EQUILIBRIUM IN n-PERSON GAME OF "SHOWCASE SHOWDOWN"

VLADIMIR MAZALOV and ANNA IVASHKO

Institute of Applied Mathematical Research,
Karelian Research Center of RAS
Petrozavodsk, Russia
E-mail: vmazalov@krc.karelia.ru, afalko@krc.karelia.ru

1. INTRODUCTION. We consider a non-cooperative n-person optimal stopping game related with the popular TV game "The price is right". In this game each of n players in turn spins the wheel once or twice attaining some total score and then waits for the results of the succeeding players' spins. The object of the game is to have the highest score, from once or two spins, without going over a given upper limit.

Kaynar, B. (2009). Optimal stopping in a stochastic game, *Probability in the Engeneering and Information Sciences* 23: 51-60.

It was constructed the optimal solution for the game with two possible attempts and two and three players. The author finds the payoff function and achieves the solution from the Nash equilibrium conditions.

2. SHOWCASE SHOWDOWN GAME WITH TWO STEPS

There are n players. Each player chooses one or two random numbers $x_1^{(k)}, x_2^{(k)}, k = 1, 2, ..., n,$ -i.i.d. random variables uniformly distributed in $[0, 1]$. After the first draw a player decides to stop or continue for a second draw. She does not know what the other players have done. The object of the game is to have the highest total score without going over 1. In case the total scores of all players exceed 1, the player whose score is closest to 1 is the winner in the game. Each player uses the threshold strategy u: if the first number x_1 is larger or equal to u the player chooses this number, otherwise, she chooses the second number x_2.

$$\tau = \begin{cases} 1, & \text{if } x_1 \geq u, \\ 2, & \text{if } x_1 < u \end{cases} \quad \text{and} \quad s_\tau = \begin{cases} x_1, & \text{if } x_1 \geq u, \\ x_1 + x_2, & \text{if } x_1 < u, \end{cases}$$

Calculate for $x \in [u, 1]$ the probability

$$P\{s_\tau \le x\} = P\{x_1 \in [u, x]\} + P\{x_1 < u, x_1 + x_2 \le x\} = x - u + \int_0^u (x - y)dy$$

and for $x > 1$ the probability

$$P\{s_\tau > x\} = P\{x - 1 < x_1 < u, x_1 + x_2 > x\} = \int_{x-1}^{u} (1 - (x - y))dy.$$

Symmetry yields the optimal strategies are equal. Suppose that the players $(2, ..., n)$ use the identical thresholds strategies $u^{(n-1)} = (u, ..., u)$ and let $x_1 = x$. The expected payoff of the first player

$$h(x|u^{(n-1)}) = \prod_{k=2}^{n} (P\{s_\tau^{(k)} < x\} + P\{s_\tau^{(k)} > 1\}).$$

$$h(x|u^{(n-1)}) = (x - u + xu)^{n-1}, x \in [u, 1].$$

Denote $Ph(x|u^{(n-1)})$ the return of the first player if she continues the process with the current score after the first step being x.

$$
\begin{aligned}
Ph(x|u^{(n-1)}) &= \int_x^1 h(y|u^{(n-1)})dy + \int_1^{x+1} \prod_{k=2}^{n} P\{s_\tau^{(k)} > y\}dy \\
&= \int_x^1 h(y|u^{(n-1)})dy + \int_1^{x+1} \left(\int_{y-1}^{u} (1-(y-z))dz \right)^{n-1} dy \\
&= \frac{1-(x-u+xu)^n}{n(u+1)} + \frac{u^{2n-1}-(u-x)^{2n-1}}{2^{n-1}(2n-1)}.
\end{aligned}
$$

The value of the optimal threshold can be achieved from the equation $h(x|u^{(n-1)}) = Ph(x|u^{(n-1)})$.

Letting $x = u$ we obtain

$$u^{2(n-1)} = \frac{1 - u^{2n}}{n(u+1)} + \frac{u^{2n-1}}{2^{n-1}(2n-1)}. \qquad (2.3)$$

From (2.3) we can find the optimal strategies for different n (see Table 1).

