An Operational Measure of Riskiness

Sergiu Hart

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Sergiu Hart
Center for the Study of Rationality
Dept of Economics    Dept of Mathematics
The Hebrew University of Jerusalem

hart@huji.ac.il
http://www.ma.huji.ac.il/hart
Dean Foster and Sergiu Hart
"An Operational Measure of Riskiness" (2009)
*Journal of Political Economy*

www.ma.huji.ac.il/hart/abs/risk.html
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www.ma.huji.ac.il/hart/abs/risk.html

Dean Foster and Sergiu Hart
"A Reserve-Based Axiomatization of the Measure of Riskiness" (2008)

www.ma.huji.ac.il/hart/abs/risk-ax.html
Sergiu Hart
"A Simple Riskiness Order Leading to the Aumann–Serrano Index of Riskiness" (2008)
www.ma.huji.ac.il/hart/abs/risk-as.html
Sergiu Hart
"A Simple Riskiness Order Leading to the Aumann–Serrano Index of Riskiness" (2008)
www.ma.huji.ac.il/hart/abs/risk-as.html

Sergiu Hart
"Comparing Risks by Acceptance and Rejection" (2009)
www.ma.huji.ac.il/hart/abs/risk-u.html
I: Introduction
A gamble

\[ g = \begin{cases} \frac{1}{2} & +$120 \\ \frac{1}{2} & -$100 \end{cases} \]
A gamble

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\[ \mathbb{E}[g] = -$10 \]
A gamble

\[ g = \begin{array}{c}
\frac{1}{2} \\
\frac{1}{2}
\end{array} \begin{array}{c}
+ \$120 \\
- \$100
\end{array} \]

\[ \mathbb{E}[g] = \$10 \]

\begin{itemize}
\item[\bullet] **ACCEPT** \( g \) or **REJECT** \( g \) ?
\end{itemize}
A gamble

\[ g = \frac{1}{2} \cdot +$120 \quad \frac{1}{2} \cdot -$100 \]

\[ \mathbb{E}[g] = $10 \]

- **ACCEPT** \( g \) or **REJECT** \( g \)?

- What is the **RISK** in accepting \( g \)?
A gamble

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- **ACCEPT** \( g \) or **REJECT** \( g \)?
- What is the **RISK** in accepting \( g \)?
- What is the **RISKINESS** of \( g \)?
What is the **RISKINESS** of a gamble?
The Riskiness of a Gamble

- What is the **RISKINESS** of a gamble?
- Is there an **OBJECTIVE** way to measure the **RISKINESS** of a gamble?
What is the **RISKINESS** of a gamble?

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**OBJECTIVE** = depends only on the gamble, not on the decision-maker
The Riskiness of a Gamble

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**OBJECTIVE** measures:
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- **RETURN** = expectation (\( E[g] \))
- **SPREAD** = standard deviation (\( \sigma[g] \))
The Riskiness of a Gamble

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- **Riskiness** = ?
The Riskiness of a Gamble

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- **SPREAD** = standard deviation ( $\sigma[g]$ )
- **RISKINESS** = ?

( $\sigma$ ? )
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**RETURN** = expectation ( $E[g]$ )

**SPREAD** = standard deviation ( $\sigma[g]$ )

**RISKINESS** = ?

($\sigma$ ? **not** monotonic !)
Seeking a \textbf{MEASURE OF RISKINESS} that is:
The Riskiness of a Gamble

Seeking a **MEASURE OF RISKINESS** that is:

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- Seeking a **MEASURE OF RISKINESS** that is:
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  - Measured in the same **units** as the outcomes (**scale-invariant**)
The Riskiness of a Gamble

- Seeking a **MEASURE OF RISKINESS** that is:
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The Riskiness of a Gamble

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  [like $E, \sigma$]

- Measured in the same **units** as the outcomes (*scale-invariant*)  
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- **Monotonic** (with respect to stochastic dominance)  
  [?]
The Riskiness of a Gamble

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  - **Objective** (depends only on the distribution of the gamble) [like \( E, \sigma \)]
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  - **Simple** interpretation, formula [ ? ]
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- . . .
The Risk of Accepting a Gamble

\[ g = \begin{cases} \frac{1}{2} & +$120 \\ \frac{1}{2} & -$100 \end{cases} \]
The Risk of Accepting a Gamble

$g = \begin{cases} \frac{1}{2} & +$120 \\ \frac{1}{2} & -$100 \end{cases}$

Accepting the gamble $g$ when the wealth $W$ is:
The Risk of Accepting a Gamble

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1/2 & +$120 \\
1/2 & -$100 
\end{cases} \]

Accepting the gamble \( g \) when the wealth \( W \) is:

- \( W = \$100 \): very risky (bankruptcy)
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- \( W = $1000000 \): not risky
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The risk of accepting a gamble depends on the current wealth.
The Risk of Accepting a Gamble

$$g = \begin{cases} \frac{1}{2} & +$120 \\ \frac{1}{2} & -$100 \end{cases}$$

Accepting the gamble $g$ when the wealth $W$ is:

- $W = $100: very risky (bankruptcy)
- $W = $1000000: not risky

Is there a “cutoff point”? 

The risk of accepting a gamble depends on the current wealth.
Given a gamble $g$: 
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1. *Identify* the wealth levels where accepting the gamble $g$ is RISKY
The Measure of Riskiness

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1. *Identify* the wealth levels where accepting the gamble $g$ is **RISKY**

2. *Define* the **RISKINESS** of the gamble $g$ as:
Given a gamble $g$:

1. **Identify** the wealth levels where accepting the gamble $g$ is **RISKY**

2. **Define** the **RISKINESS** of the gamble $g$ as:

   *the CRITICAL WEALTH level below which accepting $g$ becomes **RISKY***
II: The Bankruptcy Model
A **gamble** is a real-valued random variable $g$. 
A *gambles* is a real-valued random variable $g$

Positive expectation: $E[g] > 0$
A *gamble* is a real-valued random variable $g$

- Positive expectation: $E[g] > 0$

- Some negative values: $P[g < 0] > 0$  
  (loss is possible)
A *gamble* is a real-valued random variable $g$

- Positive expectation: $\mathbb{E}[g] > 0$

- Some negative values: $\mathbb{P}[g < 0] > 0$
  (loss is possible)

- [technical] Finitely many values:
  $g$ takes the values $x_1, x_2, \ldots, x_m$
  with probabilities $p_1, p_2, \ldots, p_m$
The initial wealth is $W_1 > 0$
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At every period $t = 1, 2, \ldots$:
Gambles and Wealth

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Gambles and Wealth

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  - a gamble $g_t$ is offered
Gambles and Wealth

- The initial wealth is $W_1 > 0$
- At every period $t = 1, 2, \ldots$:
  - let $W_t > 0$ be the **CURRENT WEALTH**
  - a gamble $g_t$ is **OFFERED**
  - $g_t$ may be **ACCEPTED** or **REJECTED**
Gambles and Wealth

The initial wealth is $W_1 > 0$

At every period $t = 1, 2, \ldots$

- let $W_t > 0$ be the CURRENT WEALTH
- a gamble $g_t$ is OFFERED
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- if ACCEPTED then $W_{t+1} = W_t + g_t$
Gambles and Wealth

