**On the extensions of the mollified Boltzmann and Smoluchovski equations to $k$-nary interacting particle systems.**

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**Abstract.** We deduce the kinetic equations describing the low density (and the large number of particles) limit of interacting particle systems with $k$-nary interaction of pure jump type supplemented by an underlying "free motion" being an arbitrary Feller process. The well posedness of the Cauchy problem together with the propagation of chaos property are proved for these kinetic equations under some reasonable assumptions. Particular cases of our general equations are given by (spatially non-trivial) Boltzmann and Smoluchovski equations with mollifier. Even for the classical binary models our analysis yield new results.

**Key words.** Interacting particles, $k$-nary interaction, measure-valued limits, kinetic equations, Boltzmann equation, coagulation and fragmentation, propagation of chaos.

**Running Head:** Kinetic equations for $k$-nary interaction.

AMS 2000 subject classification 60K35, 82C21, 82C22.

## 1. Introduction.

*1. Aims of the paper and its content.* This paper is a continuation of the author's work on measure-valued limits of $k$-nary interacting particle systems (see [Ko1], [Ko2], [Ko3]). It extends the main results from [Ko4] devoted to the interactions of only pure jump type to the case when a non-trivial spatial motion is present in the model. Major particular cases concern (i) spatially non-trivial mass exchange processes which include the Smoluchovski models with continuous mass distributions and its extensions with not necessary binary coagulations or fragmentation, (ii) processes of collisions described by the Boltzmann type kernels. The main objectives of the paper are (i) to show that as a number of particles go to infinity and under a natural scaling of interaction rates, the processes of $k$-nary interacting particle systems converge to measure-valued deterministic processes (hydrodynamic limits), (ii) to derive general kinetic equations that describe the evolution of these limiting processes, (iii) to prove the well-posedness of the Cauchy problem and the propagation of chaos property for these equations.

In [BK] we used a different method to obtain similar kinetic equations which was formal (i.e. without any rigorous convergence or existence results). In fact, we developed two such methods, one was suggested in [Be] and was based on the study of the evolution of the generating functionals and another was based on the idea of propagation of chaos (see e.g. [BM], [MT], [Sz]). In this paper we shall justify the formal calculations of [BK] for the models under consideration proving the propagation of chaos property for the solutions of our kinetic equations.

The paper is organized as follows. In the introduction, we first fix some general notations, then describe our basic Markov models for $k$-nary interacting particles and present a heuristic deduction of the corresponding kinetic equations, and finally discuss the basic assumptions on the model which are needed to the rigorous analysis that follows. In Section 2 we deduce the main properties of our Markov processes of $k$-nary interacting particles. In Section 3 we formulate and in Section 4 prove our main results on the weak convergence of these Markov processes to the deterministic processes described by the kinetic equations and on the well-posedness and the propagation of chaos property for these equations. Section 5 is devoted to the regularity of the solutions to kinetic equations obtained above outlying also a non-probabilistic approach to proving the existence of these solutions. In Section 6 the most important particular examples are discussed. In Appendix we collect the auxiliary results needed in the main text.

*2. General notations.* We list here a few notations that will be used throughout the paper without further reminder:

$\mathbf{1}$ denotes the function that equals identically 1 or the identity operator in a Banach space; $\mathbf{1}_M$ for a set $M$ denotes the indicator function of $M$ that equals 1 for $x \in M$ and vanishes otherwise; $o(1)_{x \to a}$ denotes a function depending on $x$ that tends to zero as $x \to a$;

for a measurable space $Y$, $B(Y)$ denotes the Banach space of real bounded measurable functions on $Y$ equipped with the usual sup-norm; if $Y$ is a topological space, $C_b(Y)$ denotes the Banach subspace of $B(Y)$ consisting of continuous functions; $\mathcal{M}(Y)$ is the space of finite (signed) measures on $Y$, considered as a Banach space with the norm $\|.\|$ on $\mathcal{M}(Y)$ being the total variation norm; for $\mu \in \mathcal{M}(Y)$ we shall denote by $|\mu|$ the total variation measure of $\mu$ so that $\|\mu\| = \int_Y |\mu|(dy)$;

if $X$ is a locally compact space, $C_0(X)$ (respectively $C_c(X)$) is the Banach space of continuous functions vanishing at infinity and equipped with sup-norm (respectively its subspace consisting of functions with a compact support);

the upper subscript "+" for all these spaces (e.g. $C_0^+(X)$, $\mathcal{M}^+(X)$) will denote the corresponding cones of non-negative elements;

$(f, \nu) = \int f(x)\nu(dx)$ is the usual pairing between $C_b(X)$ and $\mathcal{M}(X)$; if $A$ is a linear operator in $C_b(X)$, we denote by $A^\star$ its dual operator acting in $\mathcal{M}(X)$;

by a symmetric function of $n$ variables we shall understand a function, which is symmetric with respect to all permutations of these variables, and by a symmetric operator on the space of functions of $n$ variables we shall understand an operator that preserves the set of symmetric functions;

for a finite subset $I = \{i_1, ..., i_k\}$ of a countable set $J$, we denote by $|I|$ the number of elements in $I$, by $\bar{I}$ its complement $J \setminus I$, by $\mathbf{x}_I$ the collection of the variables $x_{i_1}, ..., x_{i_k}$ and by $d\mathbf{x}_I$ the measure $dx_{i_1} ... dx_{i_k}$.

A transitional kernel from $X$ to $Y$ means, as usual, a measurable function $\mu(x, .)$ from $x \in X$ to the cone of positive finite measures on $Y$.

We shall denote by $D_X[0, \infty)$ the Skorokhod space of càdlàg paths $[0, \infty) \mapsto X$ equipped with the standard filtration $\mathcal{F}_t$. We shall denote by bald $\mathbf{P}$ and $\mathbf{E}$ the probability and the expectation of events and functions respectively.

*3. State space and observables of systems of interacting particles.* Throughout the paper we shall denote by $X$ a locally compact metric space equipped with its Borel sigma algebra. Denoting by $X^0$ a one-point space and by $X^j$ the powers $X \times ... \times X$ ($j$-times) considered with their product topologies, we shall denote by $\mathcal{X}$ their disjoint union $\mathcal{X} = \cup_{j=0}^{\infty} X^j$, which is again a metric locally compact space. In applications, $X$ specifies the state space of one particle and $\mathcal{X} = \cup_{j=0}^{\infty} X^j$ stands for the state space of a random number of similar particles. We shall denote by $B_{sym}(\mathcal{X})$ (resp. $C_{sym}(\mathcal{X})$) the Banach spaces of symmetric bounded measurable (resp. continuous) functions on $\mathcal{X}$ and by $B_{sym}(X^k)$ (resp. $C_{sym}(X^k)$) the corresponding spaces of functions on the finite power $X^k$. The space of symmetric measures will be denoted by $\mathcal{M}_{sym}(\mathcal{X})$. The elements of $\mathcal{M}^+_{sym}(\mathcal{X})$ and $C_{sym}(\mathcal{X})$ are called respectively the (mixed) states and observables for a Markov process on $\mathcal{X}$. We shall denote the elements of $\mathcal{X}$ by bold letters, e.g. $\mathbf{x}$, $\mathbf{y}$. Sometimes it is convenient to consider the factor spaces $SX^k$ and $S\mathcal{X}$ obtained by the factorization of $X^k$ and $\mathcal{X}$ with respect to all permutations, which allows for the identifications $C_{sym}(\mathcal{X}) = C(S\mathcal{X})$ and likewise. Symmetrical laws on $X^k$ (which are uniquely defined by their projections to $SX^k$) are called exchangeable systems of $k$ particles. A key observation for the theory of measure-valued limits is the inclusion $S\mathcal{X}$ to $\mathcal{M}^+(X)$ given by

$$\mathbf{x} = (x_1, ..., x_l) \mapsto \delta_{x_1} + ... + \delta_{x_l}, \tag{1.1}$$

which defines a bijection between $S\mathcal{X}$ and the space $\mathcal{M}^+_{\delta}(X)$ of finite linear combinations of $\delta$-measures (notice that our inclusion (1.1) differs by a normalization from the form of this inclusion discussed in [Da]).

Clearly each $f \in B_{sym}(\mathcal{X})$ is defined by its components $f^k$ on $X^k$ so that for $\mathbf{x} = (x_1, ..., x_k) \in X^k \subset \mathcal{X}$, say, one can write $f(\mathbf{x}) = f(x_1, ..., x_k) = f^k(x_1, ..., x_k)$ (the upper index $k$ at $f$ is optional and is used to stress the number of variables in an expression). Similar notations are for measures. In particular, the pairing between $C_{sym}(\mathcal{X})$ and $\mathcal{M}(\mathcal{X})$ can be written as

$$(f, \rho) = \int f(\mathbf{x})\rho(d\mathbf{x}) = f^0\rho_0 + \sum_{n=1}^{\infty} \int f(x_1, ..., x_n)\rho(dx_1...dx_n),$$

$$f \in C_{sym}(\mathcal{X}),\ \rho \in \mathcal{M}(\mathcal{X}), \tag{1.2}$$

so that $\|\rho\| = (\mathbf{1}, \rho)$ for $\rho \in \mathcal{M}^+(\mathcal{X})$.

A useful class of measures (and mixed states) on $\mathcal{X}$ is given by the decomposable measures of the form $Y^{\otimes}$ and $Y^{\tilde{\otimes}}$, which are defined for an arbitrary finite measure $Y(dx)$ on $X$ by their components

$$(Y^{\otimes})_n(dx_1...dx_n) = Y^{\otimes n}(dx_1...dx_n) = Y(dx_1)...Y(dx_n) \tag{1.3}$$

and

$$(Y^{\tilde{\otimes}})_n(dx_1...dx_n) = \frac{1}{n!}Y^{\otimes n}(dx_1...dx_n) = \frac{1}{n!}Y(dx_1)...Y(dx_n). \tag{1.4}$$

Notice that unlike $Y^{\tilde{\otimes}}$ the measure $Y^{\otimes}$ need not be a finite measure on $\mathcal{X}$ even when $Y$ is finite. Similarly the decomposable observables (or exponential vectors) are defined for an arbitrary $g \in C(X)$ as

$$(g^{\otimes})^n(x_1, ..., x_n) = g^{\otimes n}(x_1, ..., x_n) = g(x_1)...g(x_n). \tag{1.5}$$

We shall use also the additively decomposable observables defined for an arbitrary $g \in C(X)$ as

$$(g^+)(x_1, ..., x_n) = g(x_1) + ... + g(x_n) \tag{1.6}$$

($g^+$ vanishes on $X^0$). In particular, if $g = \mathbf{1}$, then $g^+ = \mathbf{1}^+$ is the number of particles: $\mathbf{1}^+(x_1, ..., x_n) = n$.

*4. Basic Markov models for k-nary interacting particles.* A $k$-nary interaction is specified by a transition kernel

$$P^k(x_1, ..., x_k; d\mathbf{y}) = \{P^k_m(x_1, ..., x_k; dy_1...dy_m)\}$$

from $SX^k$ to $S\mathcal{X}$ with the intensity

$$P^k(x_1, ..., x_k) = \int P^k(x_1, ..., x_k; d\mathbf{y}) = \sum_{m=0}^{\infty} \int P^k_m(x_1, ..., x_k; dy_1...dy_m)$$

such that $P^k(\mathbf{x}; \{\mathbf{x}\}) = 0$ for all $x \in X^k$. The intensity defines the rate of decay of any collection of $k$ particles $x_1, ..., x_k$ and the measure $P^k(x_1, ..., x_k; d\mathbf{y})$ defines the distribution of possible outcomes. Supposing that any $k$ particles from a given set of $n \geq k$ particles can interact, we arrive to the following generator of $k$-nary interacting particle systems specified by the kernel $P^k$:

$$(G_k f)(x_1, ..., x_n)$$

$$= \sum_{m=0}^{\infty} \sum_{I \subset \{1,...,.n\}, |I|=k} \int (f^{l+(n-k)}(\mathbf{x}_{\bar{I}}, y_1, ..., y_m) - f(x_1, ..., x_n)) P^k_m(x_I; dy_1...dy_m)$$

$$= \sum_{I \subset \{1,...,.n\}, |I|=k} \int (f(\mathbf{x}_{\bar{I}}, \mathbf{y}) - f(x_1, ..., x_n)) P^k(x_I, d\mathbf{y}). \tag{1.7}$$

Taking into account all possible interactions of order $\leq k$ given by the kernels $P = \{P^l_m\}$, $l = 0, ..., k$, $m = 0, 1, ...$, with the intensity

$$P(\mathbf{x}) = \int P(\mathbf{x}; d\mathbf{y}) = \int \sum_{m=0}^{\infty} P^l_m(x_1, ..., x_l; dy_1, ..., dy_m) \tag{1.8}$$

whenever $\mathbf{x} = (x_1, ..., x_l)$ with $l \leq k$, and $P(\mathbf{x}) = 0$ whenever $\mathbf{x} \in X^l$ with $l > k$, yields the generator of a general pure jump Markov processes with the interaction of order $\leq k$:

$$(G_{\leq k} f)(x_1, ..., x_n) = \sum_{I \subset \{1,...,n\}} \int (f(\mathbf{x}_{\bar{I}}, \mathbf{y}) - f(x_1, ..., x_n)) P(\mathbf{x}_I, d\mathbf{y}). \tag{1.9}$$

Now let $A$ be an operator in $C_0(X)$ with domain $D(A) \subset C_c(X)$ whose closure generates a (unique) Feller semigroup in $C_0(X)$ and the corresponding Feller process $z_s^x$ with sample paths in $D_X[0,\infty)$ ($x$ stands for the initial point). It implies (see e.g. [EK]) that the corresponding martingale problem is well posed for $A$, i.e. for any $x \in X$, the process $z_s^x$ yields a unique distribution on $D_X[0,\infty)$ such that

$$M_t = f(z_t^x) - f(x) - \int_0^t Af(z_s^x)\,ds \tag{1.10}$$

is an $\mathcal{F}_t$-martingale for all $f \in D(A)$ with $M_0 = 0$ almost surely. The process $z_s^x$ describes the underlying "free" motion of particles on $X$. The operator $A$ naturally induces (second quantization procedure) the operator $A_{diag}$ on $C_b(S\mathcal{X})$ with domain $D(A_{diag})$ being the linear span of the the tensor product $D(A) \otimes ... \otimes D(A)$. This operator preserves each $C_b(SX^k)$ (i.e. it is diagonal) and is uniquely specified by its action on decomposable observables:

$$(A_{diag}g^{\otimes n})(x_1,...,x_n) = \sum_{j=1}^n g(x_1)...g(x_{j-1})(Ag)(x_j)g(x_{j+1})...g(x_n) \tag{1.11}$$

for any $g \in D(A)$. The operator $A_{diag}$ describes the independent motion of any finite collection of particles on $X$. Our basic model of interacting particles with a pure jump $k$-nary interaction and subject to a underlying free motion specified by $A$ is defined as a Markov process on $\mathcal{X}$ with the generator

$$G = A_{diag} + G_{\le k} \tag{1.12}$$

on $C_b(S\mathcal{X})$. Surely one needs some additional assumptions to ensure the existence and uniqueness of so defined Markov process (see below).

