Turnpikes for Stochastic Games and Nonlinear Markov Games

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Classical turnpike theorem

Robert Dorfman, Paul Samuelson and Robert Solow (1958) —Linear programming and economic analysis

"If the origin and destination are close together and far from the turnpike, the best route may not touch the turnpike. But if the origin and destination are far enough apart, it will always pay to get on the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at the either end."

Extensions

- Stochastic extensions in growth models
  e.g., Majumdar, Mukul (1987), Marimon, Ramon (1989), Joshi, Sumit (2003)$\cdots$
- Extension on differential games
  e.g., Carlson & Haurie (1995) $\cdots$

Games setting

Let X be a state space and μ be a reference measure on X.

A stochastic zero sum game on a state space $X = \mathbb{R}^n$

- $p(x, y, \alpha, \beta)$: the collection of the probabilities of transitions from state $x \in \mathbb{R}^n$ to $y \in \mathbb{R}^n$ for certain strategies α and β, assume the function p is bounded and continuous of α and β.
- $b(x, y, \alpha, \beta)$: the income of the second player from the transition $p(x, y, \alpha, \beta)$, assume the function b is continuous and bounded of α and β.
- $g(k) \in C(\mathbb{R}^n)$: terminal value on the position $k \in \mathbb{R}^n$.

Game setting

- 1st step: the players choose independently certain strategies α and β, and with the probability $p(x, y, \alpha, \beta)$ the game moves from the state x to the state y, and the first player pays $b(x, y, \alpha, \beta)$ to the second player.
- 2nd step: played analogously starting form the position y.
- $\cdots$
- After the last step the second player receives additional payment $g(k)$ depending on the position $k \in \mathbb{R}^n$, where the game terminates.

Bellman operator

A stochastic game with a value

$$\min_{\alpha} \max_{\beta} \int_{\mathbb{R}^n} p(x, y, \alpha, \beta)(g(y) + b(x, y, \alpha, \beta))dy$$
$$= \max_{\beta} \min_{\alpha} \int_{\mathbb{R}^n} p(x, y, \alpha, \beta)(g(y) + b(x, y, \alpha, \beta))dy \quad (1)$$

for all $g \in C(\mathbb{R}^n)$, where $x \in \mathbb{R}^n$.

Then an operator $B : C(\mathbb{R}^n) \mapsto C(\mathbb{R}^n)$ such that $B(x, g)$ is equal to equation (1) is called the Bellman operator.

Bellman operator

Properties of the game Bellman operator

$$B(a+h) = a + B(h), \quad \forall a \in \mathbb{R} \text{ and } h \in C(\mathbb{R}^n). \tag{2}$$

and

$$||B(h) - B(g)|| \leq ||h - g||, \quad \forall h, g \in C(\mathbb{R}^n) \tag{3}$$

where $||h|| = sup|h(y)|$ for $y \in \mathbb{R}^n$.

Quotient space Φ

Define the quotient space Φ of the space $C(\mathbb{R}^n)$ by the factor space with constant functions. Let $\Pi : C(\mathbb{R}^n) \mapsto \Phi$ be the natural projection.

Quotient norm on Φ

$$||\Pi(h)|| = \inf_{a\in\mathbb{R}} ||h+a|| = \frac{1}{2}(\sup_y h(y) - \inf_y h(y))$$

Continuous quotient map $\tilde{B}$

Π has a unique isometric section $S : \Phi \mapsto C(\mathbb{R}^n)$. The image $S(\Phi)$ consists of all $h \in C(\mathbb{R}^n)$ such that

$$\sup_y h(y) = -\inf_y h(y).$$

Thus one can identify $H \in \Phi$ with its image $h = S(H) \in C(\mathbb{R}^n)$.

Continuous quotient map

$$\tilde{B} : \Phi \mapsto \Phi$$

Main results

To show the main results, we need the following crucial property of the transition probability $p(x, y, \alpha, \beta)$.

Property of the transition probability $p(x, y, \alpha, \beta)$

$$\exists \delta > 0 : \forall x \in \mathbb{R}^n, \alpha, \beta : \exists A \subset \mathbb{R}^n, \mu(A) > 0 : \forall y \in A$$
$$p(x, y, \alpha, \beta) \geq \delta.$$

Main results

Lemma 1

If the above Property of the transition probability $p(x, y, \alpha, \beta)$ holds, then

- $\tilde{B}$ maps each ball of radius $R \geq C\delta^{-1}$ centered at the origin into itself.
- $$||\tilde{B}(H) - \tilde{B}(G)|| \leq (1 - \delta)||H - G||, \quad \forall H, G \in \Phi.$$

Main results

Theorem 1 (On the average income)

If the above property of the transition probability $p(x, y, \alpha, \beta)$ holds, then

- there exists a unique $\lambda \in \mathbb{R}$ and a unique $h \in C(\mathbb{R}^n)$ such that
$$B(h) = \lambda + h,$$
and for all $g \in C(\mathbb{R}^n)$ we have
$$||B^m g - m\lambda|| \leq ||h|| + ||h - g||,$$
$$\lim_{m\to\infty} \frac{B^m g}{m} = \lambda.$$
- h is unique up to equivalence, i.e., $H = \Pi(h)$ is unique, and
$$||\tilde{B}^m(G) - H|| \leq (1-\delta)^m ||G - H||, \quad \forall G \in \Phi.$$