- The initial wealth is $W_1 > 0$
- At every period $t = 1, 2, \ldots$
  - let $W_t > 0$ be the current wealth
  - a gamble $g_t$ is offered
  - $g_t$ may be accepted or rejected
  - if accepted then $W_{t+1} = W_t + g_t$
  - if rejected then $W_{t+1} = W_t$
The initial wealth is $W_1 > 0$

At every period $t = 1, 2, \ldots$ :

- let $W_t > 0$ be the CURRENT WEALTH

- a gamble $g_t$ is OFFERED

$g_t$ may be ACCEPTED or REJECTED

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if REJECTED then $W_{t+1} = W_t$
Gambles

At every period \( t = 1, 2, \ldots \) :

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Gambles

At every period $t = 1, 2, \ldots$

- a gamble $g_t$ is OFFERED:

- the sequence $G = (g_1, g_2, \ldots, g_t, \ldots)$ is arbitrary
Gambles

At every period $t = 1, 2, \ldots$

- a gamble $g_t$ is **OFFERED**:
  - the sequence $G = (g_1, g_2, \ldots, g_t, \ldots)$ is *arbitrary*
  - $g_t$ may *depend on the past* wealths, gambles, decisions
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NOTE: not i.i.d., arbitrary dependence;
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    wealths, gambles, decisions

NOTE: not i.i.d., arbitrary dependence; non-Bayesian; “adversary”

[technical] \( G \) is finitely generated: there is a finite collection of gambles such that every \( g_t \) is a multiple of one of them
CRITICAL-WEALTH function $Q$:
Critical Wealth

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$Q : \{ \text{the set of gambles} \} \rightarrow [0, \infty]$
Critical Wealth

**CRITICAL-WEALTH** function $Q$:
- $Q : \{ \text{the set of gambles} \} \rightarrow [0, \infty]$ 
- $Q(g)$ depends only on the distribution of $g$
Critical Wealth

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- $Q(\lambda g) = \lambda Q(g)$ for $\lambda > 0$ (scaling)
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SIMPLE STRATEGY $s \equiv S_Q$:
Critical Wealth and Strategies

CRITICAL-WEALTH function $Q$:

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SIMPLE STRATEGY $s \equiv s_Q$:

- $s$ rejects the gamble $g$ at wealth $W$ when $W < Q(g)$
- $s$ accepts the gamble $g$ at wealth $W$ when $W \geq Q(g)$
$W_t = 0$
\[
\lim_{t \to \infty} W_t = 0
\]
NO-BANKRUPTCY:

\[ \{ \lim_{t \to \infty} W_t = 0 \} \text{ has probability 0} \]
A strategy GUARANTEES NO-BANKRUPTCY:

\[ \{ \lim_{t \to \infty} W_t = 0 \} \] has probability 0

for every \( G = (g_1, g_2, \ldots, g_t, \ldots) \)
and every \( W_1 > 0 \)
Main Result
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For every gamble $g$ there exists a unique positive number $R(g)$ such that:
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A simple strategy $s_Q$

with critical-wealth function $Q$
For every gamble $g$ there exists a unique positive number $R(g)$ such that:

A simple strategy $s_Q$ with critical-wealth function $Q$ guarantees no-bankruptcy
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for every gamble $g$. 
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Main Result

A simple strategy $s_Q$ guarantees no-bankruptcy if and only if $Q(g) \geq R(g)$ for every $g$. 
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**Diagram:**

- Arrow from $R(g)$ to $W$.
A simple strategy $s_Q$ guarantees no-bankruptcy if and only if $Q(g) \geq R(g)$ for every $g$. 
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Examples of such strategies:
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- $Q(g) = \infty$ for all $g$: Always reject.
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- $Q(g) = \infty$ for all $g$: Always reject
- $Q(g) = R(g)$ for all $g$: Reject $\iff W < R(g)$
Main Result

A simple strategy $s_Q$ guarantees no-bankruptcy if and only if $Q(g) \geq R(g)$ for every $g$

Examples of such strategies:

- $Q(g) = \infty$ for all $g$: Always reject
- $Q(g) = R(g)$ for all $g$: Reject $\iff W < R(g)$
- Anything in between
A simple strategy $s_Q$ guarantees no-bankruptcy if and only if $Q(g) \geq R(g)$ for every $g$.
A simple strategy $s_Q$ guarantees no-bankruptcy if and only if $Q(g) \geq R(g)$ for every $g$.

A simple strategy $s$ guarantees no-bankruptcy if and only if $s$ rejects $g$ when $W < R(g)$.
A simple strategy $s_Q$ guarantees no-bankruptcy if and only if $Q(g) \geq R(g)$ for every $g$.

A simple strategy $s$ guarantees no-bankruptcy if and only if $s$ rejects $g$ when $W < R(g)$. 

\[ \text{REJECT} \quad \leftrightarrow \quad \text{ACCEPT} \]
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Main Result

\[ R(g) = \text{the RISKINESS of } g \]
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No-bankruptcy is guaranteed

if and only if

One never accepts gambles whose RISKINESS exceeds the current wealth
Main Result

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No-bankruptcy is guaranteed if and only if

One never accepts gambles whose RISKINESS exceeds the current wealth

\[ \text{RISKINESS } \sim \text{“reserve”} \]
Moreover, for every gamble \( g \), its \textbf{RISKINESS} \( R(g) \) is the unique solution \( R > 0 \) of the equation.
Moreover, for every gamble \( g \),

its RISKINESS \( R(g) \)

is the unique solution \( R > 0 \) of the equation

\[
E \left[ \log \left( 1 + \frac{1}{R} g \right) \right] = 0
\]
The Riskiness of Some Gambles

\[ g = \frac{1}{2} X + \frac{1}{2} - 100 \]
The Riskiness of Some Gambles

\[ g = \left( \frac{1}{2} \frac{1}{2} \right) + X - \$100 \]

<table>
<thead>
<tr>
<th>X</th>
<th>E [g]</th>
<th>R(g)</th>
</tr>
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<tbody>
<tr>
<td>$120</td>
<td>$10</td>
<td>$600</td>
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The Riskiness of Some Gambles

\[ g = \frac{1}{2}X + \frac{1}{2}X - 100 \]

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The Riskiness of Some Gambles

\[ g = \frac{1}{2} X + \frac{1}{2} (X - 100) \]

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<td>$300</td>
<td>$100</td>
<td>$150</td>
</tr>
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\[ g = \frac{1}{2} X + \frac{1}{2} (X - 100) \]

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<td>$50</td>
<td>$200</td>
</tr>
<tr>
<td>$120</td>
<td>$10</td>
<td>$600</td>
</tr>
<tr>
<td>$105</td>
<td>$2.5</td>
<td>$2100</td>
</tr>
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The Riskiness of Some Gambles

\[ g = \frac{1}{2} - $100 + X \frac{1}{2} \]

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<td>$120</td>
<td>$10</td>
<td>$600</td>
</tr>
<tr>
<td>$105</td>
<td>$2.5</td>
<td>$2100</td>
</tr>
<tr>
<td>$102</td>
<td>$1</td>
<td>$5100</td>
</tr>
</tbody>
</table>
The Riskiness of Some Gambles

\[ g = \begin{cases} p & \text{+ $105} \\ 1-p & \text{- $100} \end{cases} \]
The Riskiness of Some Gambles