Changing the state space by (1.1) yields the corresponding Markov process on $\mathcal{M}_\delta^+(X)$. In order to see how $A_{diag}$ acts in $B(\mathcal{M}_\delta(X))$ it is instructive to rewrite (1.11) in the form

$$[A_{diag}\exp\{(\ln g,.)\}](\nu) = (Ag,\nu)\exp\{(\ln g,\nu)\} \tag{1.13}$$

for $g \in D(A)$ taking values in $(0,1]$, where $\nu = \delta_{x_1} + ... + \delta_{x_n}$.

Formula (1.13) defines the natural extension of $A_{diag}$ to $B(\mathcal{M}(X))$, as the linear span of functions $\mu \mapsto \exp\{(\ln g,\mu)\}$ with $g$ taking value in $(0,1]$ is dense in $C_b(\mathcal{M}(X))$. For example, by differentiation, one gets from (1.13) that

$$[A_{diag}(\ln g,.)^k](\nu) = k(Ag,\nu)(\ln g,\nu)^{k-1}.$$

Choosing a positive parameter $h$, we shall perform now the following scaling: firstly, we scale the empirical measures $\delta_{x_1} + ... + \delta_{x_n}$ by a factor $h$, secondly we scale all $l$-nary interactions by a factor $h^l$, and thirdly we scale the whole generator by $1/h$ (which effectively means the scaling of time as in the theory of superprocesses, see e.g. [Dy]). After this scaling the operator (1.12) takes the form

$$\Lambda^h f(h\nu) = (A^h_{diag} + G^h_{\le k})f(h\nu) = A_{diag}f(h\nu) + \frac{1}{h}\sum_{l=0}^k h^l \sum_{I\subset\{1,...,n\},|I|=l} \sum_{m=0}^\infty$$

$$\times \int [f(h\nu - \sum_{i\in I} h\delta_{x_i} + h\delta_{y_1} + ... + h\delta_{y_m}) - f(h\nu)] P(\mathbf{x}_I; dy_1...dy_m) \tag{1.14}$$

(we have omitted the index $k$ from the notation of $\Lambda$ for brevity) and acts on the space $B(\mathcal{M}^+_{h\delta}(X))$ of functions defined on the set $\mathcal{M}^+_{h\delta}(X)$ of measures of the form $h\nu = h\delta_{x_1} + ... + h\delta_{x_n}$ . For linear functions

$$f_g(\mu) = \int g(x)\mu(dx) = (g, \mu) \tag{1.15}$$

on $\mathcal{M}(X)$, one has

$$\Lambda^h f_g(h\nu) = (A(g), h\nu) + \frac{1}{h} \sum_{l=0}^{k} h^l \sum_{I\subset\{1,...,n\}, |I|=l}$$

$$\times \int [hg^+(\mathbf{y}) - h\sum_{i\in I} g(x_i)] P(\mathbf{x}_I; d\mathbf{y}). \tag{1.16}$$

*5. Kinetic equations in weak, mild and interaction representations.* Applying formula (A2) from Appendix yields for the r.h.s. of (1.14) the formula

$$A_{diag} f(h\nu) + \frac{1}{h} \sum_{l=0}^{k} \frac{1}{l!} \sum_{m=0}^{\infty} \int [f(h\nu - h\delta_{z_1} - ... - h\delta_{z_l} + h\delta_{y_1} + ... + h\delta_{y_m}) - f(h\nu)]$$

$$\times P(z_1, ..., z_l; dy_1...dy_m) \prod_{j=1}^{l} (h\delta_{x_1} + ... + h\delta_{x_n})(z_j)$$

$$+ \frac{1}{h} \sum_{\Gamma} \alpha_\Gamma h^{l-p} \int [f(h\nu - h\gamma_1\delta_{z_1} - ... - h\gamma_p\delta_{z_p} + h\delta_{y_1} + ... + h\delta_{y_m}) - f(h\nu)]$$

$$\times P(z_1, ..., z_1, ..., z_p, ..., z_p; dy_1...dy_m) \prod_{j=1}^{p} (h\delta_{x_1} + ... + h\delta_{x_n})(z_j), \tag{1.17}$$

where $\sum_\Gamma$ is the sum over all Young schemes $\Gamma = \{1 \leq \gamma_1 \leq ... \leq \gamma_p\}$ with $\gamma_p > 1$ and $\gamma_1 + ... + \gamma_p = l \leq k$, and $P(z_1, ..., z_1, ...z_p, ...z_p; dy_1...dy_m)$ means that the first $\gamma_1$ arguments of $P$ equal $z_1$, the next $\gamma_2$ arguments equal $z_2$ etc. As $h \to 0$ and $h\delta_{x_1} + ... + h\delta_{x_n}$ tends to some finite measure $\mu$ (i.e. the number of particles tends to infinity, but the "whole mass" remains finite due to the scaling of each atom), the operator (1.17) tends (formally at least) to

$$(\Lambda f)(\mu) = A_{diag} f(h\nu) + \sum_{l=0}^{k} \frac{1}{l!} \sum_{m=0}^{\infty} \int [-\frac{\delta f}{\delta\mu(z_1)} - ... - \frac{\delta f}{\delta\mu(z_l)} + \frac{\delta f}{\delta\mu(y_1)} + ... + \frac{\delta f}{\delta\mu(y_m)}]$$

$$\times P(z_1,...,z_l;dy_1...dy_m)\prod_{j=1}^l \mu(dz_j). \tag{1.18}$$

The equation $\dot{f}=\Lambda f$ describing the evolution of functions on $\mathcal{M}^+(X)$ is a first order partial differential equation in functional derivatives. The kinetic equations we are interested in are the characteristics of this equation describing the evolution of measures themselves, i.e. the deterministic process $t\mapsto \mu_t$ such that $f(\mu_t)$ satisfies the equation $\dot{f}=\Lambda f$. In the present infinite-dimensional framework, it is convenient to write these equations in the weak form. To get them, we pick up a linear function $f_g$ of form (1.15) on $\mathcal{M}(X)$ and insert $f_g(\mu_t)$ in the evolution equation $\dot{f}=\Lambda f$. As $\frac{\delta f_g}{\delta\mu(z)}=g(z)$ and $A_{diag}f_g(\mu)=(A(g),\mu)$ this leads to the following *weak kinetic equation*

$$\frac{d}{dt}(g,\mu_t)=(A(g),\mu_t)+\sum_{l=0}^k \frac{1}{l!}\int_{z_1,...,z_l,\mathbf{y}}[g^+(\mathbf{y})-g(z_1)-...-g(z_m)]$$

$$\times P(z_1,...,z_l;d\mathbf{y})\prod_{j=1}^l \mu_t(dz_j).$$

or in the concise form (using notation (1.4))

$$\frac{d}{dt}(g,\mu_t)=(A(g),\mu_t)+\int (g^+(\mathbf{y})-g^+(\mathbf{z}))P(\mathbf{z};d\mathbf{y})\mu_t^{\tilde{\otimes}}(d\mathbf{z}), \tag{1.19}$$

which must hold for $g\in D(A)$. Integrating over time yields the following *integral version of the weak kinetic equation*

$$(g,\mu_t)-(g,\mu_0)-\int_0^t ds\left((A(g),\mu_s)+\int (g^+(\mathbf{y})-g^+(\mathbf{z}))P(\mathbf{z};d\mathbf{y})\mu_s^{\tilde{\otimes}}(d\mathbf{z})\right)=0. \tag{1.20}$$

Next, by the standard perturbation theory approach (Du Hammel formula), (1.19) is formally equivalent (see Lemma A4 for a version of the rigorous result) to the following (strong) *mild kinetic equation*

$$\mu_t=e^{A^\star t}\mu_0+\int_0^t e^{A^\star(t-s)}\pi(\mu_s)\,ds \tag{1.21}$$

where the integral is well defined as the Riemann or Lebesgue integral with respect to the norm topology on $\mathcal{M}(X)$ and where $\pi(\nu)$ is the nonlinear transformation on measures from the r.h.s. of (1.20) defined by

$$(g,\pi(\nu))=\int (g^+(\mathbf{y})-g^+(\mathbf{z}))P(\mathbf{z};d\mathbf{y})\nu^{\tilde{\otimes}}(d\mathbf{z}). \tag{1.22}$$

We shall say that $\mu_t$ is a weakly mild solution if (1.21) is satisfied in the weak sense. At last, in the particular case when the semigroup $e^{-A^\star t}$ is well defined (which holds e.g.

when $A$ is a first order partial differential operator), one can apply $e^{-A^\star t}$ to the both sides of (1.21) to get the *kinetic equation in the interaction representation* (respectively *the weak interaction representation*)

$$\nu_t = \nu_0 + \int_0^t e^{-A^\star s} \pi(e^{A^\star s} \nu_s) \, ds \tag{1.23}$$

for $\nu_t = e^{-A^\star t} \mu_t$, where the integral is well defined as the Riemann or Lebesgue integral (respectively, in the weak sense).

Equations (1.19)-(1.23) can be called regular kinetic equations, the term "regular" expressing the fact that the kernel $P(\mathbf{z}; d\mathbf{y})$ is weakly continuous in $z$. They include spatially non-trivial Boltzmann and Smoluchiovski equations with a mollifier. The main classical spatially non-trivial kinetic equations (e.g. the Boltzmann equation without a mollifier) are not regular in this sense, their transition kernel being singular due to the assumption of the locality of interaction. At the end of the paper we shall show how these equations can be obtained (formally at least) from the regular kinetic equations by a natural limiting procedure.

*6. Basic assumptions on the model.* In order to justify the above calculations one needs of course some additional assumptions. Aiming at classical models of statistical mechanics (coagulation, fragmentation, collisions), we shall reduce our consideration to the case when the phase space $X$ and the operator $A$ have an additional product structure, namely we assume the following condition:

(C1) $X = Q \times V$, where both $Q$ and $V$ are locally compact metric space (we shall denote by $\mathcal{Q}$ and $\mathcal{V}$ the unions $\cup_{j=0}^\infty Q^j$ and $\cup_{j=0}^\infty V^j$ respectively) and the operator $A$ is specified by its action on factorizable functions $(g \otimes h)(q, v) = g(q)h(v)$ as

$$(A(g \otimes h))(q, v) = h(v)(B_v g)(q), \quad q \in D(B),$$

where $B_v$, $v \in V$, is a family of linear operators in $C_0(Q)$ defined on the same domain $D(B) = D(B_v)$ such that the closure of each $B_v$ generates a conservative Feller semigroup on $C_0(Q)$; one easily sees that the closure of this $A$ defined on the domain $D(B) \times C_c(V)$ generates a Feller semigroup on $C_c(X)$.

We shall discuss now the basic assumptions on the transition kernel $P$. Essentially they represent natural extensions to $k$-nary interaction of basic assumptions used in binary models (see e.g. [No] and references therein). Let $L$ be a non-negative function on $X$. As this function can be often interpreted as a mass or a size of a particle, we shall call the number $L(x)$ the size of a particle $x$. We say that $P(\mathbf{x}; d\mathbf{y})$ in (1.9) is *$L$-preserving* (respectively *$L$-non-increasing*) if the measure $P(\mathbf{x}; d\mathbf{y})$ is supported on the set $\{\mathbf{y} : L^+(\mathbf{y}) = L^+(\mathbf{x})\}$ (respectively $\{\mathbf{y} : L^+(\mathbf{y}) \leq L^+(\mathbf{x})\}$). We say that $P(\mathbf{x}; d\mathbf{y})$ is $l$-nary $L$-preserving or $L$-non-increasing, if the corresponding property holds only for $\mathbf{x} \in X^l$. We shall say that the intensity (1.8) is *multiplicatively $L$-bounded* or *$L^\otimes$-bounded* (respectively *additively $L$-bounded* or *$L^+$-bounded*) whenever $P(\mathbf{x}) \leq cL^\otimes(\mathbf{x})$ (respectively $P(\mathbf{x}) \leq cL^+(\mathbf{x})$) for all $\mathbf{x}$ and some constant $c > 0$, where we used the notations (1.5), (1.6). In the future, we shall always take $c = 1$ here for brevity. We shall need also some conditions that forbid the creation of dust, i.e. of a large number of small particles. One can work with various conditions of this kind. We shall use the following two conditions (see some other possibilities in [Ko4]):

(C2) $P(\mathbf{x}; d\mathbf{y})$ is $l$-nary $\mathbf{1}$-non-increasing (i.e. the number of particles does not increase by $l$-nary interactions) for $l \geq 2$ and the number of particles created by an act of $l$-nary interaction with $l = 1$ is uniformly bounded, i.e. $P^1_m(x; dy_1...dy_m) = 0$ for all $x$ and $m > m_0$ with some $m_0$,

(C3) if $k > 2$, then either $P$ is $\mathbf{1}$-non-increasing (in particular, no fragmentation is allowed) or $L(x) \geq \epsilon$ for all $x$ and some $\epsilon > 0$, i.e. the size of the particles can not become arbitrary small.

