A Class of Differential Games with Random Duration

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Outline

Game duration. Main formulations

1. $t \in [t_0; T]$; duration $(T - t_0)$.
2. Infinite time horizon:
   $t \in [t_0, \infty)$; duration is ∞.

1. Integral payoff of player i:

$$K_i(x_0, u_1, \dots, u_n) = \int_{t_0}^{T} h_i(x, \tau, u) d\tau.$$

2. Integral payoff of player i:

$$K_i(x_0, u_1, \dots, u_n) = \int_{t_0}^{\infty} h_i(x, \tau, u) \cdot e^{-\rho(\tau - t_0)} d\tau.$$

Random Duration

$t \in [t_o; T]$, but T is a random variable with distribution function $F(t), \quad t \in [t_0, \infty)$! Then the duration $(T - t_0)$ is random!

References

1. Petrosjan, L.A. and Murzov, N.V. Game-theoretic problems of mechanics, 1966.
2. Yaari, M.E. Uncertain Lifetime, Life Insurance, and the Theory of the Consumer, 1965.

Integral payoff of player i:

$$K_i(x_0, u_1, \ldots, u_n) = E\left(\int_{t_0}^{T} h_i(x, \tau, u)d\tau\right) =$$

$$= \int_{t_0}^{\infty}\int_{t_0}^{t} h_i(x, \tau, u)d\tau dF(t).$$

Let $\exists \quad f(t) = F'(t)$. Then $K_i = \int_{t_0}^{\infty}\int_{t_0}^{t} h_i(x, \tau, u)d\tau f(t)dt$.

The simplification of integral functional

Integration by parts: [Burness, 1976], [Boukas, Haurie and Michel, 1990], [Chang, 2004]

$$\int_{t_0}^{\infty} \int_{t_0}^{t} h(x, u, \tau)\, d\tau\, f(t) dt. \tag{1}$$

Let $t_0 = 0$. Let us denote $h(x, u, \tau) = h(\tau)$.

$$\int_{0}^{\infty} \int_{0}^{t} h(\tau)\, d\tau\, f(t) dt. \tag{2}$$

Define

$$g(t, \tau) = f(t)h(\tau) \cdot \chi_{\{\tau \leq t\}} = \begin{cases} f(t)h(\tau), & \tau \leq t; \\ 0, & \tau > t. \end{cases} \tag{3}$$

The simplification of integral functional

$$\int_0^{+\infty} dt \int_0^t f(t)h(\tau)d\tau = \int_0^{+\infty} dt \int_0^{+\infty} g(t,\tau)d\tau \overset{!!}{=} \iint\limits_{[0,+\infty)\times[0,+\infty)} g(t,\tau)dtd\tau \overset{!!}{=}$$

$$\overset{!!}{=} \int_0^{+\infty} d\tau \int_0^{+\infty} g(t,\tau)dt = \int_0^{+\infty} d\tau \int_\tau^{+\infty} f(t)h(\tau)dt =$$

$$= \int_0^{+\infty} (1 - F(\tau))h(\tau)d\tau. \tag{4}$$

Thus, the integral payoff is

$$K_i(x_0, u_1, \ldots, u_n) = \int_{t_0}^{+\infty} (1 - F(\tau))h_i(x,\tau,u)d\tau.$$

Game $\Gamma(x_0)$

Payoff of player i: integral payoff + terminal payoff

$$K_i(x_0, u_1, \dots, u_n) = \int_{t_0}^{+\infty} (1 - F(\tau)) h_i(x, \tau, u) d\tau + \tag{5}$$

$$+ \int_{t_0}^{+\infty} f(\tau) H_i(x, \tau, u) d\tau, \qquad i = 1, \dots, n. \tag{6}$$

$$\dot{x} = g(x, u_1, \dots, u_n), \quad x \in R^n,\ u_i \in U \subseteq \text{comp}\, R^l, \tag{7}$$

$$x(t_0) = x_0.$$

Conditional distribution function

$(1 - F(\vartheta))$ is the probability to start $\Gamma(x(\vartheta))$.
The conditional distribution function $F_\vartheta(t)$:

$$F_\vartheta(t) = \frac{F(t) - F(\vartheta)}{1 - F(\vartheta)}, \quad t \in [\vartheta, \infty). \tag{8}$$

