The effect of information constraints on decision-making and economic behaviour

Roman V. Belavkin

Middlesex University

April 16, 2010

Learning Systems

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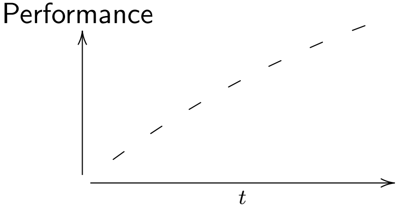

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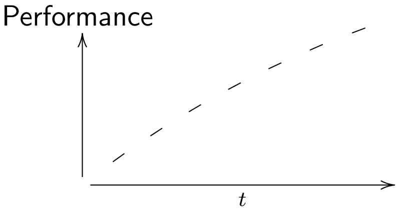

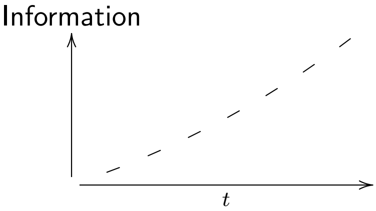

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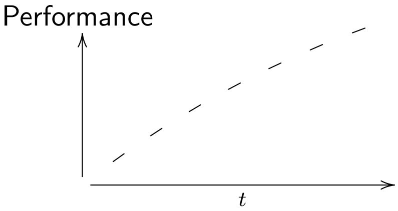

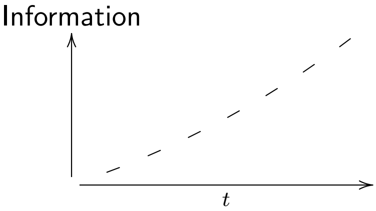

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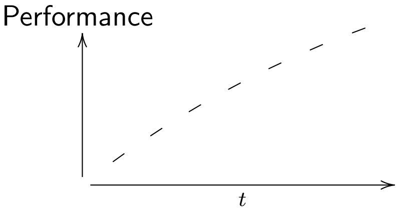

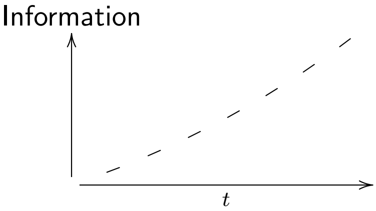

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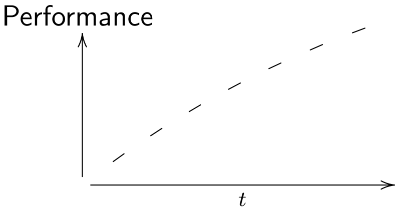

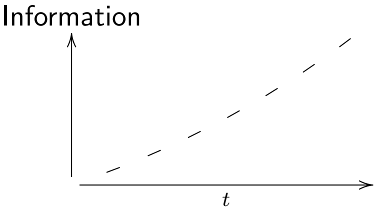

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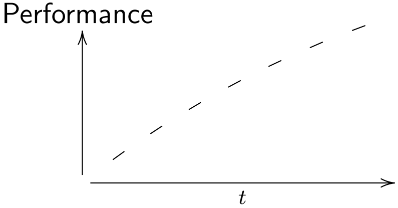

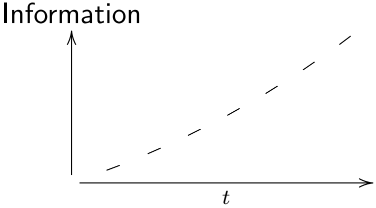

- Performance and information have orders, and the relation between them is monotonic.
- Complete partial orders, domain theory.
- Utility theory, information theory
- Allows for treating both deterministic and non-deterministic case:

$$x = f(\omega)\,, \qquad x = f(\omega) + \mathsf{rand}()$$

Expected Utility Theory

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Question (Why expected utility?)

1. $\mathbb{E}_y\{f\} = f(\omega)$ *if* $y(\Omega') = \delta_\omega(\Omega')$.
2. $x \lesssim y \quad \iff \quad \lambda x \lesssim \lambda y$, $\forall\, \lambda > 0$
3. $x \lesssim y \quad \iff \quad x + z \lesssim y + z$, $\forall\, z \in Y$.

