Characteristic Function for Games with Prohibited Coalitions

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$N = \{1, 2, \ldots, n\}$: the set of players.

$S \subseteq N$: a coalition.

$\mathbb{S}$: the set of all coalitions from N.

$\mathbb{S} = \mathbb{A} \cup \mathbb{P}$ such that $\mathbb{A} \cap \mathbb{P} = \emptyset$ with the property:

- if $S \in \mathbb{A}$ then $\forall T \subset S \Rightarrow T \in \mathbb{A}$.
- $i \in \mathbb{A}\ \forall i \in N$.

$\Delta(N)$: a partition of N such that

$$\Delta(N) = \{S_1, \ldots, S_k\} \text{ and } S_i \cap S_j = \emptyset \forall i, j = 1, \ldots, k, i \neq j.$$

$\Delta_{\mathbb{A}}(N)$: an admissible partition of N: $S_j \in \mathbb{A} \forall j = 1, \ldots, k$.

We consider an approach of an admissible partition formation in multistage games with perfect information:

- each player before acting has an option to cooperate or not to cooperate (the decision to cooperate or not is the element of players strategy). The player may enter any current coalition to form an admissible one or stay individual player;
- players who decide to cooperate, maximize the coalitional payoff, and individual players maximize their own payoffs;
- players from each admissible coalitions share the coalitional payoff in accordance with some imputation.

Graph tree G: W — the set of vertices of G

$\Gamma(O') = \langle N, G, P'_1, \ldots, P'_n, P'_{n+1}, h'_1, \ldots, h'_n \rangle$

$W = P'_1 \cup \ldots \cup P'_{n+1},\ P'_i \cap P'_j = \emptyset,\ i \neq j,\ i, j = 1, \ldots, n+1$

$h'_i : P'_{n+1} \mapsto \mathbb{R}, \quad i = 1, \ldots, n$.

Graph tree $\tilde{G}$: X — the set of vertices of $\tilde{G}$

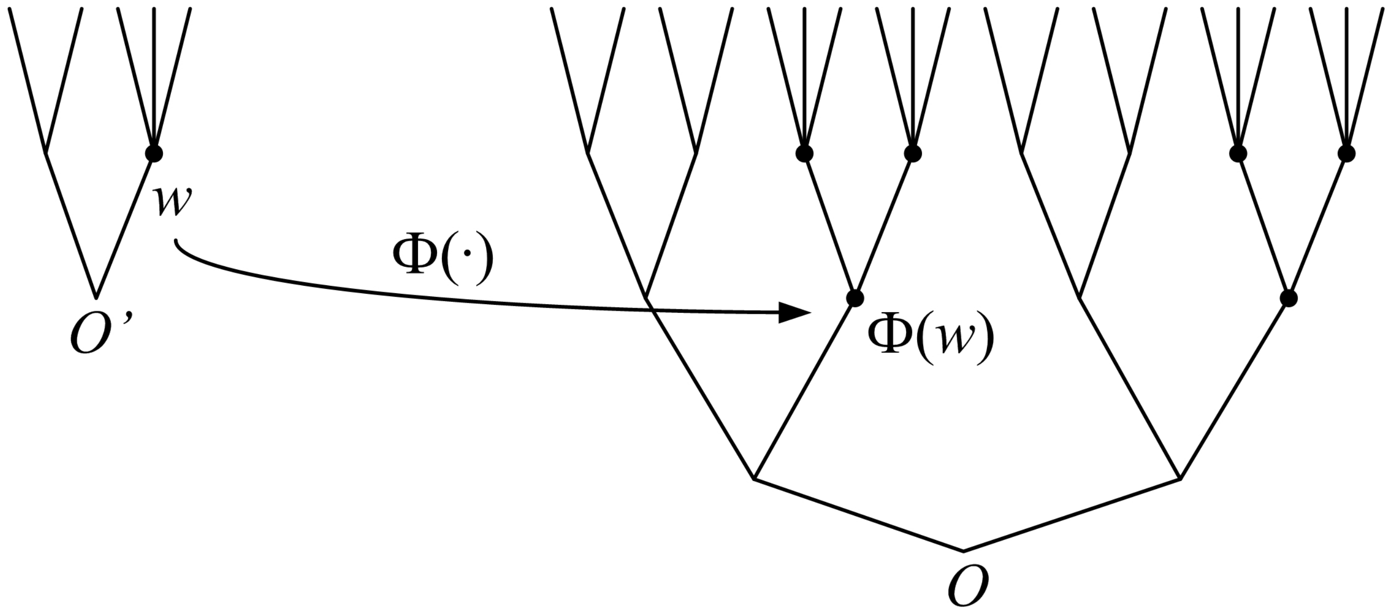
$\tilde{\Gamma}(O) = \langle N, \tilde{G}, P_1, \ldots, P_n, P_{n+1}, h_1, \ldots, h_n \rangle$