*Remark.* Condition (C3) is void for $k \leq 2$ and is the simplest (and a very strong) assumption for dealing with some additional technical difficulties arising from the interactions of higher orders.

## 2. Preliminaries.

We prove here the basic properties of our model of $k$-nary interacting particles given by operator (1.14). The first Proposition 2.1 is devoted to the general model with $L^{\otimes}$-bounded intensity. The second Proposition 2.2 shows that in case of $(1+L)^+$-boundedness, the corresponding Markov process enjoys a remarkable property: it preserves the finiteness of the moments of the convex functions of $L$. The third Proposition 2.3 is devoted to the tightness of the family of our Markov models of interacting particle systems as $h \to 0$.

**Proposition 2.1.** *Suppose*

*(i) the transition kernel $P(\mathbf{x}, .)$ is $L$-non-increasing and $(1+L)^{\otimes}$-bounded for some continuous non-negative function $L$ on $X$, and is a continuous function from $S\mathcal{X}$ to $\mathcal{M}^+(S\mathcal{X})$, where $\mathcal{M}^+(S\mathcal{X})$ is considered with its weak topology;*

*(ii) conditions (C1), (C2) hold and $L$ is a function of the second variable only, i.e. $L(q, v) = L(v)$ for all $q, v$, and such that $L(v) \to \infty$ as $v \to \infty$;*

*(iii) $P^0(dy) = 0$, i.e. no spontaneous input in the system is possible.*

*Then (i) the Markov process $Z^{h\nu}(t)$ with generator (1.14) and the starting point $h\nu$ in $\mathcal{M}^+_{h\delta}(X) \subset \mathcal{M}^+(X)$ is uniquely defined; (ii) the process $f_L(Z^{h\nu}(t))$ is a non-negative supermartingale, where $f_L$ is defined by (1.15); (iii) given arbitrary $b > 0$, $T > 0$, for a family $Z^{h\nu}(t)$ with $h \in (0, 1]$ and initial $h\nu$ with $(\mathbf{1} + L, h\nu) \leq b$, it follows that*

$$\mathbf{P}\left(\sup_{t\in[0,T]} f_{L+1}(Z^{h\nu}(t)) > r\right) \leq \frac{C(T, b, M_2)}{r} \tag{2.1}$$

*for all $r > 0$, where the constant $C(T, b, M_2)$ depends on $T$,$b$ and $M_2$ from (ND1) (but not on $h$); (iv) the moment measures of $Z^{h\nu}(t)$ are uniformly bounded, i.e. $\mathbf{E}(\|Z^{h\nu}(t)\|^r)$ are uniformly bounded for $t \in [0, T]$ for arbitrary $r \geq 1$ and $T > 0$.*

*Proof.* As $P$ is $L$-non-increasing and $L$ depends on the second variable only (and hence $A(f(L)) = 0$ for any $f \in C_c(\mathbf{R}_+)$),

$$\Lambda^h f_L(h\nu) = G^h_{\leq k} f_L(h\nu) \leq 0, \quad \nu \in \mathcal{M}^+_\delta(X),$$

and by (C2),

$$\Lambda^h f_{\mathbf{1}}(h\nu) \leq \int (\mathbf{1}^+(\mathbf{y}) - \mathbf{1}(z))P(z; d\mathbf{y})h\nu(dz) \leq m_0 f_{1+L}(h\nu).$$

Moreover, the intensity $q$ of the jumps of the operator $G^h_{\leq k}$ equals

$$q(h\nu) = \frac{1}{h}\sum_{l=1}^{k} h^l \sum_{I\subset\{x_1,...,x_n\},|I|=l} \int P(\mathbf{x}_I; d\mathbf{y}) \leq \frac{1}{h}\sum_{l=1}^{k} \frac{1}{l!}(f_{1+L}(h\nu))^l \tag{2.2}$$

for $\nu = (\delta_{x_1} + ... + \delta_{x_n})$. Hence the conditions of Lemma A3 hold with $f_{\mathbf{1}+L}$ playing the role of the barrier $\psi$ (notice that the property of $A$ implies similar properties of $A_{diag}$ by Theorem 10.1 from Chapter 4 of [EK]), which implies statements (i)-(iii). It remains to prove (iv). For this we observe that

$$\Lambda^h (f_{\mathbf{1}})^r(h\nu) = \frac{1}{h}\sum_{l=0}^{k} h^l \sum_{I\subset\{1,...,n\}} h^r \sum_{m=0}^{\infty}((n+m-l)^r - n^r)P(\mathbf{x}_I; dy_1...dy_n),$$

which by (C2) does not exceed

$$\sum_{i=1}^{n} h^r((n+m_0-1)^r - n^r)P(x_i; d\mathbf{y})$$

$$\leq (f_{\mathbf{1}}(h\nu) + f_L(h\nu))(m_0-1)rh^{r-1}(n+m_0-1)^{r-1} \leq rm_0^r(f_{\mathbf{1}}(h\nu) + f_L(h\nu))(f_{\mathbf{1}}(h\nu))^{r-1}.$$

As $P$ is $L$-non-increasing, $f_L(Z^{h\nu}(t))$ is almost surely uniformly bounded and using induction in $r$ we can conclude that

$$\mathbf{E}(\Lambda^h(f_{\mathbf{1}})^r(Z^{h\nu}(t)) \leq C(1 + \mathbf{E}((f_{\mathbf{1}})^r(Z^{h\nu}(t))$$

with some constant $C$. Consequently, using the martingale property of the process (A3) with $f = (f_{\mathbf{1}})^r$ and Gronwall's lemma implies (iv), which completes the proof of Proposition 2.1.

**Proposition 2.2.** *Under the conditions of Proposition 2.1, suppose additionally that (C3) holds and $P$ is $(1+L)^+$-bounded. Then for any $\beta \geq 1$ and $r > 0$*

$$\mathbf{E}\left(f_{L^\beta}(Z^{h\nu}(s))\|Z^{h\nu}(s)\|^r\right) \leq a(t)f_{L^\beta}(h\nu)\|h\nu\|^r + b(t) \tag{2.3}$$

*uniformly for all $s \in [0,t]$ with an arbitrary $r > 0$, $t > 0$, and with some constants $a(t), b(t)$ depending on $t$, $\beta$, $r$, but not on $h$.*

*Proof.* First let $r = 0$. As the process is $L$-non-increasing, the process $Z^{h\nu}(s)$ lives on measures with support on the set $\{x \in X : L(x) \leq c/h\}$ with some constant $c$. As the expectation of the number of particles is bounded it follows that

$$\max_{s\in[0,t]} \mathbf{E}f_{L^\beta}(Z^{h\nu}(s))$$

is bounded (not necessarily uniformly in $h$). Similarly

$$\max_{s\in[0,t]} \mathbf{E}\Lambda^h f_{L^\beta}(Z^{h\nu}(s))$$

is also bounded which allows to conclude (by Lemma A3 and by approximating $L^\beta$ by functions with a compact support) that the process

$$M_g(t) = f_g(Z^{h\nu}(t)) - f_g(h\nu) - \int_0^t \Lambda^h f_g(Z^{h\nu}(s))\, ds \tag{2.4}$$

is a martingale for any $g \in D(A)$ and also for $g = L^\beta$. Consequently

$$\mathbf{E}(f_{L^\beta}(Z^{h\nu}(t))) \le f_{L^\beta}(h\nu) + \int_0^t \mathbf{E}\Lambda^h(f_{L^\beta}(Z^{h\nu}(s)))\, ds. \tag{2.5}$$

Since (1.8) is $L$-non-increasing,

$$\int (L^\beta)^+(\mathbf{y})P(\mathbf{x}, d\mathbf{y}) \le \int (L^+(\mathbf{y}))^\beta P(\mathbf{x}, d\mathbf{y}) \le (L^+(\mathbf{x}))^\beta P(\mathbf{x}, d\mathbf{y}),$$

and consequently for $\nu = \delta_{x_1} + ... + \delta_{x_n}$

$$\Lambda^h f_{L^\beta}(h\nu) \le \frac{1}{h}\sum_{l=0}^k h^l \sum_{|I|=l} [((L^+(\mathbf{x}_I))^\beta - (L^\beta)^+(\mathbf{x}_I)]P(\mathbf{x}_I)$$

$$\le \sum_{l=1}^k \frac{1}{l!}\int [(L(y_1) + ... + L(y_l))^\beta - (L(y_1))^\beta - .... - (L(y_l))^\beta]P(y_1, ..., y_l)\prod_{j=1}^l (h\nu(dy_j)),$$

where we used Lemma A2. Using the symmetry with respect to permutations of $x_1, ..., x_n$, we conclude that $(\Lambda^h f_{L^\beta})(h\nu)$ does not exceed

$$\sum_{l=2}^k \frac{1}{(l-1)!}\int [(L(y_1) + ... + L(y_l))^\beta - (L(y_1))^\beta - .... - (L(y_l))^\beta](1 + L(y_1))\prod_{j=1}^l (h\nu(dy_j)).$$

Using the elementary inequalities

$$(a+b)^\beta - a^\beta - b^\beta \le w^1_\beta(ab^{\beta-1} + ba^{\beta-1})$$

and

$$a[(a+b)^\beta - a^\beta - b^\beta] \le w^2_\beta(ab^\beta + ba^\beta)$$

(that hold for all positive $a, b$ with some constants $w^1_\beta$ and $w^2_\beta$, see e.g. [Ca] for a proof of the second one) for $a = L(y_1)$, $b = L(y_2) + ... + L(y_l)$, and again the symmetry yields for the last expression the estimate

$$\sum_{l=2}^k \kappa^l_\beta(L^\beta(y_1) + L^{\beta-1}(y_1))L(y_2)\prod_{j=1}^l (h\nu(dy_j)) \tag{2.6}$$

with some constants $\kappa^l_\beta$. In case $k>2$, this does not exceed $\tilde a f_{L^\beta}(h\nu)+\tilde b$ with some $\tilde a,\tilde b$, as (C3) implies that $\|Z^{h\nu}(t)\|$ are uniformly bounded almost surely. From this estimate, (2.5) and Gronwall's lemma, one obtains (2.3). In case $k=2$ one sees that $\|Z^{h\nu}(t)\|$ is not involved in (2.6) and we get the same conclusion as for $k>2$ using only the boundedness of $f_L(Z^{h\nu}(t))$ (and induction in $\beta$ if necessarily).

Next, let $r>0$. Then

$$\Lambda^h(f_{L^\beta}f^r_{\mathbf{1}})=\frac{1}{h}\sum_{l=0}^k h^l\sum_{I\subset\{1,...,n\},|I|=l}\sum_{m=0}^\infty$$

$$\times\int[(f_{L^\beta}(h\nu-h\nu_I+h\delta_{y_1}+...+h\delta_{y_m})-f_{L^\beta}(h\nu))(n-l+m)^rh^r$$

$$+f_{L^\beta}(h\nu)((n-l+m)^r-n^r)]P(\mathbf{x}_I;dy_1...dy_m).$$

Estimating the first term in brackets as above and the second term as in the proof of Proposition 2.1 (iv), we get for this expression the estimate

$$[\sum_{l=2}^k\kappa^l_\beta(L^\beta(y_1)+L^{\beta-1}(y_1))L(y_2)\prod_{j=1}^l(h\nu(dy_j))+f_{L^\beta}(h\nu)]f^r_{\mathbf{1}}(h\nu),$$

and again the application of Gronwall's lemma completes the proof.

As the function $L$ may stay finite with $|x|\to\infty$, we need another barrier to get the compactness of the family $Z^{h\nu}$. In applications, the function $L$ stands for the size of the particle (energy, mass, etc) and the function $\tilde L$ introduced below stands for the magnitude of its position.

**Proposition 2.3.** *Under the assumptions of Proposition 2.1, suppose additionally that $P$ is $(1+L^\alpha)^+$-bounded for some $\alpha\in[0,1]$, (C3) holds and there exists a non-negative continuous function $\tilde L$ on $Q$ such that*

*(i) the jumps are local in $q$, i.e. there exists a constant $K$ such that $P(q_1,...,q_l,\mathbf{v};M)\neq 0$ only if*

$$\max_{i,j}|\tilde L(q_i)-\tilde L(q_j)|\le K,\qquad \sup_{y_1,y_2\in M}|\tilde L(y_1)-\tilde L(y_2)|\le 2K,$$

*(ii) $\tilde L(q)\to\infty$ as $q\to\infty$ and $\tilde L$ is a barrier for the Feller process $z^x_s$ generated by $A$, i.e. $\tilde L$ can be represented as a limit of increasing functions $\psi_n$ from $D(A)$ with uniformly bounded $A(\psi_n)$ and*

$$(A\tilde L)(q,v)\le C(\tilde L(x)+L(v))$$

*with some constant $C$,*

*(iii) $(1+L^\beta+\tilde L,h\nu)\le b$ for the initial conditions $h\nu$ with some constants $\beta>1$ and $b>0$.*

*Then, if either (1) $P(\mathbf{x},.)$ is $\mathbf{1}$-non-increasing or (2) $(A\tilde L)(q,v)\le b(\tilde L(x)+L^\alpha(v))$ and condition (iii) holds with $\beta=1+\alpha$, then the family $Z^{h\nu}(t)$ from Proposition 2.1 satisfies* *the compact containment condition, i.e. for arbitrary* $\eta > 0$, $T > 0$ *there exists a compact subset* $\Gamma_{\eta,T} \subset \mathcal{M}^+(X)$ *for which*

$$\inf_{h\nu} \mathbf{P}(Z^{h\nu}(t) \in \Gamma_{\eta,T} \quad for \quad 0 \le t \le T) \ge 1 - \eta, \tag{2.7}$$

*and moreover,* $Z^{h\nu}(t)$ *is tight as a family of processes with sample paths in* $D_{\mathcal{M}^+(X)}[0, \infty)$.