TABLE 1. Optimal Strategies u^*

n	2	3	4	5	6	7	8	9	10
u^*	0.563	0.661	0.718	0.757	0.785	0.806	0.823	0.837	0.849

For $n = 2, 3$ the optimal thresholds coincide with the values in [3]. We see also from the Table 1 that for large number of players the optimal strategy is closed to 1.

3. SHOWCASE SHOWDOWN GAME WITH INFINITE NUMBER OF STEPS

Assume now that the players observe the sequence of sums of i.i.d. random variables

$$s_t^{(k)} = \sum_{i=1}^{t} x_i^{(k)},\ t = 1, 2, ...$$

where $k = 1, ..., n$. Without loss of generality suppose that for any k the random variables $\{x_i^{(k)}\}, i = 1, 2, ...$ are uniformly distributed in $[0, 1]$. The critical value of the threshold is 1.

Let a player uses the strategy $u, 0 < u < 1$ and stops her sum of scores at the first moment as the sum s_t exceeds u. Denote this random stopping time as τ. Thus,

$$\tau = \min\{t \geq 1 : s_t \geq u\}.$$

To construct the payoff in this game let us find the distribution for the stopping sum s_τ.

For $u \le x \le 1$ we present

$$P\{s_\tau \le x\} = \sum_{t=1}^{\infty} P\{s_t \le x, \tau = t\} = \sum_{t=1}^{\infty} P\{s_1 < u, ..., s_{t-1} < u, s_t \in [u, x]\}.$$

It yields

$$P\{s_\tau \le x\} = \sum_{t=1}^{\infty} \frac{u^{t-1}}{(t-1)!}(x-u) = \exp(u)(x-u). \qquad (3.1)$$

From (3.1) we obtain that the probability of ruin is

$$P\{s_\tau > 1\} = 1 - P\{s_\tau \le 1\} = 1 - \exp(u)(1-u). \qquad (3.2)$$

Calculate the probability $P\{s_\tau > x\}$ for $x > 1$.

$$P\{s_t > x, \tau = t\} = P\{s_1 < u, s_2 < u, ..., s_{t-2} < u, s_{t-1} \in (x-1, u), s_t > x\} =$$

$$= \int_x^{u+1} dz_t \int_{z_t - 1}^{u} dz_{t-1} \int_0^{z_{t-1}} dz_{t-2} ... \int_0^{z_2} dz_1 =$$

$$\frac{u^{t-1}(u - x + 1)}{(t-1)!} - \frac{u^t - (x-1)^t}{t!}.$$

Hence,

$$P\{s_\tau > x\} = \sum_{t=2}^{\infty} P\{s_t > x, \tau = t\} = \sum_{t=2}^{\infty} \left(\frac{u^{t-1}(u - x + 1)}{(t-1)!} - \frac{u^t - (x-1)^t}{t!} \right).$$

Simplifying

$$P\{s_\tau > x\} = \exp(x - 1) - (x - u)\exp(u), \quad x > 1. \tag{3.3}$$

Now we can find the equilibrium in the game. Let $n-1$ players use the identical thresholds' strategies u and find the best reply of the first player.

Denote the current value of her sum $s_t^{(1)} = x$. If $x \geq u$, then the expected gain of the first player in case of stopping will be positive if all other $n-1$ players stop at the values which are less than x or larger than 1.

$$h(x|u) = \prod_{k=1}^{n} P\{s_\tau^{(k)} < x\} + P\{s_\tau^{(k)} > 1\} = (P\{s_\tau^{(k)} < x\} + P\{s_\tau^{(k)} > 1\})^{n-1}.$$

(3.1)–(3.2) yield

$$h(x|u) = (\exp(u)(x-u) + 1 - \exp(u)(1-u))^{n-1} = (\exp(u)(x-1) + 1)^{n-1}. \tag{3.4}$$