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- To enter the lottery, you must pay a fee of $\pounds X$
- How much is $\pounds X$?

Why is it a paradox?

- We don't want to pay more than expect to win:

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- Let $p(n)$ be the probability of $n \in \mathbb{N}$

head	1	2	3	4	$\cdots$	n	$\cdots$
win	£2	£4	£8	£16	$\cdots$	2^n	$\cdots$
$p(n)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\cdots$	$\frac{1}{2^n}$	$\cdots$

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- It is easy to see that

$$\mathbb{E}_p\{\text{win}\} = 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} + 8 \cdot \frac{1}{8} + 16 \cdot \frac{1}{16} + \cdots = \sum_{n=1}^{\infty} \frac{2^n}{2^n} = \infty$$

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- One cannot buy what is not for sale.

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- Some suggest to use only f such that

$$\|f\|_\infty := \sup |f(\omega)| < \infty$$

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- How much would you borrow? ($\pounds X =$?)

The Allais (1953) paradox

Consider two lotteries:

A : $p(\pounds 300) = \frac{1}{3}$ (and $p(\pounds 0) = \frac{2}{3}$)

B : $p(\pounds 100) = 1$

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Remark

Safety is preferred (i.e. risk averse).

The Allais (1953) paradox (2)

Consider two lotteries:

$$\text{C} : p(-£300) = \tfrac{1}{3} \qquad (\text{and } p(£0) = \tfrac{2}{3})$$
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Risk is preferred (i.e. risk taking).

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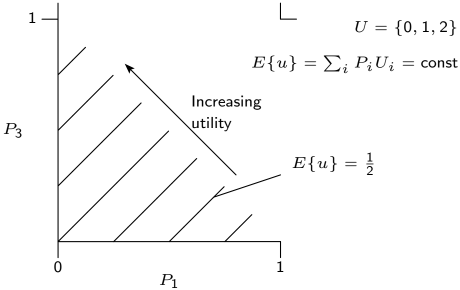

Remark

Any linear functional (e.g. $\mathbb{E}_p\{x\}$) has parallel level sets. If people use expected utility to make choices, then they are either risk-averse or risk-taking, but not both.

Prospect theory

Due to Tversky and Kahneman (1981)

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Remark

This theory is not normative (i.e. it is not derived using rational approach).

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Remark

More information is preferred.

Extreme Value Problems

Unconditional extremum

- Maximise $f(y)$:

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- Sufficient if $K(y, \alpha) := f(y) + \alpha[\lambda - g(y)]$ is concave.

Representation in Paired Spaces

- $x \in X$, $y \in Y$, $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$

$$\langle x, y \rangle := \int_\Omega x(\omega)\, dy(\omega)$$

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- Expected value

$$\mathbb{E}_p\{x\} = \langle x, y \rangle|_{\mathcal{M}}$$

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Definition (Information resource)

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Example (Relative Information (Belavkin, 2010b))

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$$F(y) := \begin{cases} \left\langle \ln \frac{y}{z}, y \right\rangle - \langle 1, y - z \rangle \,, & \text{if } y > 0 \\ \langle 1, z \rangle \,, & \text{if } y = 0 \\ \infty \,, & \text{if } y < 0 \end{cases}$$

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- The dual $F^* : X \to \mathbb{R} \cup \{\infty\}$ is

$$F^*(x) := \langle 1, e^x \, z \rangle$$

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- Related functions

$$\begin{aligned} -U_{-x}(I) &:= \inf\{\langle x, y \rangle : F(y) \leq I\} \\ I_x(U) &:= \inf\{F(y) : U_0 \leq U \leq \langle x, y \rangle\} \\ I_x(U) &:= \inf\{F(y) : \langle x, y \rangle \leq U < U_0\} \end{aligned}$$

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A solution to $U_f(I)$ and $I_f(U)$ exists if and only if set $\{x : F_q^*(x) \leq I^*\}$ absorbs function f:

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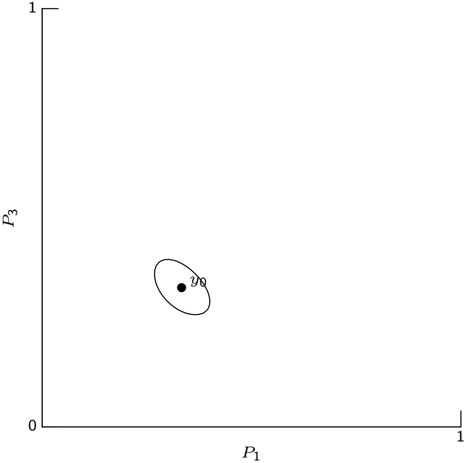
Remark (Separation of information)

For all $I \in (\inf F, \sup F)$ there exist β_1^{-1}, $\beta_2^{-1} > 0$:

$$F_q(\partial F_q^*(\beta_1 f)) < I < F_q(\partial F^*(\beta_2 f))$$

Information Topology (Belavkin, 2010a)



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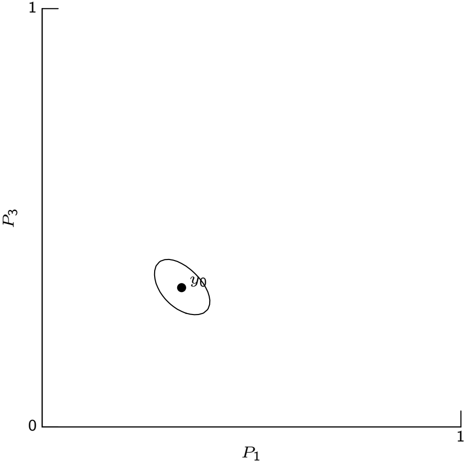

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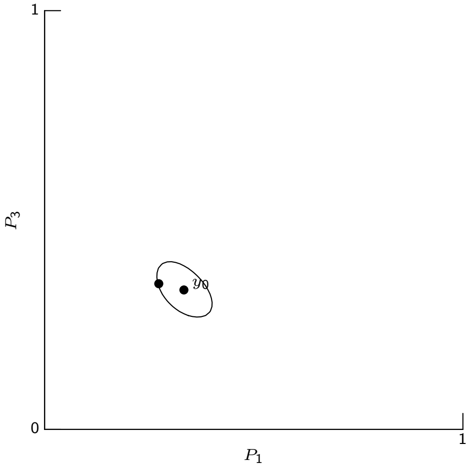

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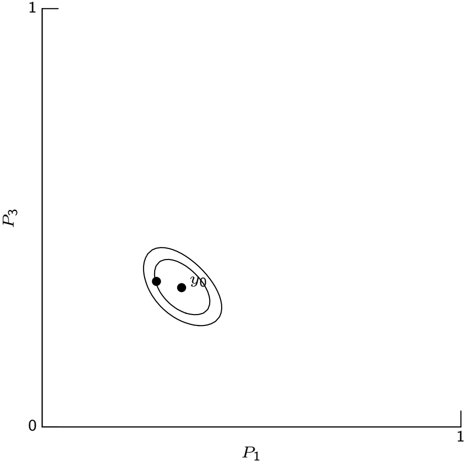

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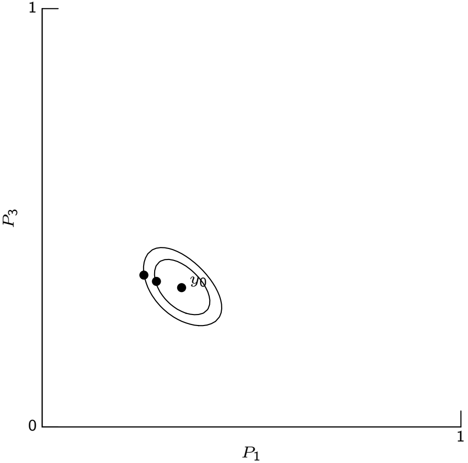