*Proof.* By conditions (ii)

$$\Lambda^h f_{\tilde{L}}(h\nu) \le C f_{\tilde{L}}(h\nu) + C f_L(h\nu)$$

$$+\frac{1}{h}\sum_{l=1}^k h^l \sum_{I \subset \{1,\ldots,n\}, |I|=l} \int [h\tilde{L}^+(\mathbf{y}) - h\sum_{i\in I} \tilde{L}(x_i)] P(\mathbf{x}_I; d\mathbf{y}),$$

and by condition (i) in case (1) of (iv) the integral in this expression does not exceed

$$\int 2K\mathbf{1}^+(\mathbf{y}) P(\mathbf{z}; d\mathbf{y})(h\nu)^{\tilde{\otimes}}(d\mathbf{z})$$

$$\le \int 2K\mathbf{1}^+(\mathbf{z})(1+L)^+(\mathbf{z})(h\nu)^{\tilde{\otimes}}(d\mathbf{z}) \le \int 2Kk^2 \sum_{l=1}^k (\|h\nu\| + f_L(h\nu))\|h\nu\|^{l-1}.$$

Hence, the martingale property of (2.4) with $g = \tilde{L}$ yields $\sup_{s\in[0,t]} \mathbf{E}(1+L+\tilde{L}, Z^{h\nu}(s)) < \infty$ and hence by Doob's martingale inequality

$$\mathbf{P}\left(\sup_{t\in[0,T]} f_{\tilde{L}+L+1}(Z^{h\nu}(t)) > r\right) \to 0, \quad r \to \infty,$$

uniformly in $h \in (0, 1]$. This implies (2.7) by the standard Prokhorov criterion of compactness for a family of measures in case (1).

In case (2) of (iv) we get

$$\Lambda^h f_{\tilde{L}(1+L)}(h\nu) \le C f_{\tilde{L}(1+L)}(h\nu) + C f_{L+L^{1+\alpha}}(h\nu)$$

$$+\frac{1}{h}\sum_{l=1}^k h^l \sum_{I \subset \{1,\ldots,n\}, |I|=l} \int (h(\tilde{L}(1+L))^+(\mathbf{y}) - h\sum_{i\in I} (\tilde{L}(1+L))(x_i)) P(\mathbf{x}_I; d\mathbf{y}),$$

and similarly to the above, the integral in this expression does not exceed

$$\int 3K((L^+(\mathbf{z}))^{1+\alpha} + kL^+(\mathbf{z}) + k)(h\nu)^{\tilde{\otimes}}(d\mathbf{z}) + ((L+1)\tilde{L}, h\nu).$$

By Proposition 2.2 and Gronwall's lemma this implies that $\sup_{s\in[0,t]} \mathbf{E}((1+L)\tilde{L}, Z^{h\nu}(s))$ is bounded which implies (2.7) as above.

It remains to prove tightness. By the well known Jacubovski criterion (see e.g. [EK] or [Da]), when the compact containment condition holds, in order to get tightness it is enough to show the tightness of the family of the real valued processes $f(Z^{h\nu}(t))$ (as a family of processes with sample paths in $D_{\mathbf{R}}[0,\infty)$) for any $f$ from a dense subset (in the topology of uniform convergence on compact sets) of $C_b(\mathcal{M}^+(X))$. By the Weierstrass theorem, it is thus enough to verify tightness of $f(Z^{h\nu}(t))$ for $f$ from the algebra generated by $f_g$ of type (1.15) with $g \in D(A) \subset C_c(X)$. But by Lemma A3, the process (2.4) is a martingale for any $g \in D(A)$ and the tightness now follows from the general Theorem 9.4 from Chapter 3 of [EK] (see details of quite similar arguments in [Je], [Ko4], [No]) .

*Remark.* Alternatively, one can prove tightness from the estimates of the quadratic variations of the martingale (2.4) that are obtained in the proof of Theorem 3.1 below.

## 3. Main results.

**Theorem 3.1.** *Under the assumptions of Propositions 2.1, 2.3 suppose additionally that the family of initial measures $h\nu$ converges (weakly) to a measure $\mu_0 \in \mathcal{M}^+(X)$. Then*

*(i) there is a subsequence of the family of processes $Z^{h\nu}(t)$ that weakly converges to a process $\mu_t$ in $\mathcal{M}^+(X)$ such that (1.20) holds for all $t$ and all $g \in D(A)$,*

*(ii) this solution enjoys the following additional property: for arbitrary $T$*

$$\sup_{t\in[0,T]} \int (\mathbf{1} + L^\beta)(x)\mu_t(dx) \leq b(T) \tag{3.1}$$

*and*

$$\sup_{t\in[0,T]} \int (\tilde{L})(x)\mu_t(dx) \leq b(T)$$

*with some constant $b(T)$,*

*(iii) if the intensity is not only $L$-non-increasing but also $L$-preserving, then the obtained solution $\mu_t$ is also $L$-preserving, i.e. for all $t$*

$$\int L(x)\mu_t(dx) = f_L(\mu_t) = f_L(\mu_0) = \int L(x)\mu_0(dx), \tag{3.2}$$

*(iv) if $\mu_t$ is any solution of (1.20) satisfying (3.1), then $\pi(\mu_t)$ is a weakly continuous function of $t$ and $\mu_t$ satisfies both the corresponding weak and weakly mild kinetic equations, i.e. (1.19) holds for all $g \in D(A)$ and (1.21) holds in the weak sense.*

*(v) At last, if $\alpha = 0$, i.e. $P$ is bounded, then statement (iv) holds without restriction (3.1) (for any $T$ including $T = 0$).*

*Remark.* Due to an extension of a theorem of de la Vallee-Poussin (see Theorem 3.5 in [LW]), $(L, \mu_0) < \infty$ implies $(U(L), \mu_0) < \infty$ with some $U$ which is a non-negative continuously differentiable convex function on $[0, \infty)$ such that $U(0) = 0$, $U'(0) \geq 0$, $U'$ is a concave function and $\lim_{r\to\infty} U'(r) = \infty$. Hence by extending Proposition 2.2 from power functions $L^\beta$ to any such functions $U(L)$ (which is more or less straightforward), one gets the existence of the solutions to (1.20) for all $\mu_0$ such that $(1 + L + \tilde{L}, \mu_0) < \infty$.

We shall obtain now two uniqueness results, one under the additional assumption of the invertibility of the semigroup $e^{tA}$, and another for general $A$, but for solutions with finite exponential moments.

**Theorem 3.2.** *(i) Under the conditions of Theorem 3.1, suppose additionally that the operator $A$ generates a deterministic process on $X$, i.e. $(e^{tA}f)(x) = f(\phi_t(x))$, where $\phi_t$, $-\infty < t < \infty$, is a group of homeomorphisms of $X$ (for example, $X$ is a differentiable manifold and $A$ is a smooth vector field defining a flow on $X$). Then any solution $\mu_t$ of (1.20) and (3.1) satisfies (1.23) in the strong sense, i.e.*

$$\frac{d}{dt}\nu_t = e^{-tA^\star}\pi(e^{tA^\star}\nu_t) \tag{3.3}$$

*for $\nu_t = e^{-tA^\star}\mu_t$, where the derivative is understood in the sense of the norm on $\mathcal{M}(X)$ and the r.h.s. of (3.3) is also norm-continuous.*

*(ii) Moreover, if $\mu_t^1$, $\mu_t^2$ are any two solutions of (1.20) satisfying (3.1), then*

$$\int (\mathbf{1}+L)|\nu_t^1 - \nu_t^2|\,(dx) \le ae^{at}\int (\mathbf{1}+L)|\nu_0^1 - \nu_0^2|\,(dx) \tag{3.4}$$

*for some constant $a = a(T)$ uniformly for $t \in [0,T]$. In particular, the uniqueness holds for the solutions of (1.20), (3.1) with given initial conditions, and consequently the whole family of processes $Z^{h\nu}(t)$ from Theorem 3.1 (not just its subsequence) converges weakly to $\mu_t$.*

**Theorem 3.3.** *(i) Under the assumptions of Theorem 3.1 suppose additionally that $\int \exp\{\omega L^\gamma(x)\}\mu_0(dx) < \infty$ for some positive $\omega$ and $\gamma$. Then there exists a unique global weak solution of the corresponding kinetic equations satisfying*

$$\sup_{t\in[0,T]}\int (1+\tilde{L}(x)+\exp\{\omega L^\alpha(x)\})\mu_t(dx) < \infty. \tag{3.5}$$

*If $\alpha = 0$, i.e. $P$ is bounded, then uniqueness holds without restriction (3.5).*

As a consequence, we shall obtain now a version of the propagation of chaos property for the interacting particle system $Z^{h\nu}$. In general, this property means (see e.g. [Sz]) that the moment measures of some random measures tend to the product measures when passing to a certain limit. The moment measures $\mu_t^m$ of the processes $Z^{h\nu}(t)$ are defined as

$$\mu_{t,h}^m(dx_1...dx_m) = \mathbf{E}\left(Z^{h\nu}(t)(dx_1)...Z^{h\nu}(t)(dx_m)\right). \tag{3.6}$$

**Corollary.** *Under the conditions of Theorems 3.2 or 3.3 suppose the the family of initial measures $h\nu = h\nu(h)$ converges weakly to a certain measure $\mu_0$ as $h \to 0$. Then for any $m = 1, 2, ...$, the moment measures $\mu_{t,h}^m$ converge weakly to the product measure $\mu_t^{\otimes m}$.*

*Proof.* By Theorems 3.2, 3.3, for any $g \in C_c(SX^m)$ the random variables

$$\eta_h = \int g(x_1,...,x_m)Z^{h\nu}(t)(dx_1)...Z^{h\nu}(t)(dx_m)$$

converge almost surely to the $\int g(x_1,...,x_m)\prod_{j=1}^m \mu_t(dx_j)$ as $h\to 0$. But by Proposition 2.1 (iv) the random variables $\eta_h$ have uniformly bounded variances (second moments), which implies that also the expectations of $\eta_h$ converge to the expectation of its point-wise limit.

## 4. Proofs.

*Proof of Theorem 3.1.*

(i) By Lemma A3, the process

$$M_g(t)=M_g(t;h)=f_g(Z^{h\nu}(t))-f_g(h\nu)-\int_0^t \Lambda^h f_g(Z^{h\nu}(s))\,ds \tag{4.1}$$

is a martingale for any $g\in C_c(X)$.

By Proposition 2.3 we can choose a sequence of positive numbers tending to zero such that the family $Z^{h\nu}(t)$ is weakly converging as $h\to 0$ and belong to this sequence. Let us denote the limit by $\mu_t$ and prove that it satisfies (1.20). The positivity of $\mu_t$ follows from the corresponding properties of $Z^{h\nu}(t)$.

By Skorokhod's theorem, we can and will assume that $Z^{h\nu}(t)$ converges to $\mu_t$ almost surely. The idea now is to pass to the limit in equation (4.1). Let us show first that martingale (4.1) tends to zero almost surely. To this end let us estimate the quadratic variation $[M_g(t)]$ of the martingale (4.1). Let $\Pi=\{s_1<s_2<...\}$ denote the (random) set of jumps of $M_g(t)$ caused by the jump part $G^h_{\leq k}$ of $\Lambda h$. As the number of these jumps is almost surely finite for any $h>0$, one has

$$[M_g(t)]=\sum_{s_j\leq t}(\Delta f_g(Z^{h\nu}(s_j)))^2+\sum_{s_j\leq t}\sum_{i=1}^{N(j)}[M_g^i(s_j)-M_g^i(s_{j-1})], \tag{4.2}$$

where $\Delta Z(s)=Z(s)-Z(s_-)$ denotes the jump of a process $Z(s)$ and $[M_g^i(s_j)-M_g^i(s_{j-1})]$ are the quadratic variations of the free motion of each particle $i=1,...,N(j)$ that is present between $s_{j-1}$ and $s_j$. By (C2), the number of particles created in one go is bounded, and hence it follows from (1.14) that

$$|\Delta f_g(Z^{h\mu}(s_j))|^2\leq \|g\|^2(m_0+2k)^2h^2 \tag{4.3}$$

for any $s_j$. As the number of jumps on the interval $[s,t]$ is controlled by the product of $(t-s)$ and the maximal intensity on $[0,T]$ (see e.g. [Br]), and the expectation of the intensity is of order $1/h$ by (2.3), it follows that the first term in (4.2) tends to zero almost surely as $h\to 0$. Next, the expectation of $[M_g^i(t)-M_g^i(s)]$ is uniformly of order $O(t-s)h^2$, because (1.10) is a martingale for $f=g$ and by Ito's formula for the quadratic variation $[M_t]=[g(z_t^x)]$ one has

$$[M_t]=\frac{1}{h^2}[M_g^i(t)]=g^2(z_t^x)-g^2(x)-2\int_0^t g(z_{s_-}^x)(Ag)(z_{s_-}^x)\,ds-2\int_0^t g(z_{s_-}^x)\,dM_t. \tag{4.4}$$

Since the maximum of the number of particles is of order $1/h$ almost surely, it implies that the second term in (4.2) also tends to zero almost surely as $h \to 0$. As $[M_g]$ tends to zero almost surely, we conclude that the martingale (4.1) tends to zero almost surely.

As the first two terms on the r.h.s. of (4.1) tend obviously to the first two terms on the l.h.s. of (1.20), to prove the statement (i) it remains to show that the integral on the r.h.s. of (4.1) tends to the last integral on the l.h.s. of (1.20).

To this end, let us observe that

$$|\Lambda^h f_g(Z^{h\nu}(s)) - \Lambda f_g(Z^{h\nu}(s))| \to 0, \quad h \to 0, \tag{4.5}$$

uniformly for $s \in [0,t]$, because by (1.16) and Lemma A2, for any $\eta = \delta_{v_1} + ... + \delta_{v_m}$

$$\Lambda^h f_g(h\eta) = (A(g), h\eta) + \sum_{l=0}^{k} \frac{1}{l!} \int [g^+(\mathbf{y}) - g(z_1) - ... - g(z_l)] P(z_1, ..., z_l; d\mathbf{y}) \prod_{j=1}^{l} (h\eta(dz_j))$$

$$+ \sum_{\Gamma} \alpha_\Gamma h^{l-p} \int (g^+(\mathbf{y}) - \gamma_1 g(z_1) - ... - \gamma_p g(z_p)) P(z_1, ..., z_1, ..., z_p, ..., z_p; d\mathbf{y}) \prod_{j=1}^{p} (h\eta(dz_j)),$$

and all terms in the $\sum_\Gamma$ tend to zero, as all

$$h^{l-p} \int_{z_1,...,z_p,\mathbf{y}} P(z_1, ..., z_1, ..., z_p, ..., z_p; d\mathbf{y}) \prod_{j=1}^{p} (h\eta(dz_j)) \tag{4.6}$$

tend to zero as $h \to 0$ uniformly for all $\eta \in \mathcal{M}_\delta^+(X)$ with uniformly bounded $f_L(h\eta)$.