$X = P_1 \cup \ldots \cup P_{n+1},\ P_i \cap P_j = \emptyset,\ i \neq j,\ i, j = 1, \ldots, n+1$

$h_i : P_{n+1} \mapsto \mathbb{R}, \quad i = 1, \ldots, n$.

$\Phi : W \to X$: a point-to-set mapping that to each $w \in W$ assigns $\Phi(w) \subset X$.

Graph tree G Graph tree $\tilde{G}$



x: a vertex of a graph tree;

x has a rank k ($k = 0, 1, 2, \ldots$) if this vertex can be reached from the root of the graph tree exactly in k stages;

F_x: the set of vertices immediately following the vertex x;

$i(x)$: player who makes decision in vertex x;

$\Delta_{\mathbb{A}_x}(N) = \{S_1, \ldots, S_{k(x)}\}$: an admissible partition of N in the vertex x.

$\Delta_{\mathbb{A}_x}(N) = \{S_1, \ldots, S_{k(x)}\}$, $x \in P_i$.
i's alternatives in x:

- individual behavior;
- announce cooperative behavior acting individually if he still individual, and acting in cooperation if some other player joints to i;
- enter to any coalition $S \in \Delta_{\mathbb{A}_x}(N)$ iff $i \cup S \in \mathbb{A}$ and acting in cooperation with players from S.

Suggestion. If player choose cooperative behavior he keeps this type of behavior till the end of the game process.

n-person game

Definition An n-person game in extensive form with perfect information and admissible partition is a graph tree $\tilde{G}$ with the following properties:

- **Graph tree.** A partition $P_1, P_2, \dots, P_n, P_{n+1}$ of the set of vertices X is given. Here P_i, $i \in N$ is the set of personal positions (vertices) of player i, and P_{n+1} is a set of terminal vetices such that $P_i = \Phi(P'_i), i = 1, \dots, n+1.$;
- **Partition.** In each vertex $x \in X$ the admissible partition $\Delta_{\mathbb{A}}(N, x) = \{S_1, \dots, S_{k(x)}\}$ is uniquely defined where the coalition $S \in \Delta_{\mathbb{A}}(N, x)$ consists of players who choose to enter to this coalition along the path leading to x;
- **Payoffs.** $\forall w \in P'_{n+1}$ and $\forall i \in N \;\; h_i(x) = h'(w), \; \forall x \in \Phi(w)$.

A strategy

Definition The strategy $u_i(\cdot)$ of player $i \in N$ is the mapping which to each position (vertex) $x \in P_i$ assigns the vertex $y \in F_x$ or probability distribution p^x over F_x

$$p^x = \{p^x(y)\}, y \in F_x, p^x(y) \geqslant 0, \sum_{y \in F_x} p^x(y) = 1.$$

Let $u_i(\cdot)$ be the strategy of player $i \in N$ and $Y_i \subset P_i$ be the set of all personal positions of i of odd rank which are possible if player i always choose non-cooperative behavior.

Denote by $B^i(u_i(\cdot))$ the subset of strategies of player i which consists of all strategies which differ from $u_i(\cdot)$ only by choices in vertices $y \in Y_i$.

If $u_i(\cdot)$ chooses cooperative behavior in some $z \in P_i$, suppose $B^i(u_i(\cdot)) = \emptyset$.

Payoff function

For each n-tuple of strategies $\bar{u}(\cdot)) = (\bar{u}_1(\cdot), \ldots, \bar{u}_n(\cdot))$ in $\tilde{G}$ define players payoff functions in the following manner. Suppose in $\bar{u}(\cdot)$ the admissible partition $\Delta_{\mathbb{A}}(N) = \{S_1, \ldots, S_k\}$ and the path $\{O, x_1, \ldots, x_\ell\}$ are realized. Then

$$K_i(O; \bar{u}(\cdot)) = h_i(x_\ell), \quad |S_j| = i,$$
$$K_i(O; \bar{u}(\cdot)) = Sh_i(y), \quad i \in S_j,$$

where $Sh_i(y), i \in S_j$ is the Shapley value computed for the coalition S_j in the subgame $\tilde{\Gamma}(y)$, where y is the first vertex on the path $\{O, x_1, \ldots, x_l\}$ in which the coalition partition $\Delta_{\mathbb{A}}(N)$ is formed.