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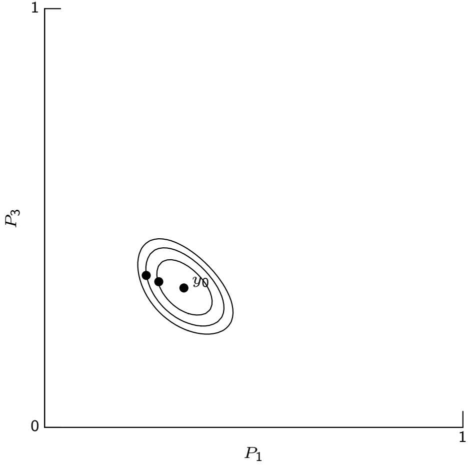

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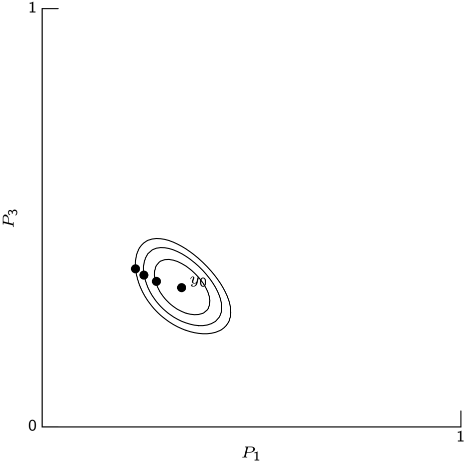

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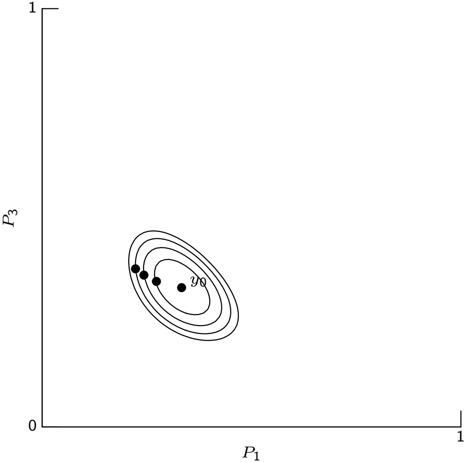

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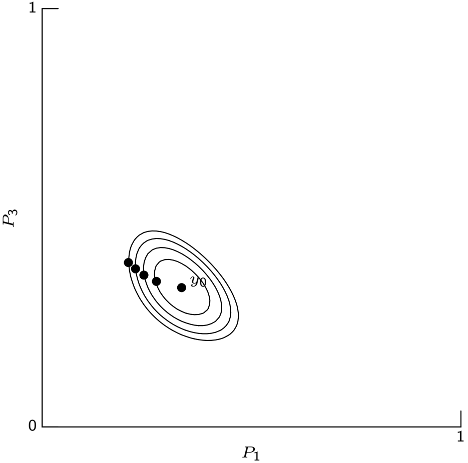

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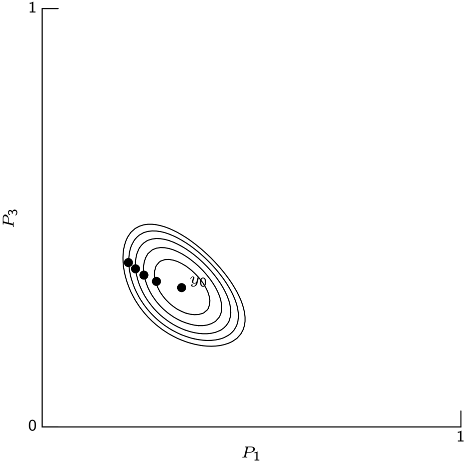

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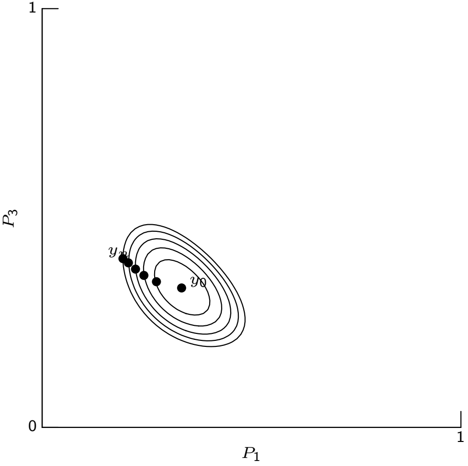