The conditional density function $f_\vartheta(t)$:

$$f_\vartheta(t) = \frac{f(t)}{1 - F(\vartheta)}, \; t \in [\vartheta, \infty). \tag{9}$$

Subgame $\Gamma(x(\vartheta))$

$(1 - F(\vartheta))$ is the probability to start $\Gamma(x(\vartheta))$.
Subgame $\Gamma(x(\vartheta))$:
Payoff

$$K_i(x, \vartheta, u_1, \dots, u_n) = \frac{1}{1 - F(\vartheta)} \int_\vartheta^\infty (1 - F(\tau)) h_i(x, \tau, u) d\tau + \frac{1}{1 - F(\vartheta)} \int_\vartheta^\infty H_i(x, \tau, u) f(\tau) d\tau. \quad (10)$$

$$\dot{x} = g(x, u_1, \dots, u_n), \quad x \in R^n, \; u_i \in U \subseteq \operatorname{comp} R^l, \qquad (11)$$
$$x(\vartheta) = x.$$

Hamilton-Jacobi-Bellman equation

Maximization problem:

$$\frac{1}{1-F(t)}\int_t^\infty \Big(h(x,u,s)(1-F(s)) + H(x,u,s)f(s)\Big)ds.$$

$$\dot{x} = g(x,u)$$
$$x(t) = x.$$

Let W be Bellman function for this problem.
Let us consider the optimization problem

$$\int_t^\infty \Big(h(x,u,s)(1-F(s)) + H(x,u,s)f(s)\Big)ds.$$

$$\dot{x} = g(x,u)$$
$$x(t) = x.$$

Let $\bar{W}$ be Bellman function for this problem.

Hamilton-Jacobi-Bellman equation

Obviously,

$$\bar{W}() = W() \cdot (1 - F(t)). \tag{12}$$

Then

$$\frac{\partial \bar{W}}{\partial t} = -f(t)W + (1 - F(t))\frac{\partial W}{\partial t}; \tag{13}$$

$$\frac{\partial \bar{W}}{\partial x} = (1 - F(t))\frac{\partial W}{\partial x}. \tag{14}$$

For the problem with Bellman function $\bar{W}$ we have the standard HJB equation:

$$\frac{\partial \bar{W}}{\partial t} + \max_u \left(h(x,u,t)(1 - F(t)) + H(x,u,t)f(t) + \frac{\partial \bar{W}}{\partial x} g(x,u) \right) = 0.$$

Using (12), (13), (14) we get HJB equation for the problem with random duration:

$$\frac{f(t)}{1 - F(t)} W = \frac{\partial W}{\partial t} + \max_u \left(h(x,u,t) + \frac{f(t)}{1 - F(t)} H(x,u,t) + \frac{\partial W}{\partial x} g(x,u) \right).$$

Hamilton-Jacobi-Bellman equation. Analysis

$$\frac{f(t)}{1-F(t)}W = \frac{\partial W}{\partial t} + \max_u\Big(h(x,u,t) + \frac{f(t)}{1-F(t)}H(x,u,t) + \frac{\partial W}{\partial x}g(x,u)\Big). \quad (15)$$

Shevkoplyas E., 2004 (without simplification to standard DP problem)

Petrosyan L.A., Murzov N.V., 1966 — Bellman-Isaacs equation for the differential pursuit game with terminal payoff at random final time instant

Game theory. Reliability theory

Game theory: T – time instant (random) when the game ends

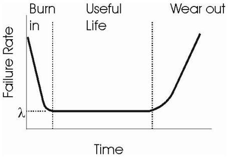
$\Downarrow$

Reliability theory: T – time of failure of the system

Hazard function (failure rate):

$$\lambda(t) = \frac{f(t)}{1 - F(t)}. \tag{16}$$

Hazard function $\lambda(t)$



The life circle and the hazard function $\lambda(t)$:

1. Burn- in ("infant"). Infant mortality or early failures. Decreasing failure rate $\lambda(t)$.
2. Normal life (useful life,"adult"). Random failures. Constant failure rate $\lambda(t) = \lambda$.
3. End of life (wear-out). Increasing failure rate $\lambda(t)$.