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<table>
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<th>( p )</th>
<th>E ( [g] )</th>
<th>R( (g) )</th>
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<tbody>
<tr>
<td>0.5</td>
<td>$2.5</td>
<td>$2100</td>
</tr>
<tr>
<td>0.6</td>
<td>$23</td>
<td>$235.23</td>
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<tr>
<td>0.8</td>
<td>$64</td>
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</tr>
<tr>
<td>0.9</td>
<td>$84.5</td>
<td>$100.16</td>
</tr>
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The Riskiness Measure R
The **Riskiness Measure R**

- is objective and universal
The **Riskiness Measure R**

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- is independent of utilities, risk aversion, ...
The **Riskiness Measure** $R$

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- is independent of utilities, risk aversion, ...
- has a clear operational interpretation
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- is defined for each gamble separately
The Riskiness Measure $R$

- is objective and universal
- is independent of utilities, risk aversion, ...
- has a clear operational interpretation
- is defined for each gamble separately
- is normalized (unit = $\)
The *Riskiness Measure* \( R \)

- is objective and universal
- is independent of utilities, risk aversion, ...
- has a clear operational interpretation
- is defined for each gamble separately
- is normalized (unit = $)

(... more to follow ...)
III: The Shares Model
May take *any proportion* of the offered $g_t$
(i.e., $\alpha_t g_t$ for $\alpha_t \geq 0$, instead of $\alpha_t = 0, 1$)
The Shares Model

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**Theorem** Let $s_Q$ be a *simple shares strategy*. 

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- **Theorem** Let $s_Q$ be a simple shares strategy.
  - $\lim_{t \to \infty} W_t = \infty$ (a.s.) for every process
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  - $\lim_{t \to \infty} W_t = 0$ (a.s.) for some process
    $\Leftrightarrow$ $Q(g) < R(g)$ for some gamble $g$. 

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The Shares Model

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The Shares Model

Therefore we may replace **NO-BANKRUPTCY** with other criteria, such as:

- **NO-LOSS**: \( \lim \inf_{t} W_{t} \geq W_{1} \)
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The Shares Model

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... 

**Corollary** A simple shares strategy \( S_Q \) guarantees **NO-LOSS**

- if \( Q(g) > R(g) \) for every gamble \( g \)
- only if \( Q(g) \geq R(g) \) for every gamble \( g \)
Example

\[ g = \frac{1}{2} + \frac{1}{2} \]

\[ \frac{1}{2} + \$120 \]

\[ \frac{1}{2} - \$100 \]
Consider an i.i.d. sequence $(g_t)_t$ with $g_t \sim g$
Example: \( Q(g) = \$200 \)

\[
g = \begin{cases} 
\frac{1}{2} + \$120 \\
\frac{1}{2} - \$100
\end{cases}
\]

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\[
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\]

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- At time \(t\) the gamble \((W_t/200)g_t\) is taken
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$$W_{t+1} = W_t + \left( \frac{W_t}{200} \right) g_t = W_t \left( 1 + \frac{g_t}{200} \right)$$
Example: $Q(g) = \$200$

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$W_{t+1} = W_t \left(1 + \frac{g_t}{200}\right)$
Example: \( Q(g) = \$200 \)

\[
\frac{g}{200} = W_{t+1} = W_t \left( 1 + \frac{g_t}{200} \right)
\]
Example: $Q(g) = $200

\[ \frac{g}{200} = \frac{1}{2} + \frac{120}{200} - \frac{1}{2} - \frac{100}{200} \]

\[ W_{t+1} = W_t \left( 1 + \frac{g_t}{200} \right) \]
Example: $Q(g) = $200

\[
\frac{g}{200} = \frac{1/2}{1/2} + \frac{120}{200} = +60\% \\
\frac{1/2}{1/2} - \frac{100}{200} = -50\%
\]

\[
W_{t+1} = W_t \left(1 + \frac{g_t}{200}\right)
\]
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These are the *relative returns* from accepting $g$ at $W = $200

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\frac{1/2}{200} - \frac{100}{200} = -50%
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These relative returns are obtained every period i.i.d.:

\[W_{t+1} = W_t \left( 1 + \frac{g_t}{200} \right)\]
Example: \( Q(g) = \$200 \)
Example: $Q(g) = \$200$

Proposition. $W_t \rightarrow 0 \text{ (a.s.)}$
Example: \( Q(g) = \$200 \)

\[
\begin{array}{c}
\frac{1}{2} \\
\downarrow \\
\frac{1}{2} \\
\end{array}
\quad + \quad 60\% \\
\quad - \quad 50\%
\]

Proposition. \( W_t \rightarrow 0 \) (a.s.)

Proof.
Example: \( Q(g) = \$200 \)

\[
\begin{align*}
1/2 & \quad + \quad 60\% \quad W_{t+1} = W_t \times 1.6 \\
1/2 & \quad - \quad 50\%
\end{align*}
\]

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\left(1/2\right) + 60\% & \quad W_{t+1} = W_t \times 1.6 \\
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Proof. The Law of Large Numbers \( \Rightarrow \)

- about half the days wealth is multiplied by 1.6
- about half the days wealth is multiplied by 0.5
Example: $Q(g) = $200

\[
\begin{align*}
\begin{array}{c}
\text{1/2} \\
\text{1/2}
\end{array}
\end{align*}
\begin{align*}
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Proof. The Law of Large Numbers $\Rightarrow$
- about half the days wealth is multiplied by 1.6
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$\Rightarrow$ A factor of $\approx \sqrt{1.6 \cdot 0.5}$ per period
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Proposition. $W_t \to 0$ (a.s.)

Proof. The Law of Large Numbers $\Rightarrow$

- about half the days wealth is multiplied by 1.6
- about half the days wealth is multiplied by 0.5

$\Rightarrow$ A factor of $\approx \sqrt{1.6 \cdot 0.5} < 1$ per period
Example: \( Q(g) = \$200 \)

\[
\begin{align*}
\frac{1}{2} + 60\% \quad W_{t+1} &= W_t \times 1.6 \\
\frac{1}{2} - 50\% \quad W_{t+1} &= W_t \times 0.5
\end{align*}
\]

**Proposition.** \( W_t \to 0 \) (a.s.)

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\( \Rightarrow \) \( W_t \to 0 \) (a.s.)
Example: $Q(g) = \$1000$
Example: \( Q(g) = $1000 \)

\[
W_{t+1} = W_t \left(1 + \frac{g_t}{1000}\right)
\]
Example: \( Q(g) = \$1000 \)

\[ \frac{g}{1000} = \]

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\[
\frac{g}{1000} = \frac{1}{2} + \frac{120}{1000} - \frac{100}{1000}
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Example: \( Q(g) = \$1000 \)

\[
\frac{g}{1000} = \left( \frac{1}{2} \right) + \frac{120}{1000} = +12\%
\]

\[
\frac{1}{2} - \frac{100}{1000} = -10\%
\]

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W_{t+1} = W_t \left( 1 + \frac{g_t}{1000} \right)
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$$\frac{g}{1000} = \frac{1}{2} + \frac{120}{1000} = +12\%$$

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These are the relative returns from accepting $g$ at $W = $1000

$$W_{t+1} = W_t \left(1 + \frac{g_t}{1000}\right)$$
Example: \( Q(g) = \$1000 \)

\[
\frac{g}{1000} = \frac{1}{2} \sqrt{1 + \frac{120}{1000}} = +12\%
\]

\[
- \frac{100}{1000} = -10\%
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These are the *relative returns* from accepting \( g \) at \( W = \$1000 \).