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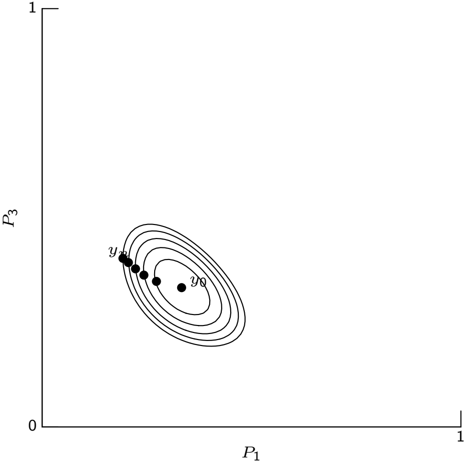

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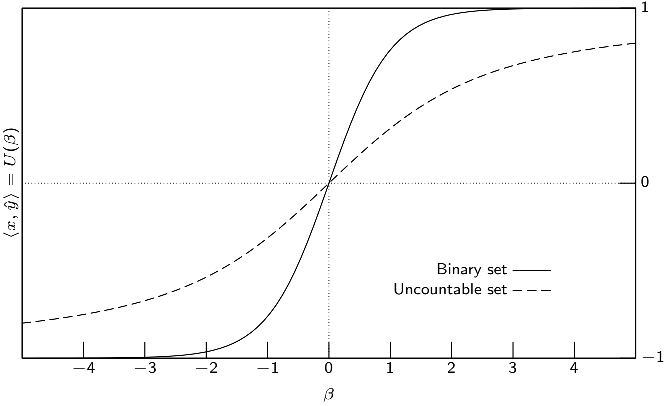
Parametrisation by the Expected Utility

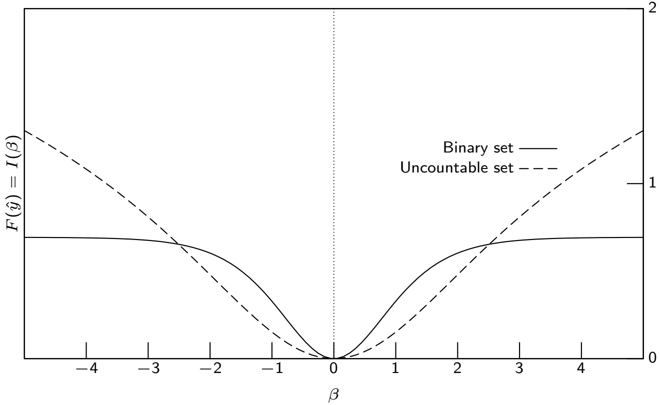


Let $F(y)$ be negative entropy (i.e. $F(y)$ is minimised at $y_0(\omega) = \text{const}$)

$$x : \Omega \to \{c - d, c + d\} \qquad U(\beta) = \Psi'(\beta) = c + d \tanh(\beta\, d)$$

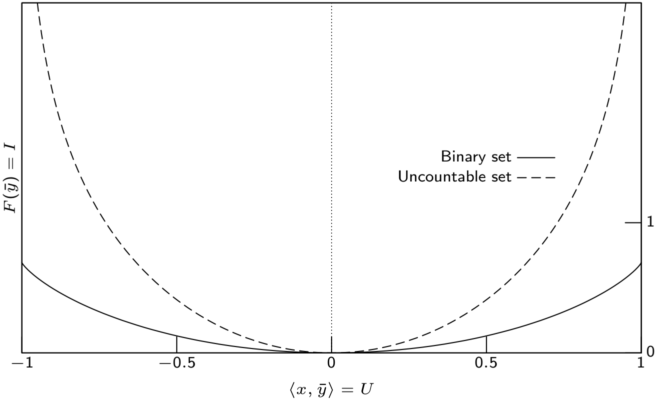
$$x : \Omega \to [c - d, c + d] \qquad U(\beta) = \Psi'(\beta) = c + d \coth(\beta\, d) - \beta^{-1}$$