When (4.5) is proved, it remains to show that

$$|\Lambda f_g(Z^{h\nu}(t)) - \Lambda f_g(\mu_t)| \to 0,$$

or more explicitly that the integral

$$\int_0^t ds \int (g^+(\mathbf{y}) - g^+(\mathbf{z})) P(\mathbf{z}; d\mathbf{y}) [\tilde{\mu}_s^{\otimes}(d\mathbf{z}) - \tilde{Z}^{h\nu}(s)^{\otimes}(d\mathbf{z})] \tag{4.8}$$

tends to zero as $h \to 0$. But from a weak convergence it follows (see e.g. [EK]) that $Z^{Nh}(s)$ converges to $\mu_s$ for almost all $s \in [0,t]$. Hence we need to show that

$$\int P(\mathbf{z}; d\mathbf{y}) [\tilde{\mu}_s^{\otimes}(d\mathbf{z}) - \tilde{Z}^{h\nu}(s)^{\otimes}(d\mathbf{z})] \tag{4.9}$$

tends to zero as $h \to 0$ under condition that $Z^{h\nu}(t)$ weakly converges to $\mu_t$ (then the convergence of (4.8) to zero would follow from the dominated convergence theorem).

To prove (4.9) decompose the integral in (4.9) into the sum of two integrals by decomposing the domain of integration into the domains $\{\mathbf{z} = (z_1, ..., z_m) : \max L(z_j) \geq K\}$ and its complement. On the second domain, the integrand is uniformly bounded and hence the weak convergence of $Z^{h\nu}(s)$ to $\mu_t$ ensures the smallness of the l.h.s. of (4.9). Hence we need only to show that for any $l$, by choosing $K$ arbitrary large, we can make

$$\int_{L(z_1)\geq K} L(z_1)[\prod_{j=1}^{l} \mu_s(dz_j) + \prod_{j=1}^{l} Z^{h\nu}(s)(dz_j)]$$

arbitrary small. But this integral does not exceed

$$\frac{K}{K^\beta} \int (L(z_1))^\beta [\prod_{j=1}^{l} \mu_s(dz_j) + \prod_{j=1}^{l} Z^{h\nu}(s)(dz_j)],$$

which is finite and tends to zero as $K \to \infty$ due to Proposition 2.3.

(ii) This follows from the corresponding estimate of the approximating process $Z^{h\nu}(s)$.

(iii) As the approximations $Z^{h\nu}$ are clearly $L$-preserving whenever the transition kernel $P$ is, we only need to show that

$$\lim_{h\to 0} \int L(x)(\mu_t - Z^{h\nu}(t))(dx) = 0. \tag{4.10}$$

But this is done as above. Decomposing the domain of integration into two parts: $\{x : L(x) \leq K\}$ and its complement, we observe that the required limit for the first integral is zero due to the weak convergence of $Z^{h\nu}(t)$ to $\mu_t$, and on the second part both integrals from (4.10) can be made arbitrary small by choosing $K$ large enough, due to the Proposition 2.3.

(iv) It follows from (1.20) that $(g, \mu_t)$ is an absolutely continuous function of $t$ for any $g \in D(A)$ and that (1.19) holds almost surely. Next, a straightforward approximation argument shows that $(g, \mu_t)$ is continuous for all $g \in C_0(X)$. And this implies that for $g \in D(A)$, the r.h.s. of (1.19) is a continuous function of $t$, because one can write $P(\mathbf{z}; d\mathbf{y}) = P(\mathbf{z}; d\mathbf{y})\chi(z) + P(\mathbf{z}; d\mathbf{y})(1 - \chi(z))$ with $\chi(z)$ is a smoothed version of the indicator $\mathbf{1}_{L(.)\leq r}$, and then the integral of the first function is continuous by the above and the integral of the second function can be made arbitrary small by choosing $r$ large enough (due to (3.1)). This implies (1.19). Equation (1.21) follows then by general Lemma A4 from the Appendix.

(v) This is the same as (iv).

*Proof of Theorem 3.2.* (i) A straightforward calculation shows that if $(e^{tA}f)(x) = f(\phi_t(x))$, where $\phi_t$, $-\infty < t < \infty$, is a group of homeomorphisms of $X$, then

$$(f, e^{-tA^\star}\pi(e^{tA^\star}\nu)) = \int\int (f^+(\phi_t^{-1}(\mathbf{y})) - f^+(\mathbf{z}))P(\phi_t(\mathbf{z}); d\mathbf{y})\nu^{\tilde{\otimes}}(d\mathbf{z}). \tag{4.11}$$

Hence, if $\nu_t$ is norm-continuous in $t$, then $e^{-tA^\star}\pi(e^{tA^\star}\nu_t)$ is also norm-continuous, because the mappings $\nu(dx) \mapsto g(x)\nu(dx)$ and $\nu \mapsto \int p(x,.)\nu(dx)$ are both norm continuous for any bounded measurable $g$ and any weakly continuous transition kernel $p(x,.)$ with a bounded intensity (our transition kernel $P(\mathbf{z};.)$ is of course not bounded, but all tails vanish uniformly due to (3.1)).

Next, from (1.23) with the integral understood in the weak sense it follows that $\nu_t$ is absolutely continuous in the sense of the norm-topology on $\mathcal{M}(X)$, and hence by the above, the function under the integral on the r.h.s. of (1.23) is also continous in the sense of the norm-topology. This implies (3.3) by a remark given after Lemma A1 of the Appendix.

By (i), Lemma A1 from the Appendix (see also Remark (ii) to this Lemma) can be applied to the measure $(\mathbf{1}+L)(x)(\nu_t^1-\nu_t^2)(dx)$. Consequently, denoting by $f_t$ a version of the density of $\nu_t^1-\nu_t^2$ with respect to $|\nu_t^1-\nu_t^2|$ from Lemma A1 yields

$$\int (\mathbf{1}+L)(x)|\nu_t^1-\nu_t^2|(dx) = \|(\mathbf{1}+L)(\nu_t^1-\nu_t^2)\|$$

$$= \int (\mathbf{1}+L)(x)|\nu_0^1-\nu_0^2|(dx) + \int_0^t ds \int_X f_s(x)(\mathbf{1}+L)(x)(\dot{\nu}_s^1-\dot{\nu}_s^2)(dx).$$

By (3.3) the last integral equals

$$\int_0^t ds \sum_{l=1}^k \int\int \left([e^{-sA}(f_s(\mathbf{1}+L))]^+(\mathbf{y}) - [e^{-sA}(f_s(\mathbf{1}+L))]^+(\mathbf{z})\right) P(\mathbf{z};d\mathbf{y})$$

$$\times \sum_{j=1}^l e^{sA^\star}(\nu_s^1-\nu_s^2)(dz_j) \prod_{i=1}^{j-1}(e^{sA^\star}\nu_s^2)(dz_i) \prod_{i=j+1}^l (e^{sA^\star}\nu_s^1)(dz_i). \tag{4.12}$$

Let us pick up arbitrary $l \le k$ and $j \le l$ and estimate the corresponding term in the sum (4.12). It equals

$$\int\int \left([e^{-sA}(f_s(\mathbf{1}+L))]^+(\mathbf{y}) - [e^{-sA}(f_s(\mathbf{1}+L))]^+(\mathbf{z})\right) P(\mathbf{z};d\mathbf{y})$$

$$\times (e^{-sA}f_s)(z_j)|e^{sA^\star}(\nu_s^1-\nu_s^2)|(dz_j) \prod_{i=1}^{j-1}(e^{sA^\star}\nu_s^2)(dz_i) \prod_{i=j+1}^l (e^{sA^\star}\nu_s^1)(dz_i), \tag{4.13}$$

where we used Remark (iii) after Lemma A1. As $L$ is non-increasing by $P(\mathbf{z};d\mathbf{y})$,

$$\left([e^{-sA}(f_s(\mathbf{1}+L))]^+(\mathbf{y}) - [e^{-sA}(f_s(\mathbf{1}+L))]^+(\mathbf{z})\right)(e^{-sA}f_s)(z_j)$$

$$\le (e^{-sA}(\mathbf{1}+L))^+(\mathbf{y}) - (e^{-sA}f_s)(z_j)[e^{-sA}(f_s(\mathbf{1}+L))]^+(\mathbf{z})$$

$$\le m_0 + 2k + (e^{-sA}L)^+(\mathbf{z}) - (e^{-sA}L)(z_j) - \sum_{i\neq j}(e^{-sA}f_s)(z_j)(e^{-sA}f_s)(z_i)(e^{-sA}L)(z_i)$$

$$\le m_0 + 2k + 2\sum_{i\neq j} L(z_i),$$

where the constant $m_0$ is from condition (C2) and where we used that $e^{-sA}L = L$ (because $L(q,v)$ depends on $v$ only). Hence (4.13) does not exceed

$$\int (m_0 + 2k + 2\sum_{i\neq j} L(z_i))(k + L^\alpha(z_j) + \sum_{i\neq j} L^\alpha(z_i))|\mu_s-\nu_s|(dz_j) \prod_{i=1}^{j-1}\nu_s(dz_i) \prod_{i=j+1}^l \mu_s(dz_i).$$

Consequently, by (3.1), the integral (4.12) does not exceed

$$a(r,T)\int_0^t ds \int (\mathbf{1}+L)(x)|\mu_s-\nu_s|(dx)$$

with some constant $a(r,T)$, which implies (3.4) by Gronwall's lemma. Proof of Theorem 3.2 is complete.

*Proof of Theorem 3.3.* Consider only the case $\alpha > 0$ (the case $\alpha = 0$ being similar). Let $\mu_t^1$ and $\mu_t^2$ be any two solutions having the same initial conditions $\mu_0^1 = \mu_0^2$. By Theorem 3.1 (iv) they are also weakly mild solutions. Hence

$$\|\mu_t^1-\mu_t^2\| = \sup_{\|g\|=1}(g,\mu_t^1-\mu_t^2) \le \sup_g \int_0^t (e^{(t-s)A}g, \pi(\mu_s^1)-\pi(\mu_s^2))\,ds$$

$$\le \sup_g \sum_{l=1}^k \int_0^t ds \int_{\mathbf{z},\mathbf{y}} ((e^{(t-s)A}g)^+(\mathbf{y})-(e^{(t-s)A}g)^+(\mathbf{z}))P(\mathbf{z},d\mathbf{y})$$

$$\times \sum_{j=1}^l (\mu_s^1-\mu_s^2)(dz_j)\prod_{i=1}^{j-1}\mu_s^2(dz_i)\prod_{i=j+1}^{l}\mu_s^1(dz_i)$$

$$\le C\sum_{l=1}^k \int_0^t ds \int_{\mathbf{z}} (\mathbf{1}+L)^+(\mathbf{z}))\sum_{j=1}^l |\mu_s^1-\mu_s^2|(dz_j)\prod_{i=1}^{j-1}\mu_s^2(dz_i)\prod_{i=j+1}^{l}\mu_s^1(dz_i)$$

with some constant $C$, and by symmetry and since $(L,\mu_s^j)$ are uniformly bounded, it follows that

$$\|\mu_t^1-\mu_t^2\| \le C\int_0^t (\mathbf{1}+L^\alpha, |\mu_s^1-\mu_s^2|)\,ds$$

with some other constant $C$. Dividing the integral into two parts with $L(z) \ge K$ and $L(z) < K$ and using (3.5) yields

$$\|\mu_t^1-\mu_t^2\| \le C\int_0^t (1+K^\omega)\|\mu_s^1-\mu_s^2\|\,ds + \tilde{C}(1+K^\alpha)\exp\{-\omega K^\alpha\},$$

which by Gronwall's lemma yields

$$\|\mu_t^1-\mu_t^2\| \le \tilde{C}(1+K^\alpha)\exp\{-\omega K^\alpha\}\exp\{C(1+K^\alpha)t\}.$$

For $t \le \omega/C$, we can pass to the limit $K\to\infty$ to get zero on the r.h.s. of this inequality, which proves the uniqueness stated in Theorem 3.3. To get the existence in the required class of solutions, we notice that the statement and the proof of Proposition 2.3 remains true if one takes the function $\exp\{\omega L^\gamma\}$ instead of $L^\beta$ there. Hence (3.5) holds for any $T$ whenever it holds for $T=0$.

## 5. Regularity of solutions.

We shall give here two results on the regularity of the solutions to kinetic equations first being devoted to the case of a smoothing $B_v$ and another to the case of $B_v$ generating a deterministic flow. We shall assume the following additional assumption on our model:

(C4) $Q = \mathbf{R}^d$ (most of the results are easily generalized to the case when $Q$ is a finite-dimensional connected Riemannian manifold with the Lebesgue volume form $dq$, or even for more general models with $X$ being a fibre bundle over a manifold $Q$) and the jumps are deterministic in $q$, i.e.

$$P(q_1, v_1, ..., q_l, v_l; dp_1 dw_1 ... dp_m dw_m)$$

$$= \eta_l(q_1, ..., q_l)\delta(p_1 - \omega_1^l(q_1, ..., q_l))...\delta(p_m - \omega_m^l(q_1, ..., q_l))\Pi(v_1, ..., v_l; dw_1 ... dw_m), \quad (5.1)$$

where $\Omega(\mathbf{q}) = \{\omega_j^l\}$ is a continuous mapping from $S\mathcal{Q}$ to itself such that all points in $\Omega(q_1, ..., q_l)$ always belong to the convex hull of the set $\{q_1, ..., q_l\}$ and $\eta = \{\eta_l\}$ is a collection of non-negative continuous symmetric functions having support on the set $\{\mathbf{q} = (q_1, ..., q_l) : \max |q_i - q_j| < K\}$ with some $K < \infty$.