These relative returns are obtained every period i.i.d.:

\[
W_{t+1} = W_t \left(1 + \frac{g_t}{1000}\right)
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Example: \( Q(g) = $1000 \)
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Proposition. $W_t \to \infty$ (a.s.)
Example: \( Q(g) = \$1000 \)

Proposition. \( W_t \to \infty \) (a.s.)

Proof.
Example: $Q(g) = \$1000$

\[ \begin{align*} 
\left( \frac{1}{2} + 12\% \right) & \quad W_{t+1} = W_t \times 1.12 \\
\left( \frac{1}{2} - 10\% \right) & 
\end{align*} \]

Proposition. $W_t \rightarrow \infty$ (a.s.)

Proof.
Example: $Q(g) = $1000

\[
\begin{align*}
\frac{1}{2} + 12\% & \quad W_{t+1} = W_t \times 1.12 \\
\frac{1}{2} - 10\% & \quad W_{t+1} = W_t \times 0.90
\end{align*}
\]

Proposition. $W_t \to \infty \ (a.s.)$

Proof.
Example: $Q(g) = $1000

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\frac{1}{2} & + 12\% \quad W_{t+1} = W_t \times 1.12 \\
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Proof. The Law of Large Numbers $\Rightarrow$

- $\approx$ half the days wealth is multiplied by $1.12$
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Example: $Q(g) = $1000

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Example: \( Q(g) = \$1000 \)

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- \( \approx \) half the days wealth is multiplied by 1.12
- \( \approx \) half the days wealth is multiplied by 0.90

\( \Rightarrow \) A factor of \( \approx \sqrt{1.12 \cdot 0.90} > 1 \) per period
Example: $Q(g) = \$1000$

\[ W_{t+1} = W_t \times 1.12 \]
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Example: Riskiness $R(g) =$?
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$1 + \frac{g}{600} =$
Example: Riskiness \( R(g) = ? \)

\[
1 + \frac{g}{600} = \frac{1/2 \cdot 720}{600} = \frac{1/2 \cdot 500}{600}
\]
Example: Riskiness $R(g) = ?$

\[
1 + \frac{g}{600} = \frac{1/2}{1/2} \frac{720}{600} = \frac{6}{5} \quad \frac{1/2}{1/2} \frac{500}{600} = \frac{5}{6}
\]
Example: Riskiness $R(g) =$?

\[
1 + \frac{g}{600} = \sqrt[1/2]{\frac{720}{600}} = \frac{6}{5}
\]

\[
\sqrt[1/2]{\frac{500}{600}} = \frac{5}{6}
\]

$\Rightarrow$ Factor of $\sqrt{\frac{6}{5} \cdot \frac{5}{6}} = 1$ per period
Example: Riskiness $R(g) = ?$

\[
1 + \frac{g}{600} = \sqrt{\frac{720}{600}} = \frac{6}{5}
\]

\[
1/2 \quad \frac{500}{600} = \frac{5}{6}
\]

$\Rightarrow$ Factor of $\sqrt{\frac{6}{5} \cdot \frac{5}{6}} = 1$ per period

$\Leftrightarrow$ $E \left[ \log \left( 1 + \frac{1}{600} g \right) \right] = 0$
Example: Riskiness \( R(g) = $600 \)

\[
1 + \frac{g}{600} = \sqrt{\frac{720}{600}} = \frac{6}{5}
\]

\[
1/2
\]

\[
500/600 = \frac{5}{6}
\]

\[
\Rightarrow \text{Factor of } \sqrt{\frac{6}{5} \cdot \frac{5}{6}} = 1 \text{ per period}
\]

\[
\Leftrightarrow E \left[ \log \left( 1 + \frac{1}{600} g \right) \right] = 0
\]

The **RISKINESS** of the gamble \( g \) is

\[
R(g) = $600
\]
The critical wealth level = $600

The RISKINESS of the gamble $g$ is $R(g) = $600
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Accepting the gamble $g$ when the wealth is $W < $600 gives "bad" returns
(a regime where $W_t \to 0$ a.s.)

The **RISKINESS** of the gamble $g$ is

$$R(g) = $600$$
The critical wealth level = $600

- Accepting the gamble $g$ when the wealth is $W < $600 gives “bad” returns (a regime where $W_t \to 0$ a.s.)

- Accepting the gamble $g$ when the wealth is $W > $600 gives “good” returns: (a regime where $W_t \to \infty$ a.s.)

The **RISKINESS** of the gamble $g$ is

$$R(g) = $600$$
Up to now: limit as $t \to \infty$
Finite Time

- Up to now: limit as $t \to \infty$
- \textbf{FINITE $t$}: the distribution of wealth is quite different in the two regimes
Finite Time

- Up to now: limit as $t \to \infty$
- **FINITE $t$:** the distribution of wealth is quite different in the two regimes
- Example: Probability of no-loss after $t$ periods

<table>
<thead>
<tr>
<th>$t$</th>
<th>$Q(g)$</th>
<th>$P[W_{t+1} \geq W_1]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td></td>
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<tbody>
<tr>
<td>100</td>
<td>$200$</td>
<td>2.7%</td>
</tr>
<tr>
<td>100</td>
<td>$1000$</td>
<td>64%</td>
</tr>
<tr>
<td>1000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td></td>
<td></td>
</tr>
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</table>
Finite Time

Up to now: limit as $t \to \infty$

FINITE $t$: the distribution of wealth is quite different in the two regimes

Example: Probability of no-loss after $t$ periods

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<td>$1000$</td>
<td>64%</td>
</tr>
<tr>
<td>1000</td>
<td>$200$</td>
<td>$10^{-7}$%</td>
</tr>
<tr>
<td>1000</td>
<td>$1000$</td>
<td>87%</td>
</tr>
</tbody>
</table>
Finite Time

- Up to now: limit as $t \to \infty$
- **FINITE $t$:** the distribution of wealth is quite different in the two regimes
- Example: $\text{MED} := \text{Median of } \frac{W_{t+1}}{W_1}$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$Q(g)$</th>
<th>$P[W_{t+1} \geq W_1]$</th>
<th>MED</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>$200$</td>
<td>2.7%</td>
<td>0.0014%</td>
</tr>
<tr>
<td>100</td>
<td>$1000$</td>
<td>64%</td>
<td>148%</td>
</tr>
<tr>
<td>1000</td>
<td>$200$</td>
<td>$10^{-7}$%</td>
<td>$10^{-46}$%</td>
</tr>
<tr>
<td>1000</td>
<td>$1000$</td>
<td>87%</td>
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Properties of $R$

- **Homogeneity**: $R(\lambda g) = \lambda R(g)$ for $\lambda > 0$
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- **Subadditivity**: $R(g + h) \leq R(g) + R(h)$
Properties of $R$

- **Homogeneity**: $R(\lambda g) = \lambda R(g)$ for $\lambda > 0$
- **Subadditivity**: $R(g + h) \leq R(g) + R(h)$
- **Convexity**: For $0 \leq \lambda \leq 1$
  \[ R(\lambda g + (1 - \lambda)h) \leq \lambda R(g) + (1 - \lambda)R(h) \]
Properties of $R$

- **Homogeneity**: $R(\lambda g) = \lambda R(g)$ for $\lambda > 0$
- **Subadditivity**: $R(g + h) \leq R(g) + R(h)$
- **Convexity**: For $0 \leq \lambda \leq 1$
  \[ R(\lambda g + (1 - \lambda)h) \leq \lambda R(g) + (1 - \lambda)R(h) \]
- **First order stochastic dominance**: If $g \prec_{st1} h$ then $R(g) > R(h)$
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**Properties of** $\mathbf{R}$

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- **Subadditivity**: $\mathbf{R}(g + h) \leq \mathbf{R}(g) + \mathbf{R}(h)$
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... 

