth
- A strategy is a portfolio of wealth fractions (CRRA)
- Walrasian market clearing
- Intertemporal budget constraint
- Market dynamics is formalized as a random dynamical system
A toy market

- Two states of the world, \( s = 1, 2 \), which occur with probability \( \pi \) and \( 1 - \pi \). Bernoulli process \( \omega = (\ldots, \omega_t, \ldots, \omega_0) \in \Omega \).
- Two (short-lived) Arrow's securities, \( k = 1, 2 \), paying \( D_{k,s} = \delta_{k,s} \).
- Fraction of consumption is constant and uniform, \( \alpha_0 = c \). All the rest is invested.
- Define normalized prices \( p_{s,t} = \frac{P_{s,t}}{W_t} \) so that \( p_{1,t} + p_{2,t} = 1 - \alpha_0, \forall t \).
- Two agents, \( i = 1, 2 \), with wealth fractions \( \phi_t \) and \( 1 - \phi_t \).
- Endogenous strategies with one memory lag, \( L = 1 \),
  - \( \alpha_{1,t} = \alpha_{1}^1(p_{1,t-1}) \) describes the portfolio choice of the first agent,
  - \( \alpha_{2,t} = \alpha_{2}^1(p_{1,t-1}) \) describes the portfolio choice of the second agent.
Evolutionary finance literature shows that, among constant investment rules, $\alpha_s^* = \pi_s$ dominates and

$$l_\pi(\alpha) = \sum_{s=1}^{S} \pi_s \log \left( \frac{\pi_s}{\alpha_s} \right)$$

can be used to establish an ordering relationship.
Strategy $i$ dominates strategy $j$, $i > j$, if

$$\forall \epsilon > 0, \quad \exists T \text{ s.t. } \text{Prob} \left\{ \frac{\phi^j_t}{\phi^i_t} < \epsilon, \quad \forall t > T \right\} = 1.$$
Two agents: the random dynamical system

Given $x_t = (\phi_t, \rho_t, q_t = \rho_{t-1})$, the state of our market at time $t$, the random dynamical system is the composition of the following maps

$$
\begin{align*}
\phi_{t+1} &= \begin{cases} 
\frac{\alpha_1(q_t)\phi_t}{\rho_t} & \text{with probability } \pi \\
\frac{(1-\alpha_0-\alpha_1(q_t))\phi_t}{1-\alpha_0-\rho_t} & \text{with probability } 1-\pi
\end{cases}, \\
\rho_{t+1} &= \alpha_1(\rho_t)\phi_{t+1} + \alpha_2(\rho_t)(1 - \phi_{t+1}), \\
q_{t+1} &= \rho_t.
\end{align*}
$$

That is, $x_{t+1} = f_{\pi}(x_t)$ with probability $\pi$ and $x_{t+1} = f_{1-\pi}(x_t)$ with probability $1-\pi$, depending on the realization of $\omega_t$. 
Fixed points

Definition

The state $x^* = (\phi^*, p^*, q^* = p^*)$ is a deterministic fixed point of the random dynamical system generated by the maps $f_\pi$ and $f_{1-\pi}$, that is, $\varphi(t, \omega, x) = \ldots f_\pi \circ \ldots \circ f_{1-\pi} \ldots$ if it holds

$$\varphi(t, \omega, x^*) = x^* \quad \forall \omega \in \Omega \quad (1)$$

or, in terms of the maps, if it holds both

$$f_\pi(x^*) = x^* \quad \text{and} \quad f_{1-\pi}(x^*) = x^* \quad (2)$$
Fixed points
In our toy market

Theorem

Fixed points of the random dynamical system that represents the toy market dynamics are given by

\[ x_1^* = (\phi^* = 1, p^* = \alpha_1(p^*), q^* = p^*) \]
\[ x_2^* = (\phi^* = 0, p^* = \alpha_2(p^*), q^* = p^*) \]
\[ x_{1/2}^* = (\phi^*, p^* = \alpha_1(p^*) = \alpha_2(p^*), q^* = p^*) \]
Fixed points on a plot: the Equilibrium Market Curve

\[ \alpha_1(p) \]

\[ E_1 \]

\[ E_2 \]