In the first case we shall look for the solutions in the form $\mu_t(q, dv)dq$, where each $\mu_t$ is a transition kernel from $Q$ to $V$, i.e. $\mu_t$ belongs to the positive cone $L^1(Q, \mathcal{M}^+(V))$ of the Banach space $L^1(Q, \mathcal{M}(V))$ with the norm

$$\|\mu(.,.)\| = \int_Q \|\mu(q,.)\| \, dq = \int_Q \int_V |\mu(q, dv)| \, dq.$$

The Banach space $L^1(Q, \mathcal{M}(V))$ is isometrically embedded in the Banach space $\mathcal{M}(X)$ and in the future, with some abuse of notation, we shall identify the elements of $L^1(Q, \mathcal{M}(V))$ with their images in $\mathcal{M}(X)$. Let us denote by $pr_Q$ the projection in $\mathcal{M}(Q \times V)$ on the first variable, i.e. $(pr_Q \mu)(dq) = \int_V \mu(dq \, dv)$. Clearly $\mu \in L^1(Q, \mathcal{M}(V))$ if and only if $pr_Q \mu \in L^1(Q)$.

**Theorem 5.1.** *Under the conditions of Theorem 3.1, suppose additionally that for all $t > 0$ and $v \in V$,*

$$e^{tB_v^\star}(\mathcal{M}(Q)) \subset L^1(Q),$$

*$e^{tB_v^\star}$ defines a strongly continuous semigroup (of contraction) on $L^1(Q)$ and $pr_Q e^{tB_v^\star}$ is a compact operator $\mathcal{M}(X) \mapsto L^1(Q)$. Then any solution $\mu_t$ of (1.20), (3.1) with an initial condition $\mu_0$ from $L^1(Q, \mathcal{M}(V))$ belongs to $L^1(Q, \mathcal{M}(V))$ for all $t$, and $t \mapsto pr_Q \mu_t$ is a norm continuous function. Moreover, under conditions of Theorem 3.3, there exists a unique solution satisfying (3.5).*

*Remark.* For the conditions on $B_v$ of the Theorem to hold, it is enough to assume that for all $v$ operator $B_v$ defines a Feller semigroup on $Q$ with a continuous Green function $G(t, q, q_0; v)$ such that its derivative with respect to $q_0$ exists and is uniformly bounded for any $t$. The basic examples of such processes are given by non-degenerate diffusions, regular degenerate diffusions in the sense of [Ko5] and stable-like jump diffusions whose Green function was constructed in [Ko5], [Ko6], see also the appendix to [K1] and review [JS].

*Proof.* Uniqueness is a consequence of Theorem 3.4, and all other statements follow from Lemma A4 (ii), (iii), or more precisely its simple modification with $pr_Q\mu_t$ considered instead of $\mu_t$ everywhere in the arguments of this Lemma.

**Theorem 5.2.** *Suppose the conditions of Theorem 3.2 hold, $B_v$ generates a flow of diffeomorphisms on $Q$, i.e.*

$$(e^{tB_v^\star}(g\otimes h))(q,v)=h(v)g(\phi_t^v(q)), \tag{5.2}$$

*where $\phi_t^v$ is a group of diffeomorphisms of $Q$ and moreover the mapping $\pi$ preserves the Banach subspace $L^1(X)\subset\mathcal{M}(X)$. Then for any $\mu_0(q,v)$ from $L^1(X)\subset\mathcal{M}(X)$) such that*

$$\int\int(\mathbf{1}+L^{1+\alpha})(v)\mu_0(q,v)\,dvdq<\infty, \tag{5.3}$$

*there exists a unique weak solution of the corresponding kinetic equations with initial data $\mu_0(q,v)$ and it belongs to $L^1(X)$ for all times.*

*Proof.* Since uniqueness follow from Theorem 3.2, one only needs to show that there exists a solution that belongs to $L^1(X)$. This can be done in two steps by the following well known procedure (see e.g. [LM]). First consider an approximation to our transition kernel $P$ by the kernels

$$P_M(z_1,...,z_l;.)=\begin{cases}P(z_1,...,z_l;.), & \forall j\, L(z_j)\le M\\ (L^+(z_1,...,z_l))^{-1}P(z_1,...,z_l;.)\ \textit{otherwise.}\end{cases}$$

For any finite $M$, the transition kernel $P_M$ is uniformly bounded, and hence the r.h.s. of (3.3) is a uniformly (in $t$) Lipshitz continuous function of $\nu_t$, which one sees by inspection from the explicit formula (4.11). Consequently, any standard approximation procedure leading to a solution of (3.3) (Picard's method, or Euler's broken lines) yields a unique solution of (3.3). Hence, as the initial condition belongs to $L^1(X)$ and both mappings $\exp\{tA^\star\}$ and $\pi$ preserve this space, it follows that all approximations and hence the solution $\mu_t^M(q,v)dqdv$ itself belong to this space for all times. At last, condition (3.1) ensures that the family $\mu_t^M$ is uniformly bounded and moreover, from (3.3) one concludes that $\|\mu_t^M-\mu_s^M\|$ is of order $O(t-s)$ uniformly for all $M$. Consequently, by Askoli's theorem, there exists a sequence $M_n\to\infty$ as $n\to\infty$ such that the corresponding solutions $\mu_t^{M_n}$ converge as a family of continuous functions on $[0,T]$ with values in the Banach space $L^1(X)$. The limit clearly solves (3.3) and belongs to $L(X)$. Theorem is proved.

*Remark.* The corresponding results in $L^1(Q,\mathcal{M}(V))$ are not clear for $B_v$ from (5.2), because $L^1(Q,\mathcal{M}(V))$ is not preserved by $\exp\{tB_v^\star\}$ with such $B_v$.

## 6. Examples: $k$-nary Boltzmann and Smoluchovski equations.

As a basic example, let us distinguish the processes that combine pure coagulations of no more than $k$ particles, spontaneous fragmentation in no more than $k$ pieces, and collisions (or collision breakages) of no more than $k$ particles (with an arbitrary $k$). These processes are specified by a Feller generator $A$, by the transition kernels

$P_1^l(z_1,...,z_l) = K_l(z_1,...,z_l;dy)$, $l = 2,...,k$, called the coagulation kernels, the transition kernels $P_m^1(z;dy_1...dy_m)= F_m(z;dy_1...dy_m)$, $m = 2,...,k$, called the fragmentation kernels and the kernels

$$P_l^l(z_1,...,z_l;dy_1...dy_l) = C_l(z_1,...,z_l;dy_1...dy_2), \quad l = 2,...,k,$$

called the collision kernels, all other $P_l^m$ are supposed to vanish. Then equation (1.19) takes the form

$$\frac{d}{dt}(g,\mu_t) = (Ag,\mu_t) + \sum_{l=2}^k \frac{1}{l!}\int_{z_1,...,z_l,y} [g(y) - g(z_1) - ... - g(z_l)]K_l(z_1,...,z_l;dy)\prod_{j=1}^l \mu_t(dz_j)$$

$$+\sum_{m=2}^k \int_{z,y_1,...,y_m} [g(y_1) + ... + g(y_m) - g(z)]F_m(z;dy_1...dy_m)\mu_t(dz)$$

$$+\sum_{l=2}^k \frac{1}{l!}\int [g(y_1) + ... + g(y_l) - g(z_1) - ... - g(z_l)]C_l(z_1,...,z_l;dy_1...dy_l)\prod_{j=1}^l \mu_t(dz_j). \quad (6.1)$$

Moreover, suppose $\tilde{L}(q,v)$ is $|q|^2$ (or some smoothed version of $|q|$), and with some function $L$ on $V$ that is preserved by the kernels $K, F, C$, which means that, say, the measures $K_l(z_1,...,z_l;.)$ are supported on the set $\{y : L(y) = L(z_1) + ... + L(z_l)\}$. For this model condition (C2) is clearly satisfied. The following two basic particular cases of this model are to be distinguished.

(i) $Q = \mathbf{R}^d$, $V = \mathbf{R}_+$, $A = B_v$ is a smoothing operator from Theorem 5.1 (see also Remark after this Theorem) and $L(v) = v$ is interpreted as a mass of a particle. Then our model describes a $k$-nary analog of a spatially non-trivial Smoluchovski's coagulation-fragmentation model (with possible collision-breakage specified by the kernels $C_l$, see [CR], [KK], [Sa] for a physical discussion of collision breakage and [LW], [Ko3] for a mathematical treatment of the corresponding problem with a discrete mass distribution) with a mollifier. To write it in a more familiar form let us assume for simplicity that no collision breakage takes place, i.e. all $C_l$ vanish and that the the measures $K_l$ and $F_l$ have densities with respect to Lebesgue measure on the manifolds in the phase space arising from the conservation of mass property. Taking into account also (5.1), one can then rewrite (6.1) in the form

$$\frac{d}{dt}(g,\mu_t) = (B_v g,\mu_t) + \sum_{l=2}^k \int_0^\infty ... \int_0^\infty \prod_{j=1}^l \mu_t(q_j;dv_j)\int_{\mathbf{R}^{dl}} dq_1...dq_l$$

$$\eta_l(\mathbf{q})[g(\Omega(\mathbf{q}), v_1 + ... + v_l) - g(q_1,z_1) - ... - g(q_l,z_l)]K_l(v_1,...,v_l)+$$

$$\sum_{m=2}^k \int_\Sigma dv_1...dv_m \int dq\mu_t(q;dv)\gamma(q)[g(q,v_1)+...+g(q,v_m)-g(q,v_1+...+v_m)]F_m(v_1,...,v_m), \quad (6.2)$$

where $\int_\Sigma dv_1...dv_m$ denotes the Lebesgue integral over the symplex

$$\Sigma = \{\{v_i \geq 0\}_{i=1}^m : v_1 + ... + v_m = v\}$$

and where $\gamma, K_l, F_m$ are some non-negative symmetric functions. Assuming $\mu_t$ have densities with respect to Lebesgue measure $dv$ one can easily rewrite this equation in the strong form used e.g. in [Am] or [LW]. The results on the well-posedness and the propagation of chaos property that follow from Theorem 5.1 and Corollary to Theorem 3.3 are new for equation (6.2) even for the case of a binary pure coagulation- fragmentation model, i.e. if $K_l, F_l$ do not vanish only for $l = 2$. For the binary pure coagulation-fragmentation model without a mollifier (the corresponding equation of type (6.5) below) some results on the existence of the solution are obtained in [LW] for bounded domains and under some additional monotonicity assumptions on the fragmentation kernel (the arguments of [LW] being based on the approach from [PL]), and the well-posedness is proved for uniformly bounded $K_2, F_2$ in [Am]. In both [LW] and [Am], operator $A = B_v$ is a non-degenerate diffusion with a very restrictive dependence on $v$. The corresponding spatially homogeneous model is of course much better understood (see [DS], [Ko4], [No]).

(ii) $Q = V = \mathbf{R}^d$, $L = v^2$ is the kinetic energy of a particle, all $K_l$ and $F_l$ vanish, $A = B_v = v\frac{\partial}{\partial q}$, (5.1) holds and transition kernels $\Pi$ in (5.1) preserve the total momentum and energy and are invariant under shifts and permutations of $v_j$. Then equation (6.1) takes the form (see (A22)):

$$\frac{d}{dt}(g, \mu_t) = (v\frac{\partial}{\partial q}g, \mu_t) + \sum_{l=2}^{k} \frac{1}{l!} \int_{S_{\Gamma,\mathbf{v}}^{d(l-1)-1}} \int_{\mathbf{R}^{dl}}$$

$$\times (g^+(\Omega(\mathbf{q}), \mathbf{v} - 2(\mathbf{v}, \mathbf{n})\mathbf{n}) - g^+(\mathbf{q}, \mathbf{v}))\eta(\mathbf{q}) B_l(\{v_i - v_j\}_{i,j=1}^l; d\mathbf{n})\mu_t(q_1; dv_1)...\mu_t(q_l; dv_l). \tag{6.3}$$

One can also write it in the strong form using (A24). Equation (6.3) is a $k$-nary analog of the (generalized, see e.g. [LM],[MT]) Boltzmann equation with a mollifier. Theorem 5.2 implies that if (C4) holds for $\Omega$ and $\eta$, and $B_l$ in (6.3) has the form $B_l(\{v_i - v_j\}_{i,j=1}^l, n)\, d\mathbf{n}$, where $d\mathbf{n}$ is Lebesgue measure on $S_{\Gamma,\mathbf{v}}^{d(l-1)-1}$ and $B_l$ is a continuous function such that

$$|B_l(\{v_i - v_j\}_{i,j=1}^l, n)| \le C \sum_{i \ne j} \|v_i - v_j\|^{2\alpha}$$

with some constants $C > 0$, $\alpha \in [0, 1]$, then for any $\mu_0(q, v) \in L^1(X)$ such that

$$\int\int (1 + |v|^{2(\alpha+1)})\mu_0(q, v)\, dvdq < \infty$$

there exists a unique global solution of the weak kinetic equation (6.3) such that

$$\sup_{t\in[0,T]} \int\int (1 + |v|^{2(\alpha+1)})\mu_t(q, v)\, dvdq < \infty \tag{6.4}$$

for any $T > 0$. For the case of a usual binary Boltzmann equation with a mollifier ($k = 2$, $\alpha = 1/2$ in (6.3)) the uniqueness condition (6.4) is a slight improvement to the classical theorem, where the existence of the fourth moment is required, see reviews and references in [LM], and a more recent treatment in [AG], where mollified Boltzmann is called (a bit misleadingly) the generalized Boltzmann . For recent developments in the theory of spatially homogeneous Boltzmann equation we refer to [Vi] and [MW].