...
Properties of $R$

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- **First order stochastic dominance**: If $g \prec_{st1} h$ then $R(g) > R(h)$
- **Second order stochastic dominance**: If $g \prec_{st2} h$ then $R(g) > R(h)$

...
Utility function $u(x)$
Utility function $u(x)$:

- Accept $g$ at $W$ if and only if

$$E[u(W + g)] \geq u(W)$$
Utility function $u(x)$:

Accept $g$ at $W$ if and only if

$$E[u(W + g)] \geq u(W)$$

**LOG UTILITY:**

$$u(x) = \log(x)$$
Utility function $u(x)$:
- Accept $g$ at $W$ if and only if
  $$E[u(W + g)] \geq u(W)$$

LOG UTILITY:

$$u(x) = \log(x)$$

- Constant Arrow–Pratt Relative Risk Aversion coefficient $= 1$ (CRRA-1)
The Riskiness Measure $R$

$$E \left[ \log \left( 1 + \frac{1}{R(g)} g \right) \right] = 0$$
The Riskiness Measure $R$

\[
E \left[ \log \left( 1 + \frac{1}{R(g)} g \right) \right] = 0
\]

$\iff$

\[
E \left[ \log (R(g) + g) \right] = \log(R(g))
\]
The Riskiness Measure $R$

\[ E \left[ \log \left( 1 + \frac{1}{R(g)} g \right) \right] = 0 \]

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\[ \iff \]

LOG UTILITY rejects $g$ when $W < R(g)$

LOG UTILITY accepts $g$ when $W \geq R(g)$
The Riskiness Measure $R$

$$E \left[ \log \left( 1 + \frac{1}{R(g)} \right) \right] = 0$$

$\iff$

$$E \left[ \log (R(g) + g) \right] = \log(R(g))$$

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LOG UTILITY rejects $g$ if and only if $W < R(g)$
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No-bankruptcy

LOG UTILITY rejects $g$ if and only if $W < R(g)$

OUR RESULT:
LOG UTILITY rejects $g$ if and only if $W < R(g)$

OUR RESULT:

No-bankruptcy is guaranteed

$\iff$ reject when $W < R(g)$
No-bankruptcy

LOG UTILITY rejects $g$ if and only if $W < R(g)$

OUR RESULT:

No-bankruptcy is guaranteed

$\iff$ reject when $W < R(g)$

$\iff$ reject at least as much as LOG UTILITY
No-bankruptcy

LOG UTILITY rejects $g$ if and only if $W < R(g)$

OUR RESULT:

No-bankruptcy is guaranteed

$\iff$ reject when $W < R(g)$

$\iff$ reject at least as much as LOG UTILITY

LOG UTILITY $\iff$ relative risk aversion $\equiv 1$
No-bankruptcy

LOG UTILITY rejects $g$ if and only if $W < R(g)$

OUR RESULT:

No-bankruptcy is guaranteed

$\iff$ reject when $W < R(g)$

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$\approx$ relative risk aversion $\geq 1$

LOG UTILITY $\iff$ relative risk aversion $\equiv 1$
LOG UTILITY rejects $g$ if and only if $W < R(g)$

**OUR RESULT:**

No-bankruptcy is guaranteed

$\iff$ reject when $W < R(g)$

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$\approx$ relative risk aversion $\geq 1$

LOG UTILITY $\iff$ relative risk aversion $\equiv 1$
IV: Reserve
Every gamble $g$ has a **RESERVE** $Q(g) > 0$
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**DISTRIBUTION**: If $g$ and $h$ have the same distribution then $Q(g) = Q(h)$
Every gamble $g$ has a **RESERVE** $Q(g) > 0$

- **DISTRIBUTION**: If $g$ and $h$ have the same distribution then $Q(g) = Q(h)$

- **SCALING**: $Q(\lambda g) = \lambda Q(g)$ for every $\lambda > 0$
Reserve: Axioms

Every gamble \( g \) has a **RESERVE** \( Q(g) > 0 \)

- **DISTRIBUTION**: If \( g \) and \( h \) have the same distribution then \( Q(g) = Q(h) \)

- **SCALING**: 
  \( Q(\lambda g) = \lambda Q(g) \) for every \( \lambda > 0 \)

- **MONOTONICITY**: 
  If \( g \leq h \) and \( g \neq h \) then \( Q(g) > Q(h) \)
Every gamble $g$ has a **RESERVE** $Q(g) > 0$

**DISTRIBUTION:** If $g$ and $h$ have the same distribution then $Q(g) = Q(h)$

**SCALING:**
$Q(\lambda g) = \lambda Q(g)$ for every $\lambda > 0$

**MONOTONICITY:**
If $g \leq h$ and $g \neq h$ then $Q(g) > Q(h)$

**COMPOUND GAMBLE**
COMPOUND GAMBLE:
COMPOUND GAMBLE:

\[ +$200 \]

\[ -$100 \]
COMPOUND GAMBLE:

+$200$

−$100$

Notation: $Q$
COMPOUND GAMBLE:

\[ +$200 \]

\[ -$100 \]

\[ -$500 \]

Notation: \( Q \)
COMPOUND GAMBLE:

+$200 + $700

$500

−$100

Notation: $Q$
Reserve: Axioms

COMPOUND GAMBLE:

\[ +\$200 \quad +\quad \$700 \]
\[ \quad +\$500 \quad -\$100 \quad +\quad \$400 \]

Notation: \( Q \)
COMPOUND GAMBLE:

\[ +$200 + $700 \]
\[ -$100 + $400 \]
\[ \Rightarrow $500 \]

Notation: \( Q \)
COMPOUND GAMBLE:

$+$200 $+$700

$-$100 $+$400

$500$ $500$

$g, h_1, h_2, \ldots$ independent gambles
**COMPOUND GAMBLE:**

\[ (+\$200) + (+\$700) = +\$500 \]

\[ (-\$100) + (+\$400) = +\$300 \]

\[ \Rightarrow +\$500 \]

- \( g, h_1, h_2, \ldots \) independent gambles
- \( f = g + \sum_i 1_{[g=x_i]} h_i \)
Reserve: Axioms

COMPOUND GAMBLE:

\[ (+$200) + (+$700) \]
\[ (-$100) + (+$400) \]
\[ \Rightarrow +$500 \]

- \(g, h_1, h_2, \ldots\) independent gambles
- \(f = g + \sum_i 1_{[g=x_i]} h_i\)
- for every \(i\): \(Q(h_i) = Q(g) + x_i\)
Reserve: Axioms