Weak equilibrium

Definition The n-tuple of strategies $u^*(\cdot) = (u_1^*(\cdot), \dots, u_i^*(\cdot), \dots, u_n^*(\cdot))$ is called to be a weak equilibrium in $\tilde{G}(O)$ if

$$K_i(O; u^*(\cdot)||u_i(\cdot)) \leqslant K_i(O; u^*(\cdot))$$

for $i \in N \setminus (\cup S_j)$, $|S_j| > 1$, $u_i \in B^i(u_i^*(\cdot))$.

Constructing the weak equilibrium

Assumption: the player once entering a coalition cannot leave it till the end of the game.

$2\ell + 1$: game length.

Step 0. $\gamma_i^t(x)$: the Bellman function with the conditions

$$\gamma_i^0(x) \equiv h_i(x), \quad \forall x \in P_{n+1}.$$

Step $t \geqslant 1$.

C a s e 1. $x, y \in P_{i(x)},\ y \in F_x,\ i(x) \equiv i(y)$ and $i(y) \in S$, but $i(x) \notin S\ \ \forall S \in \Delta_{\mathbb{A}_y}(N)$.

$$\max_{z \in F_y} \sum_{i \in S} \gamma_i^{t-1}(z) = \sum_{i \in S} \gamma_i^{t-1}(\bar{x}).$$

- $v(S, y) = \sum_{i \in S} \gamma_i^{t-1}(\bar{x})$;
- $v(T, y) = \max\limits_{u_T^y} \min\limits_{u_{N\setminus T}^y} \sum_{i \in T} h_i(y^*), \quad T \subset S$;
- $v(\emptyset, y) = 0$.

$Sh_i(y)\ \forall S \in \Delta_{\mathbb{A}_y}(N),\ \forall y \in F_x$.

$$\gamma_i^t(x) = \max_{y \in F_x} Sh_i(y) = Sh_i(\bar{y}), \quad i \in S,$$

$$\gamma_j^t(x) = Sh_j(\bar{y}), \quad j \notin S,$$

Case 2. $x, y \in P_i(x), i(x) \equiv i(y)$ and $i(x) \in S,\ S \in \Delta_{\mathbb{A}_y}(N)$.

$$\gamma_i^t(x) = Sh_i(y), \quad i \in S,$$

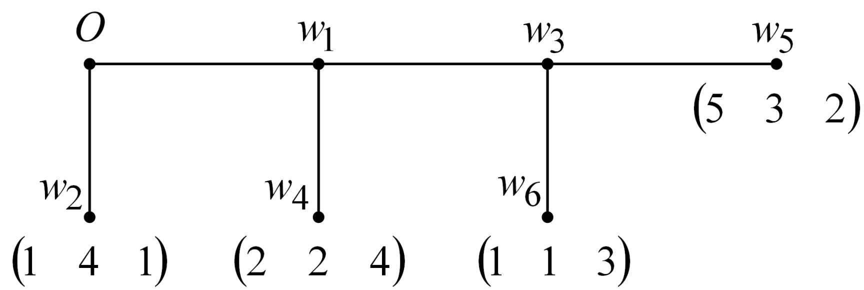
$$\gamma_j^t(x) = Sh_j(y), \quad j \notin S,$$

The vector $(\gamma_1^\ell(O), \ldots, \gamma_n^\ell(O))$ can be considered as the value of the multistage game with admissible partitions.

Theorem

There exists a weak subgame perfect equilibrium in the finite multistage game with admissible partitions.

Example



$$P_1 = \{O\}, P_2 = \{w_1\}, P_3 = \{w_3\}, P_4 = \{w_2, w_4, w_5, w_6\}.$$

- $\mathbb{A} = \{\{1\}, \{2\}, \{3\}\}$. $\Delta_{\mathbb{A}}(N) = \{\{1\}, \{2\}, \{3\}\}$. Payoffs: $(2, 2, 4)$.
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- $\mathbb{A} = \mathbb{S}$. $\Delta_{\mathbb{A}}(N) = \{\{1, 3\}, \{2\}\}$. Payoffs: $(2.5, 3, 4.5)$.