To conclude, let us show how the classical kinetic equations are obtained (heuristically) from the regular kinetic equations considered in this paper. Suppose instead of $\eta_l$ we are given a family $\eta_l^\epsilon$ of functions such that for each $l$, this family converges weakly to the measure $\sigma\delta_{Diag}$ on $SQ^l$, where $\delta_{Diag}$ is the $\delta$-measure of the diagonal set of $Q^l$ and $\sigma$ is a continuous function on $Q$, i.e.

$$\lim_{\epsilon\to 0}\int_{Q^l} f(q_1,...,q_l)\eta_l^\epsilon(q_1,...,q_l)dq_1...dq_l$$

$$=\int_{Q^l} f(q_1,...,q_l)\sigma(q_1)\delta_{Diag}(dq_1...dq_l)=\int_Q f(q,...,q)\sigma(q)dq$$

for all continuous $f\in C_b(S\mathcal{Q})$. Performing a formal limit as $\epsilon\to 0$ in the kinetic equation (1.19) leads to the following kinetic equation in weak form

$$\frac{d}{dt}\int_V g(q,v)\mu_t(q,dv)$$

$$=\int_V (B_v g)(q,v)\mu_t(q,dv)+\int_{\mathbf{v},\mathbf{w}}(g^+(q,\mathbf{w})-g^+(q,\mathbf{v}))\Pi(\mathbf{v};d\mathbf{w})\sigma(q)\mu_t^{\tilde{\otimes}}(q,d\mathbf{v}). \qquad (6.5)$$

A rigorous analysis of this limiting procedure is far from being trivial (even for classical Boltzmann equation the well-posedness is still open, see e.g. [Ma], [PL], [AV] for a discussion of this problem) and will be discussed in a separate publication.

## Appendix.

We collect here some auxiliary results used in the main text.

*1. A tool for dealing with the problem of uniqueness.* This is based on the following result that is proved in [Ko4].

**Lemma A1.** *Let $Y$ be a measurable space and the mapping $t\mapsto\mu_t$ from $[0,T]$ to $\mathcal{M}(Y)$ is continuously differentiable in the sense of the norm in $\mathcal{M}(Y)$ with a (continuous) derivative $\dot\mu_t=\nu_t$. Let $\sigma_t$ denote a density of $\mu_t$ with respect to its total variation $|\mu_t|$, i.e. the class of measurable functions taking three values $-1,0,1$ and such that $\mu_t=\sigma_t|\mu_t|$ and $|\mu_t|=\sigma_t\mu_t$ almost surely with respect to $|\mu_t|$. Then there exists a measurable functions $f_t(x)$ on $[0,T]\times Y$ such that $f_t$ is a representative of class $\sigma_t$ for any $t\in[0,T]$ and*

$$\|\mu_t\|=\|\mu_0\|+\int_0^t ds\int_Y f_s(y)\nu_s(dy). \qquad (A1)$$

*Remarks.* (i) To facilitate the application of this result, it is worth noticing that (as one easily checks) if $Y$ is a locally compact set, $\dot\mu_t=\nu_t$ holds in the weak sense and $\nu_t$ is continuous in the sense of the norm topology of $\mathcal{M}(Y)$, then $\dot\mu_t=\nu_t$ holds also in the strong sense. (ii) Suppose the assumptions of Lemma A1 hold and $L(y)$ is a measurable, non-negative and everywhere finite function on $Y$ such that $\|L\mu_s\|$ and $\|L\nu_s\|$ are uniformly bounded for $s \in [0,t]$. Then (A1) holds with $L\mu_t$ and $L\nu_t$ instead of $\mu_t$ and $\nu_t$ respectively. In fact, though $s \mapsto L\nu_s$ may be discontinuous in the sense of norm, one can write the required identity first with the space $Y_m$ instead of $Y$, where $Y_m = \{y : L(y) \leq m\}$, and then pass to the limit as $m \to \infty$. (iii) We shall need also the following obvious transformation property of densities. Let $Y$ be a measurable space and $s : Y \mapsto Y$ is a measurable bijection with a measurable inverse $s^{-1}$. Denote by $S$ and $S^\star$ the induced transformations on bounded measurable functionds and finite measures defined by

$$(Sf)(x) = f(s(x)), \quad (f, S^\star\mu) = (Sf, \mu).$$

Let $X + A^+(\mu) \cup A^-(\mu)$ denote the Hahn decomposition of $X$ defined by the signed measure $\mu$. Then clearly

$$A^\pm(S^\star\mu) = s(A^\pm(\nu)).$$

Consequently, if $\sigma(\mu)$ denote a density of $\mu$ with respect to its total variation $|\mu|$, then $(\sigma(S^\star\mu))(x) = (S^{-1}\sigma(\mu))(x) = (\sigma(\mu))(s^{-1}(x))$.

*2. A combinatorial identity.* The following identity is used in our derivation of kinetic equations and its (very simple) proof can be found again in [Ko4]. We shall denote by $\delta_x$ the Dirac measure at $x$. By a Young scheme $\Gamma$ we mean a collection $\Gamma = \{\gamma_1, ..., \gamma_l\}$ of natural numbers such that $1 \leq \gamma_1 \leq ... \leq \gamma_l$. Let $h > 0$ be a positive parameter.

**Lemma A2.** *For any natural $k$, there exist constants $\alpha_\Gamma$ parametrized by all Young schemes $\Gamma = \{\gamma_1, ..., \gamma_l\}$ with $\gamma_1 + ... + \gamma_l = k$ such that for any natural $n$, $f \in B_{sym}(X^k)$ and a collection of points $x_1, ..., x_n$ in $X$*

$$h^k \sum_{I \subset \{1,...,n\}, |I|=k} f(x_I) = \frac{1}{k!} \int f(y_1, ..., y_k) \prod_{j=1}^k (h\delta_{x_1} + ... + h\delta_{x_n})(y_j)$$

$$+ \sum_\Gamma \alpha_\Gamma h^{k-l} \int f(y_1, ..., y_1, y_2, ..., y_2, ..., y_l, ..., y_l) \prod_{j=1}^l (h\delta_{x_1} + ... + h\delta_{x_n})(y_j), \qquad (A2)$$

*where $\sum_\Gamma$ is the sum over all Young schemes $\Gamma = \{\gamma_1, ..., \gamma_l\}$ such that $\gamma_1 + ... + \gamma_l = k$ and $\gamma_l > 1$ (or, equivalently, $l < k$), and $f(y_1, ..., y_1, y_2, ..., y_2, ..., y_l, ..., y_l)$ means that the first $\gamma_1$ arguments of $f$ are equal to $y_1$, the next $\gamma_2$ arguments are equal to $y_2$, etc. Moreover, if $f$ is non-negative, then the l.h.s. of (A2) does not exceed the first term of the r.h.s. of (A2).*

*3. On the generators of Markov processes.* To give a correct description of our basic model of $k$-nary interactions, we need the following corollary of the general theory of Markov processes, whose proof we sketch here for completeness.

**Lemma A3.** *Suppose*

*(i) $X$ is a locally compact metric space,*

*(ii) $A$ is an operator in $C_b(X)$ with a separable and separating domain $D(A)$ such that the martingale problem is well posed for it, i.e. for any $x \in X$ there exists a unique process $z_s^x$ with sample paths in $D_X[0,\infty)$ such that*

$$M_t = f(z_t^x) - f(x) - \int_0^t Af(z_s^x)\, ds \tag{A3}$$

*is an $\mathcal{F}_t$-martingale for all $f \in D(A)$ with $M_0 = 0$ almost surely, and $z_s^x$ is a strong Markov process on $X$,*

*(iii) $q(x,.)$ is a transition kernel on $X$ which is continuous with respect to the weak topology of $\mathcal{M}^+(X)$ and such that $q(x,\{x\}) = 0$ for all $x$,*

*(iv) there exists a continuous non-negative function $\psi$ on $X$ such that the intensity $q(x) = q(x,X)$ is uniformly bounded on all sets $U_a = \{x : \psi(x) \le a\}$, and moreover $G\psi(x) \le b + c\psi(x)$ for all $x$ and some constants $b, c \ge 0$, where*

$$Gf(x) = \int (f(y) - f(x)) q(x,dy), \tag{A4}$$

*(v) there exists a non-decreasing sequence of non-negative functions $\psi_m \in D(A)$ that converges point-wise to $\psi$ as $m \to \infty$ and such that with $A\psi_m \le 0$ for all $m$ and the sequence $A\psi_m$ converges point-wise to some function (which can be naturally denoted by) $A\psi$.*

*Then*

*(i) for the operator $A+G$ on $C_b(X)$ with domain $D(A)$, the martingale problem is well posed, i.e. for any $x \in X$ there exists a unique process $Z_s^x$ with sample paths in $D_X[0,\infty)$ such that*

$$M_t = f(Z_t^x) - f(x) - \int_0^t (A+G)f(Z_s^x)\, ds \tag{A5}$$

*is an $\mathcal{F}_t$-martingale for all $f \in D(A)$ with $M_0 = 0$ almost surely, and this $Z_s^x$ is a strong Markov process on $X$; the corresponding semigroup $T_t f(x) = Ef(Z_t^x)$ is a semigroup of contractions on $C_b(X)$;*

*(ii) the process*

$$\psi(Z_t^x) - \psi(x) - \int_0^t (A+G)\psi(Z_s^x)\, ds \tag{A6}$$

*is an $\mathcal{F}_t$-martingale vanishing at $t = 0$ almost surely, and for arbitrary positive $T$ and $r$*

$$\mathbf{P}\left(\sup_{t\in[0,T]} \psi(Z_t^x) > r\right) \le \frac{C(T)}{r}, \tag{A7}$$

*where $C(T)$ depends only on $T$,$b$, $c$ and $\psi(x)$.*

*Sketch of the proof.* Instead of $G$, consider its approximation $G_n$ defined as

$$(G_n f)(x) = \int (f(y) - f(x)) \tilde{\mathbf{1}}_{\psi(.)(x)\le n} q(x,dy),$$

where $\tilde{\mathbf{1}}_{\psi(.)\leq n}$ is a continuous function $X \mapsto [0,1]$ that coincides with $\mathbf{1}_{\psi(.)\leq n}$ on the set of all $x$ where $\psi(x) \notin [n, n+1]$. As each $G_n$ is bounded, the statement (i) holds for $A + G_n$ and the corresponding process $Z_t^{x,n}$ by the standard perturbation theory (see Theorems 10.3, 4.6 and 4.2 (c) from Chapter 4 of [EK]). Moreover, each $G_n$ enjoys the same property $G_n\psi(x) \leq b + c\psi(x)$. Let us show first that

$$\psi(Z_t^{x,n}) - \psi(x) - \int_0^t (A + G_n)\psi(Z_s^{x,n})\, ds \tag{A8}$$

is a martingale and

$$\mathbf{E}\psi(Z_t^{x,n}) \leq (\psi(x) + tb)e^{ct}. \tag{A9}$$

To this end, let us observe that $G_n\psi_m(x) \leq \omega(n)$ with some constant $\omega(n)$ (depending on $n$). As $\psi_m \in D(A)$, and hence

$$\mathbf{E}\psi_m(Z_t^{x,n}) \leq \psi_m(x) + \int \mathbf{E}(A + G_n)\psi_m(Z_s^{x,n})\, ds. \tag{A10}$$

it follows that $\mathbf{E}\psi_m(Z_t^{x,n})$ is uniformly bounded (for a given $n$). Consequently we can pass to the limit as $m \to \infty$ in (A10) (using monotone and dominated convergence theorems on the l.h.s. and r.h.s. of (A10) respectively) to obtain

$$\mathbf{E}\psi(Z_t^{x,n}) \leq \psi(x) + \int \mathbf{E}G_n\psi(Z_s^{x,n})\, ds \leq \psi(x) + \int_0^t \mathbf{E}(c + b\psi(Z_s^{x,n})\, ds,$$

which leads to (A9) by Gronwall's lemma. By a straightforward limiting argument one shows now that (A7) is a martingale for any $n$.

Hence, for any $T$ and with probability arbitrary close to one, the processes $\psi(Z_t^{x,n})$ are uniformly bounded for $t \in [0,T]$, and hence $A + G_n$ form a localizing sequence for $A + G$ such that Theorem 6.3 from Chapter 4 of [EK] can be applied to get the well-posedness of the martingale problem for $A + G$ and hence the existence and uniqueness of the corresponding Markov process. At last, passing to the limit as $n \to \infty$ in (A8),(A9) yields the same properties for $\psi(Z_t^x)$, and hence using the Doob maximal inequality for martingales yields (A7).

*4. Weak equations on dual semigroups.* Suppose $A$ is a generator of a strongly continuous contraction semigroup $e^{tA}$ on a Banach space $B$ with the (dense in $B$) domain $D(A)$. Let $D(\pi)$ be a subspace of the dual Banach space $B^\star$ and let $\pi : D(\pi) \mapsto B^\star$ be a (possibly non-linear) measurable mapping. We want to discuss the two basic notions of the generalized solutions to the formal equation

$$\dot{\mu}_t = A^\star\mu_t + \pi(\mu_t) \tag{A11}$$

in $B^\star$. One says that a mapping $t \mapsto \mu_t$ from $[0,T]$ to $B^\star$ is a weak solution of (A11) if

$$\frac{d}{dt}(g, \mu_t) = (Ag, \mu_t) + (g, \pi(\mu_t)) \tag{A12}$$

for all $g \in D(A)$, and a mild (respectively weakly mild) solution to (A11) if

$$\mu_t = e^{tA^\star}\mu_0 + \int_0^t e^{(t-s)A^\star}\pi(\mu_s)\,ds, \tag{A13}$$

where $e^{tA^\star}$ denotes of course the dual semigroup on $B^\star$ and the integral is well defined in the sense of Riemann or Lebesgue (respectively if the integral is understood in the weak sense, i.e. if

$$(g, \mu_t) = (g, e^{tA^\star}\mu_0) + \int_0^t (g, e^{(t-s)A^\star}\pi(\mu_s))\,ds \tag{A14}$$

holds for all $g \in B$).

*Remarks.* (i) By the weak topology on $B^\star$ we always mean the topology generated by the duality $(B, B^\star)$. More precisely, this topology is usually called the $\star$-weak topology. (ii) If $e^{tA}$ is a strongly continuous contraction semigroup on $B$, then $e^{tA^\star}$ is clearly a weakly (but not necessarily strongly) continuous contraction semigroup on $B^\star$.