COMPOUND GAMBLE:

\[ g, h_1, h_2, \ldots \text{ independent gambles} \]

\[ f = g + \sum_i 1_{[g=x_i]} h_i \]

for every \( i \):

\[ Q(h_i) = Q(g) + x_i \]

\[ \Rightarrow Q(f) = Q(g) \]
Reserve: Axioms

COMPOUND GAMBLE:

\[ \begin{align*}
+\$200 & + \$700 \\
-\$100 &
\end{align*} \]

\$500

\[ \Rightarrow \$500 \]

- \[g, h_1, h_2, \ldots\] independent gambles

- \[f = g + \sum_i 1_{[g=x_i]} h_i\]

- for every \(i\): \(Q(h_i) = Q(g) + x_i\)

\[ \Rightarrow Q(f) = Q(g) \]
COMPOUND GAMBLE:

+$200$

$500$

$-100$

$g, h_1, h_2, ...$ independent gambles

$f = g + \sum_i 1_{[g=x_i]} h_i$

for every $i$: $Q(h_i) = Q(g) + x_i$

$\Rightarrow Q(f) = Q(g)$
Reserve: Axioms

**COMPOUND GAMBLE:**

- $+200$ + $700$ = $500$
- $-100$ + $400$ = $500$

- $g, h_1, h_2, ...$ independent gambles
- $f = g + \sum_i 1_{[g=x_i]} h_i$
- for every $i$: $Q(h_i) = Q(g) + x_i$
- $Q(f) = Q(g)$
THEOREM

The \textit{minimal} reserve function $Q$ that satisfies the four axioms DISTRIBUTION, SCALING, MONOTONICITY, COMPOUND GAMBLE
THEOREM

The *minimal* reserve function $Q$ that satisfies the four axioms DISTRIBUTION, SCALING, MONOTONICITY, COMPOUND GAMBLE

*is the riskiness measure* $R$
THEOREM

The *minimal reserve* function $Q$ that satisfies the four axioms 
DISTRIBUTION, SCALING, 
MONOTONICITY, COMPOUND GAMBLE 
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THEOREM

The minimal reserve function $Q$ that satisfies the four axioms DISTRIBUTION, SCALING, MONOTONICITY, COMPOUND GAMBLE is the riskiness measure $R$.

$Q = R$ satisfies the four axioms
THEOREM

The \textit{minimal reserve} function $Q$ that satisfies the four axioms
DISTRIBUTION, SCALING, MONOTONICITY, COMPOUND GAMBLE

\textit{is the riskiness measure} $R$

- $Q = R$ satisfies the four axioms
- If $Q \neq R$ satisfies the four axioms then $Q(g) > R(g)$ for every gamble $g$
**Critical Wealth**

- **CRRA(\(\gamma\)):** Utility function \(u_\gamma\) with constant relative risk aversion \(= \gamma\)
CRRA(\(\gamma\)): Utility function \(u_\gamma\) with constant relative risk aversion = \(\gamma\)

- \(u_\gamma(x) = -x^{-(\gamma-1)}\) for \(\gamma > 1\)
- \(u_\gamma(x) = \log(x)\) for \(\gamma = 1\)
- \(u_\gamma(x) = x^{1-\gamma}\) for \(0 < \gamma < 1\)
**Critical Wealth**

- **CRRA(γ):** Utility function $u_\gamma$ with constant relative risk aversion $= \gamma$

- **γ-CRITICAL WEALTH $R_\gamma(g)$** of the gamble $g$:
CRRA(γ): Utility function $u_\gamma$ with constant relative risk aversion $= \gamma$

γ-CRITICAL WEALTH $R_\gamma(g)$ of the gamble $g$:

$$E \left[ u_\gamma(R_\gamma(g) + g) \right] = u_\gamma(R_\gamma(g))$$
**Critical Wealth**

- **CRRA(\(\gamma\))**: Utility function \(u_\gamma\) with constant relative risk aversion \(= \gamma\)

- **\(\gamma\)-CRITICAL WEALTH \(R_\gamma(g)\)** of the gamble \(g\):

\[
E [u_\gamma(R_\gamma(g) + g)] = u_\gamma(R_\gamma(g))
\]

\(\text{CRRA(\(\gamma\)) accepts } g \text{ at } W \iff W \geq R_\gamma(g)\)
Critical Wealth

- **CRRA(γ)**: Utility function \( u_\gamma \) with constant relative risk aversion = \( \gamma \)

- **γ-CRITICAL WEALTH** \( R_\gamma(g) \) of the gamble \( g \):
  \[
  \mathbb{E}[u_\gamma(R_\gamma(g) + g)] = u_\gamma(R_\gamma(g))
  \]
  CRRA(γ) accepts \( g \) at \( W \Leftrightarrow W \geq R_\gamma(g) \)

- \( R_1(g) = R(g) \) (for \( \gamma = 1 \): \( u_1 = \log \))
Critical Wealth

- **CRRA(γ):** Utility function $u_\gamma$ with constant relative risk aversion $= \gamma$

- **γ-CRITICAL WEALTH $R_\gamma(g)$** of the gamble $g$:

  $E [u_\gamma(R_\gamma(g) + g)] = u_\gamma(R_\gamma(g))$

  **CRRA(γ) accepts $g$ at** $W \iff W \geq R_\gamma(g)$

- $R_1(g) = R(g)$ (for $\gamma = 1$: $u_1 = \log$)

- $R_\gamma(g)$ increases with $\gamma$
THEOREM
THEOREM

The reserve function $Q$ satisfies the four axioms
THEOREM

The reserve function $Q$ satisfies the four axioms if and only if

$$Q = R_\gamma \text{ for some } \gamma \geq 1$$
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The reserve function $Q$ satisfies the four axioms if and only if

$Q = R_\gamma$ for some $\gamma \geq 1$

$\Rightarrow Q \geq R_1 = R$
THEOREM

The reserve function \( Q \) satisfies the four axioms if and only if

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THE MINIMAL RESERVE

= the critical wealth \( R_1 \) for CRRA(1)

= THE RISKINESS MEASURE \( R \)
THEOREM
The reserve function $Q$ satisfies the four axioms if and only if $Q = R_\gamma$ for some $\gamma \geq 1$.

$\Rightarrow Q \geq R_1 = R$

THE MINIMAL RESERVE
= the critical wealth $R_1$ for CRRA(1)
= THE RISKINESS MEASURE $R$
V: Connections
Index of Riskiness $Q$
Index of Riskiness $\mathcal{Q}$

**Duality:** For gambles $g, h$ and agents $u, v$
Index of Riskiness $Q$

**Duality:** For gambles $g$, $h$ and agents $u$, $v$

If

- $u$ is uniformly more risk-averse than $v$
- $u$ accepts $g$ at wealth $W$
- $Q(g) > Q(h)$
Index of Riskiness $Q$

**Duality:** For gambles $g, h$ and agents $u, v$

If
- $u$ is uniformly more risk-averse than $v$
- $u$ accepts $g$ at wealth $W$
- $Q(g) > Q(h)$

Then
- $v$ accepts $h$ at wealth $W$
Index of Riskiness $Q$

**Duality:** For gambles $g, h$ and agents $u, v$

If

- $u$ is uniformly more risk-averse than $v$
- $u$ accepts $g$ at wealth $W$
- $Q(g) > Q(h)$

Then

- $v$ accepts $h$ at wealth $W$

**Homogeneity:** $Q(\lambda g) = \lambda Q(g)$ for $\lambda > 0$
For each gamble $g$: 
For each gamble $g$:

Let $\alpha^* \equiv \alpha^*(g)$ be the Arrow–Pratt coefficient of absolute risk-aversion of that agent $u(x) = -\exp(-\alpha^*x)$ with constant absolute risk aversion (CARA) who is indifferent between accepting and rejecting $g$. 
For each gamble $g$:

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- Let $R^{AS}(g) = 1/\alpha^*$.
For each gamble $g$:

- Let $\alpha^* \equiv \alpha^*(g)$ be the Arrow–Pratt coefficient of absolute risk-aversion of that agent $u(x) = -\exp(-\alpha^* x)$ with constant absolute risk aversion (CARA) who is indifferent between accepting and rejecting $g$.