Hazard function $\lambda(t)$ + HJB equation

From (15) we get

$$\lambda(t)W = \frac{\partial W}{\partial t} + \max_u \left(h(x,u,t) + \lambda(t)H(x,u,t) + \frac{\partial W}{\partial x} g(x,u) \right). \quad (17)$$

Moreover, $(1 - F(t)) = e^{-\int_{t_o}^{t} \lambda(\tau)d\tau}$. Then

$$K_i(x_0, u_1, \ldots, u_n) = \int_{t_0}^{+\infty} e^{-\int_{t_o}^{\tau} \lambda(s)ds} h_i(x, \tau, u) d\tau + \quad (18)$$
$$+ \int_{t_0}^{+\infty} f(\tau) H_i(x, \tau, u) d\tau, \qquad i = 1, \ldots, n.$$

First term — problem with non-constant discounting, HJB: [Chang,2004], [Karp,2007], [Marin-Solano, 2008]

Constant hazard rate

Hazard rate is constant only for exponential distribution of failure time.

$$\frac{f(t)}{1-F(t)} = \lambda(t) = \lambda = const$$

only for density probability distribution

$$f(t) = \lambda e^{-\lambda(t-t_0)}, \qquad t > t_0.$$

Then

the problem with random exponential final time is equivalent to problem with constant discounting

[Haurie, 2005] (overlapping generations model)

"End of life"distributions

Reliability theory

For technical systems:

- Exponential
- Weibull
- Normal
- Logarithmic-normal
- Gamma

Actuarial mathematics, gerontology

For biological systems:

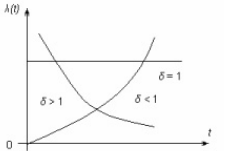
- Gompertz-Makeham
- Weibull

[Cox, Lewis, 1966]: time intervals between accidents in coal-pits and even strike actions of coalminers are well defined by Weibull

Weibull distribution

Hazard rate

$$\lambda(t) = \lambda \delta t^{\delta - 1}; \qquad (19)$$
$$t \geq 0;\ \lambda > 0;\ \delta > 0.$$



$\delta < 1$ Burn-in. $\lambda(t)$ is decreasing.
$\delta = 1$ Normal life. $\lambda(t)$ is constant λ. Weibull distribution equivalents to exponential distribution.
$\delta > 1$ Wear-out. $\lambda(t)$ is increasing. $\delta = 2$ Raleigh distribution

Next extensions

Different random durations for players $(F_i(t), i = 1, \dots, n)$
or
different types of players
$\delta < 1$ — "young"player
$\delta = 1$ — "adult"player
$\delta > 1$ — "old"player

Model of non-renewable resource extraction

E.J. Dockner, S. Jorgensen, N. van Long, G. Sorger, 2000

Players: firms, countries,...
The set of the players: $I = \{1, 2, ..., n\}$.
Let $x(t)$ be the stock of the nonrenewable resource such as an oil field.
Let $c_i(t)$ be player i's rate of resource extraction at time t.
The transition equation:

$$\dot{x}(t) = -\sum_{i=1}^{n} c_i(t); \tag{20}$$

$$\lim_{t->\infty} x(t) \geq 0; \tag{21}$$

$$x(t_0) = x_0. \tag{22}$$

Each player i has a utility function $h(c_i)$, defined for all $c_i > 0$. The utility function:

$$h(c_i) = ln(c_i). \tag{23}$$

Model of non-renewable resource extraction

Let $t_0 = 0$. The contract date has never equal to real period of fields exploitation, because either the exploitation is prematurely finished by accident or unprofitability or the period of exploitation is extended. Let T be a random variable with Weibull distribution. Expected payoff of the player i:

$$K_i(x_0, u_1, \dots, u_n) = \int_0^{\infty} (1 - F(\tau)) h_i(\cdot) d\tau, \qquad (24)$$

$$F(t) = 1 - e^{-\lambda t^{\delta}}.$$

Life-circle of resource explotation:

1 Initial phase (run-in)

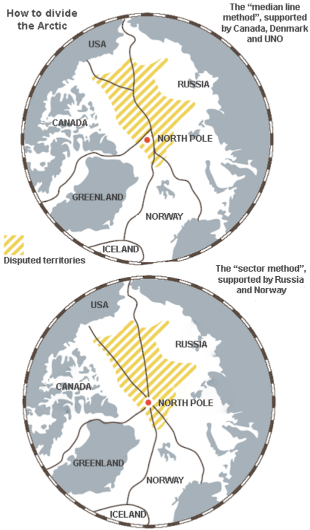
2 Regime of normal exploitation

3 Wear-out phase

0 Phase of earth resources survey

5 Phase of conservation

Oil exploitation on the continental shelf



Nash equilibrium

$$c_i^N = \frac{x \cdot e^{-\lambda(t)t}}{\int_t^{\infty} e^{-\lambda(s)s} ds}. \quad (25)$$

$$\lambda(t) = \lambda\delta t^{\delta-1}; \quad (26)$$
$$t \geq 0;\ \lambda > 0;\ \delta > 0.$$

If $\delta = 1$ (exponential distribution, "useful life")

$$c_i^N = \lambda x, \qquad i = 1, \ldots, n; \quad (27)$$

$$x^N(t) = x_0 * e^{-n\lambda t}; \quad (28)$$
$$c_i^N(t) = \lambda x_0 * e^{-n\lambda t}.$$

The results of Dockner et al. for the case with discounting rate λ.
$x^N(t)$ satisfies Lyapunov stability and asymptotic stability!

$$V(\{i\}, x(\vartheta)) = \frac{ln(\lambda x(\vartheta))}{\lambda} - \frac{n}{\lambda}. \quad (29)$$

Nash equilibrium

If $\delta = 2$ (Raileigh distribution, wear-out)

$$c_i^N = \frac{x \cdot e^{-2\lambda t^2}}{\int_t^\infty e^{-2\lambda s^2} ds}. \tag{30}$$

Then

$$c_i^N = \frac{2\sqrt{2}\sqrt{\lambda} \cdot e^{-2\lambda t^2}}{1 - erf(\sqrt{2\lambda} t)} x = \tag{31}$$

$$= \frac{2\sqrt{2}\sqrt{\lambda} \cdot e^{-2\lambda t^2}}{1 - 2\Phi_0(2\sqrt{\lambda} t)} x, \quad where \tag{32}$$

$$erf(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} ds, \tag{33}$$

$\Phi_0(t)$ – integral Laplace function.

Nash equilibrium

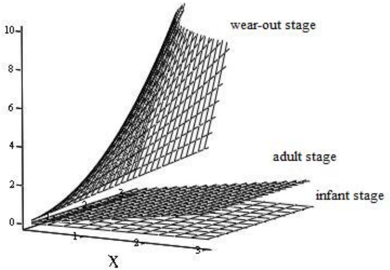
If $\delta = \frac{1}{2}$ (burn-in)

$$c_i^N = \frac{x \cdot e^{-\frac{\lambda}{2} t^{1/2}}}{\int_t^\infty e^{-\frac{\lambda}{2} s^{1/2}} ds}. \tag{34}$$

Then

$$c_i^N = \frac{\lambda^2}{4(\lambda\sqrt{t} + 2)} x. \tag{35}$$

Nash equilibrium



The optimal rate of resource extraction at an initial stage is low than for an "adult"stage (it is equivalent for accuracy of players at an early stage), but the highest rate is for wear-out stage. Of course, the resource stock $x(t)$ at the last stage tends to zero, but the optimal behavior according to our simple model is to excavate "like grim death".

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