Bottazzi-Dindo (LEM, Sant’Anna, Pisa)
Local stability

Definition

A fixed point $x^*$ of the random dynamical system $\varphi(t, \omega, x)$ is called locally stable if $\lim_{t \to \infty} ||\varphi(t, \omega, x) - x^*|| \to 0$ for all $x$ in a neighborhood $U(\omega)$ of $x$ and for all $\omega \in \Omega$. 
Local stability
In our toy market

Theorem

Provided that the eigenvalues of the iterated map are inside the unit circle the deterministic fixed point is locally stable (use Multiplicative Ergodic Theorem and Local Hartman-Grobman Theorem). For fixed points of the type \((1, \alpha_1^1(p^*), p^*)\) eigenvalues are

\[
\mu = \exp(l_\pi(\alpha^1) - l_\pi(\alpha^2)) \quad \text{and} \quad \lambda = \left. \frac{\partial \alpha_1^1(p)}{\partial p} \right|_{p^*} \quad (3)
\]

and for fixed points of the type \((\phi^*, \alpha_1^1(p^*) = \alpha_2^2(p^*), p^*)\)

\[
\mu = 1 \quad \text{and} \quad \lambda = \phi^* \left. \frac{\partial \alpha_1^1(p)}{\partial p} \right|_{p^*} + (1 - \phi^*) \left. \frac{\partial \alpha_2^2(p)}{\partial p} \right|_{p^*} \quad (4)
\]
Local stability on the EMC plot
Ordering is **not** complete

Coexistence of stable equilibria

![Graphs showing coexistence of stable equilibria](image-url)
Ordering is not complete

Coexistence of stable equilibria
Ordering is not complete

Multiple unstable equilibria
Ordering is not complete
Multiple unstable equilibria
Ordering is not transitive

\( I > \text{III} > V \sim I \)
Ordering is not transitive

\[ I > III \]
Ordering is not transitive

\[ III > V \]
Ordering is not transitive

$V \sim I$

Wealth share

Time

Bottazzi-Dindo (LEM, Sant’Anna, Pisa)
Does it exist a dominant strategy?
Yes, but not strictly
Beyond toy market

Same type of results holds with \( I \) agents, \( L \) memory lag, \( S = K \) assets. For \( x^* \) with \( \phi^I = 1 \) and \( p^* = \alpha^I(p^*) \), eigenvalues are

\[
\Lambda = (\mu_1, \ldots, \mu_{I-1}, \lambda_1, \ldots, \lambda_k, \ldots, \lambda_{K-1,L}),
\]

with

\[
\mu_i = \prod_{k=1}^{K} \left( \frac{\alpha^l_k(p^*)}{\alpha^l_k(p^*)} \right)^{\pi_k}
\]  

and, for a any given \( k \), \( \lambda_{k,l} \) one of the \( L \) solutions of the following equation

\[
\lambda^L + \sum_{l=0}^{L-1} \lambda^l (\alpha^I_k)^{(L-1-l,k)} = 0,
\]

where

\[
(\alpha^l_k)^{(0,k)} = \left. \frac{\partial \alpha^l_k}{\partial p_k} \right|_{p^*}, \quad (\alpha^l_k)^{(l,k)} = \left. \frac{\partial \alpha^l_k}{\partial p_k} \right|_{p^*}
\]  

\[
I = 1, \ldots, L, \ k = 1, \ldots, K-1.
\]
Conclusion

- Many fixed points, located on the Equilibrium Market Curve, whose local stability depends both on
  - Entropy w.r.t. dividend payment process
  - Price feedbacks being not too strong

⇒ No ordering relation based on market dominance can be established

⇒ Constant investment rule that minimize entropy $I_{\pi}(\alpha)$ is (locally) dominating all others.