- Let $R^{AS}(g) = 1/\alpha^*$

---

$R^{AS}(g)$ is the unique solution $R > 0$ of

$$
E \left[ \exp \left( -\frac{1}{R} g \right) \right] = \exp(0) = 1
$$
Theorem
Theorem

$Q$ satisfies **DUALITY and HOMOGENEITY** if and only if
Theorem

$Q$ satisfies DUALITY and HOMOGENEITY if and only if $Q$ is a positive multiple of $R^{AS}$.
Theorem

$Q$ satisfies **Duality** and **Homogeneity** if and only if $Q$ is a positive multiple of $R^{AS}$:

There is $c > 0$ such that $Q(g) = c \cdot R^{AS}(g)$ for every gamble $g$. 
\( u \ \text{ACCEPTS} \ g \)

\( v \ \text{ACCEPTS} \ h \)
$u$ ACCEPTS $g$

$u$ ACCEPTS $h$

$v$ ACCEPTS $g$

$v$ ACCEPTS $h$
\( u \succ v \Rightarrow u \text{ is uniformly more risk-averse than } v \)
\( u \succ v = "u \text{ is uniformly more risk-averse than } v" \)
Alternative approach:
Alternative approach:

Define a "more risky than" ORDER between gambles
Riskiness Order

Alternative approach:

- Define a "more risky than" ORDER between gambles
- Represent it by an "INDEX"
Alternative approach:

- Define a "more risky than" ORDER between gambles
  ↔ preference order

- Represent it by an "INDEX"
Alternative approach:

- Define a "more risky than" **ORDER** between gambles
  \[\leftrightarrow\] preference order

- Represent it by an "**INDEX**"
  \[\leftrightarrow\] utility function
An agent $u$ is **MONOTONIC** if his decisions are monotonic relative to wealth.
An agent $u$ is **MONOTONIC** if his decisions are monotonic relative to wealth:

- If $u$ accepts a gamble $g$ at wealth $W$
- Then $u$ accepts $g$ at any wealth $W' > W$
An agent $u$ is **MONOTONIC** if his decisions are monotonic relative to wealth:

- If $u$ accepts a gamble $g$ at wealth $W$,
- Then $u$ accepts $g$ at any wealth $W' > W$.

($\iff$ coefficient of absolute risk-aversion is nonincreasing in wealth)
An agent $u$ is **MONOTONIC** if his decisions are monotonic relative to wealth.
An agent $u$ is **MONOTONIC** if his decisions are monotonic relative to wealth.

An agent $u$ **TOTALLY REJECTS** $g$ if $u$ rejects $g$ at *every* wealth $W$. 
Riskiness Order

- An agent $u$ is **MONOTONIC** if his decisions are monotonic relative to wealth.

- An agent $u$ **TOTALLY REJECTS** $g$ if $u$ rejects $g$ at *every* wealth $W$.

- A gamble $g$ is **RISKIER THAN** a gamble $h$. 
An agent \( u \) is **MONOTONIC** if his decisions are monotonic relative to wealth.

An agent \( u \) **TOTALLY REJECTS** \( g \) if \( u \) rejects \( g \) at *every* wealth \( W \).

A gamble \( g \) is **RISKIER THAN** a gamble \( h \) if for any monotonic agent \( u \):

- If \( u \) totally rejects \( h \)
- Then \( u \) totally rejects \( g \)
An agent $u$ is **MONOTONIC** if his decisions are monotonic relative to wealth.

An agent $u$ **TOTALLY REJECTS** $g$ if $u$ rejects $g$ at *every* wealth $W$.

A gamble $g$ is **RISKIER THAN** a gamble $h$ if for any monotonic agent $u$:
- If $u$ totally rejects $h$
- Then $u$ totally rejects $g$

\[ g \succeq h \]
Theorem. The riskiness order is represented by the Aumann–Serrano index of riskiness:

\[ g \succeq h \iff R^{AS}(g) \geq R^{AS}(h) \]
Theorem. The riskiness order is represented by the Aumann–Serrano index of riskiness:

\[ g \gtrsim h \iff R^{AS}(g) \geq R^{AS}(h) \]

Corollary

\[ \gtrsim \text{ is a complete order} \]
Theorem. The riskiness order is represented by the Aumann–Serrano index of riskiness:

\[ g \succeq h \iff R^{AS}(g) \geq R^{AS}(h) \]

Corollary

- \succeq is a complete order
- \( R^{AS} \) is unique up to a monotonic transformation
Riskiness Order

Theorem. The riskiness order is represented by the Aumann–Serrano index of riskiness:

\[ g \succeq h \iff R^{AS}(g) \geq R^{AS}(h) \]

Corollary

- \( \succeq \) is a complete order
- \( R^{AS} \) is unique up to a monotonic transformation
- Together with homogeneity: \( R^{AS} \) is unique up to multiplication by a positive constant
$R^{AS}(g)$ is the unique solution $R > 0$ of

$$E \left[ 1 - \exp \left( -\frac{1}{R} g \right) \right] = 0$$
Comparing $R$ and $R^{AS}$

$R(g)$ is the unique solution $R > 0$ of

$$E \left[ \log \left( 1 + \frac{1}{R} g \right) \right] = 0$$

$R^{AS}(g)$ is the unique solution $R > 0$ of

$$E \left[ 1 - \exp \left( -\frac{1}{R} g \right) \right] = 0$$
Comparing $R$ and $R^{AS}$

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$$E \left[ 1 - \exp \left( -\frac{1}{R} g \right) \right] = 0$$

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$
Comparing $R$ and $R^{AS}$

$R(g)$ is the unique solution $R > 0$ of

$$E \left[ \log \left( 1 + \frac{1}{R} g \right) \right] = 0$$

$R^{AS}(g)$ is the unique solution $R > 0$ of

$$E \left[ 1 - \exp \left( -\frac{1}{R} g \right) \right] = 0$$

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

$$1 - \exp(-x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \ldots$$
Comparing $R$ and $R^{AS}$

Proposition

If $E[g]$ is small relative to $g$ then $R(g) \sim R^{AS}(g)$
Comparing $R$ and $R^{AS}$