Main results

Remarks

For per unit cost, this theorem has been much developed in many settings

- In differential games
  e.g., Kushner and Martino Bardi, $\cdots$
- In discrete setting
  e.g., Sylvain Sorin $\cdots$

Main results

- $E(g) = E(G)$: the set of equilibrium (minimax) strategies for $g \in C(\mathbb{R}^n)$ and $G \in \Phi$
- $E(B^{T-t}g)$: the set of equilibrium strategies on the step t of the T-step game with terminal income $g \in C(\mathbb{R}^n)$ of the second player

Theorem 2 (Turnpikes on the set of strategies)

Let the property of the transition probability $p(x, y, \alpha, \beta)$ holds, then for arbitrary $\Omega > 0$ and a neighborhood $U(E(h))$ of the set $E(h)$, there exists an $M \in \mathbb{N}$ such that for $T > M$ and $||\Pi(g)|| \leq \Omega$, then

$$E(B^{T-t}g) \subset U(E(h))$$

for all $t < T - M$.

Main results

Theorem 3 (Turnpikes on the sate space)

- Assume additionally that for each $x \in \mathbb{R}^n$, $E_x(h)$ contains only one pair of strategies $\tilde{\alpha}, \tilde{\beta}$.
- Let $\tilde{q}(x)$ denote the stationary distribution for the stochastic game defined on the state space $\mathbb{R}^n$ by these strategies.
- Assume also that the equilibrium transitions depend locally Lipschitz continuous on g and h, i.e., that there exists a constant k such that

$$|p(x, y, \alpha, \beta) - p(x, y, \tilde{\alpha}, \tilde{\beta})| \leq k||g - h||$$

for all $x, y \in \mathbb{R}^n$, $(\alpha, \beta) \in E_x(g)$ and g sufficiently close to h.

Main results

Theorem 3 (Turnpikes on the sate space)–Cont.

Then for all $\epsilon > 0$ and $\Omega > 0$ there exists an $M \in \mathbb{R}$ such that for each T-step game, $T > 2M$, with terminal income $g \in C(\mathbb{R}^n)$, $||\Pi(g)|| < \Omega$, of the second player we have

$$||q(x,t) - \tilde{q}(x)|| < \epsilon$$

for $t \in [M, T - M]$, where $q(x,t)$ is the probability that the game is in a state x at time t if the game is carried out with the equilibrium strategies.

Remark

These resluts are extensions of the results on a finite state space $X = (1, \cdots, n)$ considered by V.N. Kolokoltsov in On linear, additive, and homogeneous operators in Idempotent Analysis. *Advances in Soviet Mathematics 13 (1992), Idempotent Analysis, Ed. V.P.Maslov et S.N. Samborski, 87-101.*

Extension to nonlinear Markov games

- Peter Caines: consider stochastic control problems in nonlinear Markov context, but there is no games in their setting;
- Tomas Bjork: consider nonlinear Markov control (distribution dependence via its variance with underlying jump-diffusion process)

Our aim

Consider long-time nonlinear Markov games with a general setting

Extension to nonlinear Markov games

Nonlinear Markov games on a state space $\mathbb{R}^n$

- U, V: metric spaces of the control parameters of two players
- $$\Sigma = \{\mu(x) : \mathbb{R}^n \to \mathbb{R}^+ | \int_{\mathbb{R}^n} \mu(x)dx = 1\}$$
  represents the set of probability laws
- continuous mapping $\nu : \Sigma \mapsto \Sigma$: $\nu(y, u, v, \mu)$ the new state obtained from μ once the players had chosen their strategies $u \in U$, $v \in V$, $x, y \in \mathbb{R}^n$
- $g(x, y, u, v, \mu)$: a continuous transition cost function, $u \in U$, $v \in V$ and $\mu \in \Sigma$
- S: a given final cost function on Σ

Extension to nonlinear Markov games

Bellman operator

$$(BS)(\mu) = \min_u \max_v [g(u, v, \mu) + S(\nu(u, v, \mu))]$$

Stochastic representation for ν

$$\nu(y, u, v, \mu) = \int_{\mathbb{R}^n} P(x, y, u, v, \mu)\mu(x)dx$$

for $y \in \mathbb{R}^n$ where $P(x, y, \mu)$ is a family of transition probabilities depending on μ (nonlinearity).

Representation for g

$$g(u, v, \mu) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} P(x, y, u, v, \mu)\mu(x)g(x, y)dxdy$$

where $g(x, y) : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$.

Extension to nonlinear Markov games

Bellman operator

$$(BS)(\mu) = \min_u \max_v \Big[\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} P(x, y, u, v, \mu)\mu(x)g(x, y)dxdy + S\Big(\int_{\mathbb{R}^n} P(x, y, \mu)\mu(x)dx \Big) \Big] \tag{4}$$

Extension to nonlinear Markov games

Remark

This extension to a class of nonlinear Markov games with a locally compact state space is introduced in
V.N. Kolokoltsov. Nonlinear Markov processes and kinetic equations. To appear in Cambridge Univ. Press, August 2010.

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