Parametrisation by Information



$$I = \Phi'(\beta^{-1})\,, \qquad I = \beta\,\Psi'(\beta) - \Psi(\beta)$$

Parametric Dependency



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- $f : \mathbb{N} \to \mathbb{R}$ is information bounded iff for some $\beta^{-1} > 0$:

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- Using $U = \Psi'_f(\beta)$ obtain

$$U = \frac{1}{1 - 2^{\beta-1}}\,, \qquad U_0 = 2$$

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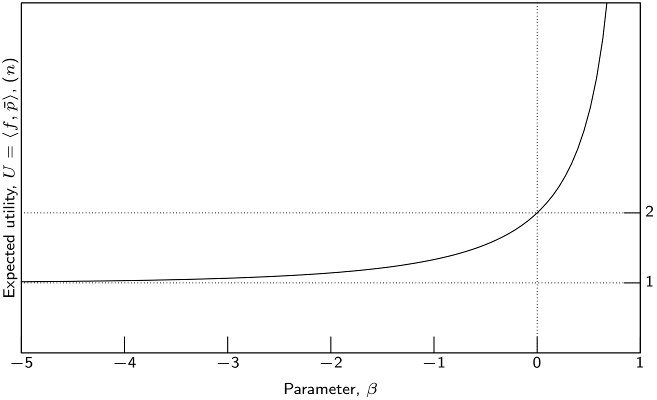
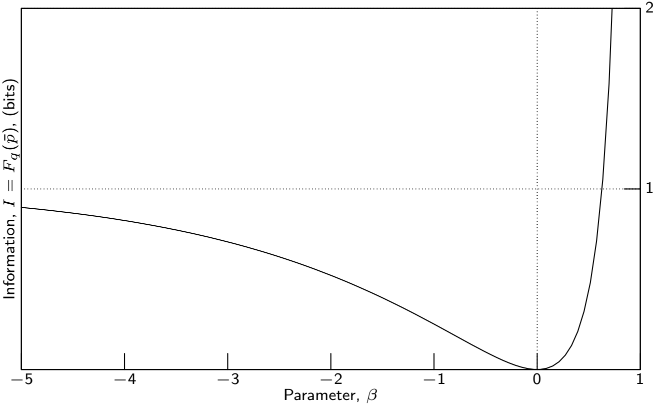
- For the information amount of 0 bits, the optimal entrance fee is $c \leq U_0 = 2$.

SPB Lottery


Expected utility, $U = \langle f, \bar{p} \rangle$, (n)
Parameter, β
−5
−4
−3
−2
−1
0
1
2
1

SPB Lottery



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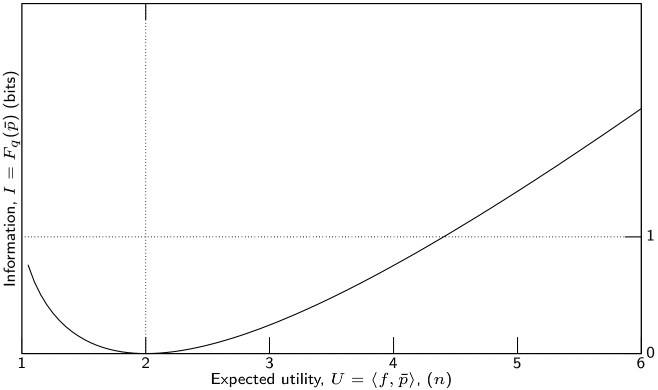
Information, $I = F_q(\bar{p})$ (bits)
1
0
1
2
3
4
5
6
Expected utility, $U = \langle f, \bar{p} \rangle$, (n)

Allais, M. (1953). Le comportement de l'homme rationnel devant le risque: Critique des postulats et axiomes de l'École americaine. *Econometrica*, *21*, 503–546.

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Belavkin, R. V. (2010b). Utility and value of information in cognitive science, biology and quantum theory. In L. Accardi, W. Freudenberg, & M. Ohya (Eds.), *Quantum Bio-Informatics III* (Vol. 26). World Scientific.

Ellsberg, D. (1961, November). Risk, ambiguity, and the Savage axioms. *The Quarterly Journal of Economics*, *75*(4), 643–669.

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