Proposition

If $E[g]$ is small relative to $g$ then $R(g) \sim R^{AS}(g)$

Example

$g = \frac{1}{2} + \frac{1}{2}$

$\begin{array}{c}
\frac{1}{2} \\
\frac{1}{2}
\end{array}$

$+$ $105$

$-$ $100$
Comparing $R$ and $R^{AS}$

**Proposition**

If $E[g]$ is small relative to $g$ then $R(g) \sim R^{AS}(g)$

**Example**

\[
g = \begin{cases} 
  +$105, & \text{1/2 probability} \\
  -$100, & \text{1/2 probability} 
\end{cases}
\]

\[
R(g) = $2100
\]
Comparing $R$ and $R^{AS}$

Proposition

If $E[g]$ is small relative to $g$ then $R(g) \sim R^{AS}(g)$

Example

$$g = \sqrt[1/2]{1/2} + $105 - $100$$

$R(g) = $2100 \quad R^{AS}(g) = $2100.42...
Comparing R and $R^{AS}$
Comparing $R$ and $R^{AS}$

- $R$: critical wealth for any risk aversion
Comparing $R$ and $R^{AS}$

- $R$: critical wealth for any risk aversion
- $R^{AS}$: critical risk aversion for any wealth
Comparing $R$ and $R^{AS}$

- $R$: critical wealth for any risk aversion
- $R^{AS}$: critical risk aversion for any wealth
- $R$: measure (one gamble)
Comparing $R$ and $R^{AS}$

- $R$: critical wealth for any risk aversion
- $R^{AS}$: critical risk aversion for any wealth

- $R$: measure (one gamble)
- $R^{AS}$: index (comparing gambles)
Comparing $R$ and $R^{AS}$

- $R$: critical wealth for any risk aversion
- $R^{AS}$: critical risk aversion for any wealth
- $R$: measure (one gamble)
- $R^{AS}$: index (comparing gambles)
- $R$: no-bankruptcy, no-loss
Comparing $R$ and $R^{AS}$

- $R$: critical wealth for any risk aversion
- $R^{AS}$: critical risk aversion for any wealth
- $R$: measure (one gamble)
- $R^{AS}$: index (comparing gambles)
- $R$: no-bankruptcy, no-loss
- $R^{AS}$: expected utility, risk aversion
Comparing $R$ and $R^{AS}$

- $R$: critical wealth for any risk aversion
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- $R$: measure (one gamble)
- $R^{AS}$: index (comparing gambles)

- $R$: no-bankruptcy, no-loss
- $R^{AS}$: expected utility, risk aversion

- unit and operational interpretation
Comparing $R$ and $R^{AS}$

- $R$: critical wealth for any risk aversion
- $R^{AS}$: critical risk aversion for any wealth

- $R$: measure (one gamble)
- $R^{AS}$: index (comparing gambles)

- $R$: no-bankruptcy, no-loss
- $R^{AS}$: expected utility, risk aversion

- unit and operational interpretation
- continuity and "black swans"
Comparing $R$ and $R^{AS}$

- **$R$:** critical wealth for any risk aversion
- **$R^{AS}$:** critical risk aversion for any wealth

- **$R$:** measure (one gamble)
- **$R^{AS}$:** index (comparing gambles)

- **$R$:** no-bankruptcy, no-loss
- **$R^{AS}$:** expected utility, risk aversion

- Unit and operational interpretation
- Continuity and "black swans"

Nevertheless: similar in many respects!!
Rabin (2000): Calibration
If a risk-averse expected-utility agent rejects the gamble $g = [+$105, 1/2; − $100, 1/2]$ at all wealth levels $W < $300 000.
If a risk-averse expected-utility agent rejects the gamble $g = [+\$105, 1/2; - \$100, 1/2]$ at all wealth levels $W < \$300,000$

Then he must reject the gamble $h = [+\$5,500,000, 1/2; - \$10,000, 1/2]$ at wealth level $W = \$290,000$
Rabin (2000): Calibration

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**OUR RESULT:** reject $g$

*at all wealth levels* $W < R(g) = $2100
If a risk-averse expected-utility agent rejects the gamble \( g = [+105, 1/2; -100, 1/2] \) at all wealth levels \( W < 300000 \)

Then he must reject the gamble \( h = [+5500000, 1/2; -10000, 1/2] \) at wealth level \( W = 290000 \)

**OUR RESULT:** reject \( g \) at all wealth levels \( W < R(g) = 2100 \)

no friction, no cheating
Rabin (2000): Calibration

If a risk-averse expected-utility agent rejects the gamble

\[ g = [+$105, 1/2; - $100, 1/2] \]

at all wealth levels \( W < $300,000 \)

Then he must reject the gamble

\[ h = [+$$5,500,000, 1/2; - $10,000, 1/2] \]

at wealth level \( W = $290,000 \)

\[ \text{OUR RESULT: reject } g \]

at all wealth levels \( W < R(g) = $2100 \)

- no friction, no cheating
- what is “wealth”?
What is Wealth?
What is Wealth?

Rejecting $g$ when $W < W + R(g)$
Guarantees a minimal wealth level of $W$
What is Wealth?

Rejecting $g$ when $W < W + R(g)$

Guarantees a minimal wealth level of $W$

(Proof: replace 0 with $W$)
What is Wealth?

Rejecting $g$ when $W < W + R(g)$
Guarantees a minimal wealth level of $W$

Back to *calibration*:
What is Wealth?

Rejecting $g$ when $W < W + R(g)$
Guarantees a minimal wealth level of $W$

Back to *calibration*:

- If $W =$ “gambling / risky investment wealth”,
  then $300,000$ seems *excessive* for $g$
  (since $R(g) = 2100$)
What is Wealth?

Rejecting \( g \) when \( W < W + R(g) \)
Guarantees a minimal wealth level of \( W \)

Back to *calibration*:

- If \( W = \) “gambling / risky investment wealth”, then $300 000 seems *excessive* for \( g \) (since \( R(g) = $2100 \))
- If \( W = \) total wealth, then rejecting \( g \) at all \( W < $300 000 \) is consistent with a required minimal wealth level \( W \geq $297 900 \),
What is Wealth?

Rejecting \( g \) when \( W < W + R(g) \)
Guarantees a minimal wealth level of \( W \)

Back to *calibration*:

- If \( W \) = “gambling / risky investment wealth”,
  then $300 000 seems *excessive* for \( g \)
  (since \( R(g) = 2100 \))

- If \( W \) = total wealth, then rejecting \( g \) at all
  \( W < 300 000 \) is consistent with a required
  minimal wealth level \( W \geq 297 900 \),
  and then one rejects \( h \) at $290 000
Summary
The Riskiness measure $R$
The Riskiness measure $R$ (recall)

- is objective and universal
- is independent of utilities, risk aversion, ...
- has a clear operational interpretation
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- may replace measures of risk ($\sigma$-based, ...)  
  - Markowitz, CAPM, ... : $E$ vs $\sigma$ $\rightarrow$ $E$ vs $R$
  - Sharpe ratio: $E/\sigma$ $\rightarrow$ $E/R$
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- may replace measures of risk ($\sigma$-based, ...)
  - Markowitz, CAPM, ... : $E$ vs $\sigma$ → $E$ vs $R$
  - Sharpe ratio: $E/\sigma$ → $E/R$
- may replace reserve measures (VaR, ...)

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"We’re recommending a risky strategy for you; so we’d appreciate if you paid before you leave."