If the first player continues the choice and makes the stop on the next step with score y then her payoff will be equal to $h(x+y|u)$ if $x+y \leq 1$ and $P\{s_\mathcal{T} > x+y\}^{n-1}$ if $x+y > 1$. So, the expected payoff under continuation is equal to

$$Ph(x|u) = \int_u^1 h(z|u)dz + \int_1^{u+1} P\{s_\mathcal{T} > z\}^{n-1}dz.$$

From (3.3) and (3.4) after simplification we obtain

$$Ph(x|u) = \int_u^1 (\exp(u)(z-1)+1)^{n-1}dz + \int_1^{u+1} (\exp(z-1) - (z-u)\exp(u))^{n-1}dz$$

$$= \frac{\exp(-u)\,(1-(\exp(u)(u-1)+1)^n)}{n} + \int_1^{u+1} (\exp(z-1) - (z-u)\exp(u))^{n-1}dz.$$

Function $h(x|u)$ for $x \geq u$ is increasing in x but $Ph(x|u)$ is constant. So, if the first player knows the strategy u of other players then she can find her best reply comparing the gains in case of stopping and continuing. Optimal threshold u_n is determined by the equation

$$h(x|u) = Ph(x|u),$$

$$\frac{e^{-u}\,(1 - (e^u(u-1)+1)^n)}{n} + \int_1^{u+1} (e^{z-1} - (z-u)e^u)^{n-1} dz =$$

$$= (e^u(x-1)+1)^{n-1}. \qquad (3.5)$$

From monotonicity of the functions $h(x|u)$ and $Ph(x|u)$ it follows that if this threshold exists it is unique.

Symmetry yields $x = u$. From the equation (3.5) we obtain

$$\frac{e^{-u}\,(1-(e^{u}(u-1)+1)^{n})}{n} + \int_{1}^{u+1}(e^{z-1}-(z-u)e^{u})^{n-1}dz =$$

$$(e^{u}(u-1)+1)^{n-1}. \tag{3.6}$$

The solution u^{*} of the equation (3.6) exists. It follows from the fact that for $u = 0$ L.H.S. of (3.6) is greater than R.H.S. $(1/n > 0)$ and for $u = 1$ L.H.S. is less than R.H.S. ($\int_1^2 (e^{z-1} - (z-1))^{n-1}dz < 1$, since $e^{z-1} - (z-1) < 1$ for $1 \leq z \leq 2$).

For instance, for $n = 2$ we find $u^{*} \approx 0.634$. The probability of ruin here is equal to

$$P\{s_{\tau} > 1\} = 1 - \exp(u^{*})(1-u^{*}) \approx 0.310.$$

In Table 2 the optimal strategies for various n are presented. In case of large n the optimal thresholds also tends to 1.

TABLE 2. Optimal Strategies u^*

n	2	3	4	5	6	7	8	9	10
u^*	0.633	0.718	0.767	0.780	0.823	0.841	0.856	0.867	0.877

Remark. This problem belongs to the class of optimal stopping problems for which the one-stage look-ahead (OLA) stopping rule, which compares the current gain with the expected return of continuing one stage and then stopping, is optimal. That is so called monotone stopping rule case [1].

Acknowledgement. The work is supported by the Division of Mathematical Sciences of Russian Academy of Sciences.

References

1. Chow, Y. S., Robbins, H. and Siegmund, D. (1971). *Great Expectations: The Theory of Optimal Stopping*, Houghton Mifflin, Boston, MA.

2. Coe, P.R. and Butterworth, W. (1995). Optimal stopping in "The Showcase Showdown", *The American Statistician* 49: 271–275.

3. Kaynar, B. (2009). Optimal stopping in a stochastic game, *Probability in the Engeneering and Information Sciences* 23: 51-60.

4. Mazalov, V. V. (1996). A Game related to optimal stopping of two sequences of independent random variables having different distributions, *Mathematica Japonica* 43, N 1: 121-128.

5. Sakaguchi, M. (2005). Equilibrium in two-player games of "Showcase Showdown", *Scientiae Mathematicae Japonicae* 61, 145-151.

6. Sakaguchi, M. (2007). Players' information in two-player games of "Score Showdown", *Game Theory and Applications* 11,

Edit. Petrosjan and Mazalov, N.Y., Nova Science Publ.: 111-124.

7. Tijms, H.C. (2007). *Understanding probability*, 2nd ed. Cambridge: Cambridge Univ. Press.