**Lemma A4.** *(i) Let $t \mapsto \mu_t$ be a weakly continuous mapping from $[0, T]$ to $B^\star$ (with a given $\mu_0$) such that $\pi(\mu_t)$ is well defined and is also weakly continuous. Then $\mu_t$ is a weak solution to (A11) if and only if it is a weakly mild solution to (A11).*

*(ii) Suppose additionally that $e^{tA^\star}$ is a compact operator in the Banach space $B^\star$ for any $t > 0$ (i.e, it takes bounded sets to relatively compact sets and hence transforms weakly convergent sequences into strongly convergent ones), then if $\mu_t$ and $\pi(\mu_t)$ are both weakly continuous, then any weakly mild solution of (A11) is also a mild solution.*

*(iii) If additionally to the assumptions of (i) and (ii), there exists a closed subspace $L \subset B^\star$ such that $e^{tA^\star}(B^\star) \subset L$ for all $t > 0$ and $e^{tA^\star}$ is a strongly continuous semigroup on $L$, then any solution $\mu_t$ of weak and mild equation from (ii) with $\mu_0 \in L$ belongs to $L$ for all $t$ and is a norm continuous function of $t$.*

*Proof.* (i) In one direction the statement is straightforward. Namely, differentiating (A14) with a $g \in D(A)$ yields (A12), because

$$\frac{d}{dt}\int_0^t (e^{(t-s)A}g, \nu_s)\,ds = (g, \nu_t) + \int_0^t (Ae^{(t-s)A}g, \nu_s)\,ds \tag{A15}$$

for any weakly continuous $\nu_t$ and a $g \in D(A)$. Consequently, in order to show that (A12) implies (A14) one needs to show that if $t \mapsto \mu_t$ and $t \mapsto \nu_t$ are two weakly continuous mappings from $[0, T]$ to $B^\star$ such that

$$\frac{d}{dt}(g, \mu_t) = (Ag, \mu_t) + (g, \nu_t) \tag{A16}$$

for all $g \in D(A)$, then

$$\mu_t = e^{tA^\star}\mu_0 + \int_0^t e^{(t-s)A^\star}\nu_s\,ds \tag{A17}$$

in the weak sense. To show this we first observe that one shows as above (by means of (A15)) that (A17) implies (A16). Hence it remains to show that a solution $\mu_t$ to (A16) is unique for given $\mu_0$ and $\nu_t$. But to do this, it is clearly enough to show that a solution to the homogeneous equation

$$\frac{d}{dt}(g,\mu_t) = (Ag,\mu_t), \quad g \in D(A), \tag{A18}$$

is unique for a given $\mu_0$. To see this, one observes that if $t \mapsto \mu_t$ is a weakly continuous mapping from $[0,T]$ to $B^\star$ satisfying (A18) then for any $t \leq T$

$$\frac{d}{ds}(e^{(t-s)A}g,\mu_s) = 0, \quad g \in D(A), \quad s \in [0,t]$$

(the weak continuity of $\mu_t$ and the strong continuity of the semigroup $e^{tA}$ are essential when proving that this derivative is well defined). Consequently $(g,\mu_t) = (e^{tA}g,\mu_0) = (g, e^{tA^\star}\mu_0)$, which implies $\mu_t = e^{tA^\star}\mu_0$, because $D(A)$ is dense in $B$.

(ii) The function under the integral in (A13) is uniformly bounded as $\pi(\mu_t)$ is weakly continuous (and hence bounded) and $e^{tA^\star}$ is a contraction. Moreover, this function is a norm-continuous function of $s$ for $0 \leq s < t$ (as $e^{tA^\star}$ is a compact operator), which implies that the integral in (A13) is a well defined Riemann integral.

(iii) The Riemann integral in (A13) belongs to $L$, if $\mu_0 \in L$, as $L$ is closed and the function under the integral belongs to $L$ for $0 \leq s < t$. This implies that $\mu_t \in L$ for all $t$ and moreover that $\mu_t$ is norm continuous, because

$$\mu_{t+\tau} - \mu_t = \int_t^{t+\tau} e^{(t+\tau-s)A^\star}\pi(\mu_s)\,ds + (e^{\tau A^\star} - 1)\int_0^t e^{(t-s)A^\star}\pi(\mu_s)\,ds,$$

and $e^{tA^\star}$ is strongly continuous on $L$.

5. *Boltzmann collision kernels for k-nary interactions.* Here we are interested in mappings that take any collection $\mathbf{v} = \{v_1,...,v_l\}$ of $l$ vectors from $\mathbf{R}^d$ to another collection $\mathbf{w} = \{w_1,...,w_l\}$ of $l$ vectors from $\mathbf{R}^d$ in such a way that the total momentum and kinetic energy are preserved, i.e.

$$v_1 + ... + v_l = w_1 + ... + w_l, \quad |v_1|^2 + ... + |v_l|^2 = |w_1|^2 + ... + |w_l|^2. \tag{A19}$$

Let a $d(l-1)$-dimensional plane in $\mathbf{R}^{dl}$ be defined by

$$\Gamma = \{\mathbf{u} = (u_1,...,u_l) \in \mathbf{R}^{dl} : u_1 + ... + u_l = 0\},$$

and let

$$S_\Gamma^{d(l-1)-1} = \{\mathbf{n} \in \Gamma : \|\mathbf{n}\| = |n_1|^2 + ... + |n_l|^2 = 1\}.$$

One sees by inspection that in terms of $\mathbf{u} = \{u_1,...,u_l\}$ defined by $\mathbf{w} = \mathbf{u} + \mathbf{v}$ conditions (A19) mean that $u \in \Gamma$ and

$$\|\mathbf{u}\|^2 = \sum_{j=1}^l u_j^2 = 2\sum_{j=1}^l (w_j,u_j) = 2(\mathbf{w},\mathbf{u}) = -2\sum_{j=1}^l (v_j,u_j) = -2(\mathbf{v},\mathbf{u}),$$

or equivalently that $\mathbf{u} = \|\mathbf{u}\|\mathbf{n}$, $\mathbf{n} \in S_\Gamma^{d(l-1)-1}$ and

$$\|\mathbf{u}\| = 2(\mathbf{w}, \mathbf{n}) = -2(\mathbf{v}, \mathbf{n}). \qquad (A20)$$

In particular $(\mathbf{v}, \mathbf{n}) \leq 0$ and denoting

$$S_{\Gamma,\mathbf{v}}^{d(l-1)-1} = \{\mathbf{n} \in S_\Gamma^{d(l-1)-1} : (\mathbf{n}, \mathbf{v}) \leq 0\},$$

we conclude that for any transition kernel $P(\mathbf{v}; d\mathbf{w})$ from $(\mathbf{R}^d)^l$ to itself that preserves the total momentum and kinetic energy, is invariant under shifts in $\mathbf{R}^d$ and is symmetric with respect to the permutations of $\mathbf{v}$ one has

$$\int_{\mathbf{R}^{dl}} g(\mathbf{w}) P(\mathbf{v}; d\mathbf{w}) = \int_{S_{\Gamma,\mathbf{v}}^{d(l-1)-1}} g(\mathbf{v} - 2(\mathbf{v}, \mathbf{n})\mathbf{n}) B(\{v_i - v_j\}_{i,j=1}^l; d\mathbf{n}), \qquad (A21)$$

where $B(\{v_i - v_j\}_{i,j=1}^l; .)$ is a measure on $S_{\Gamma,\mathbf{v}}^{d(l-1)-1}$ that is invariant with respect to all permutations of $\{v_1, ..., v_l\}$. Consequently, the $k$-nary extension of the spatially homogeneous Boltzmann equation in the weak form (equation (1.19) with $P$ as above and without spatial motion) has the form

$$\frac{d}{dt}(g, \mu_t) = \sum_{l=2}^k \frac{1}{l!} \int_{S_{\Gamma,\mathbf{v}}^{d(l-1)-1}} \int_{\mathbf{R}^{dl}}$$

$$\times (g^+(\mathbf{v} - 2(\mathbf{v}, \mathbf{n})\mathbf{n}) - g^+(\mathbf{v})) B_l(\{v_i - v_j\}_{i,j=1}^l; d\mathbf{n}) \mu_t(dv_1) ... \mu_t(dv_l). \qquad (A22)$$

To write this equation in a more familiar form we have to reduce our attention to the solutions $\mu_t$ being absolutely continuous with respect to Lebesgue measures on $\mathbf{R}^d$, i.e. having the form $\mu_t(v)\, dv$. For simplicity, we shall assume also that the measures $B_l$ are absolutely continuous with respect to Lebesgue measure on $S_{\Gamma,\mathbf{v}}^{d(l-1)-1}$. In this case, one can rewrite (A22) (using also the symmetry) as

$$\frac{d}{dt}(g, \mu_t) = \sum_{l=2}^k \frac{1}{(l-1)!} \int_{S_{\Gamma,\mathbf{v}}^{d(l-1)-1}} d\mathbf{n} \int_{\mathbf{R}^{dl}}$$

$$\times (g(v_1 - 2(\mathbf{v}, \mathbf{n})n_1) - g(v_1)) B_l(\{v_i - v_j\}_{i,j=1}^l, \mathbf{n}) \mu_t(v_1) ... \mu_t(v_l)\, dv_1 ... dv_l.$$

Changing the variables of integration, the r.h.s. of this equation can be written also as

$$\sum_{l=2}^k \frac{1}{(l-1)!} \int_{S_{\Gamma,-\mathbf{w}}^{d(l-1)-1}} d\mathbf{n} \int_{\mathbf{R}^{dl}} g(w_1) \tilde{B}_l(\{w_i - w_j\}_{i,j=1}^l, \mathbf{n}) \prod_{i=1}^l \mu_t(w_i - 2(\mathbf{w}, \mathbf{n})n_i)\, dw_i$$

$$- \sum_{l=2}^k \frac{1}{(l-1)!} \int_{S_{\Gamma,\mathbf{v}}^{d(l-1)-1}} d\mathbf{n} \int_{\mathbf{R}^{dl}} g(v_1) B_l(\{v_i - v_j\}_{i,j=1}^l, \mathbf{n}) \prod_{i=1}^l \mu_t(v_i)\, dv_i, \qquad (A23)$$

where we used that $|\det(\frac{\partial w}{\partial v})| = 1$ (as $\mathbf{v} \mapsto \mathbf{w}$ is an orthogonal transformation) and where

$$\tilde{B}_l(\{w_i - w_j\}_{i,j=1}^l, \mathbf{n}) = B_l(\{w_i - w_j - 2(\mathbf{w}, \mathbf{n})(n_i - n_j)\}_{i,j=1}^l, \mathbf{n}).$$

Notice that $(\mathbf{w}, \mathbf{n})$ depends only on the projection of $\mathbf{w}$ on $\Gamma$, i.e. on the collection of differences $\{w_i - w_j\}$. It is convenient now to make a different parametrization of $B$. Namely, let us denote by $S^{d(l-1)-1}$ the unit sphere in $\mathbf{R}^{d(l-1)}$ and for a $\mathbf{u} = \{u_2, ..., u_l\} \in \mathbf{R}^{d(l-1)}$ let $\bar{u} = u_2 + ... + u_l \in \mathbf{R}^d$ and

$$S_{\mathbf{u}}^{d(l-1)-1} = \{\mathbf{e} = \{e_2, ..., e_l\} \in S^{d(l-1)-1} : (\mathbf{e}, \mathbf{u}) = \sum_{j=2}^{l} (e_j, u_j) \leq 0\}.$$

Then each $\mathbf{n}$ from $S_\Gamma^{d(l-1)-1}$ or $S_{\Gamma,\mathbf{v}}^{d(l-1)-1}$can be written as

$$\mathbf{n} = (1 + |\bar{e}|)^{-1/2}\{-\bar{e}, \mathbf{e}\}$$

with $\mathbf{e} \in S^{d(l-1)-1}$ or $\mathbf{e} \in S_{\{v_2 - v_1, ..., v_l - v_1\}}^{d(l-1)-1}$ respectively. Then the measures $B_l$ and $\tilde{B}_l$ can be written respectively as $\sigma_l(v_2 - v_1, ..., v_l - v_1; \mathbf{e})\, d\mathbf{e}$ and $\tilde{\sigma}_l(w_2 - w_1, ..., w_l - w_1; \mathbf{e})\, d\mathbf{e}$, e.g.

$$B_l(\{v_i - v_j\}_{i,j=1}^l, \mathbf{n}) d\mathbf{n} = \sigma_l(v_2 - v_1, ..., v_l - v_1; \mathbf{e}) d\mathbf{e}.$$

Using also the natural notation $\mathbf{u} - z = \{u_2 - z, ..., u_l - z\}$ for $\mathbf{u} \in \mathbf{R}^{d(l-1)}$ and $z \in \mathbf{R}^d$ one deduces from equation (A23) that

$$\dot{\mu}_t(z) = \sum_{l=2}^{k} \frac{1}{(l-1)!} \int_{S_{z-\mathbf{w}}^{d(l-1)-1}} d\mathbf{e} \int_{\mathbf{R}^{dl}} \tilde{\sigma}_l(\mathbf{u} - z, \mathbf{e})$$

$$\mu_t\left(z + \frac{2(\mathbf{u} - z, \mathbf{e})}{1 + |\bar{e}|^2}\bar{e}\right) \prod_{i=2}^{l} \mu_t\left(u_i - \frac{2(\mathbf{u} - z, \mathbf{e})}{1 + |\bar{e}|^2} e_i\right) du_i$$

$$-\mu_t(z) \sum_{l=2}^{k} \frac{1}{(l-1)!} \int_{S_{\mathbf{u}-z}^{d(l-1)-1}} d\mathbf{e} \int_{\mathbf{R}^{d(l-1)}} \sigma_l(\mathbf{u} - z, \mathbf{e}) \prod_{i=2}^{l} \mu_t(u_i)\, du_i. \qquad (A24)$$

In case of binary interaction, i.e. when $k = 2$, one has $\mathbf{n} = \frac{1}{\sqrt{2}}(-e, e)$, $e \in \mathbf{R}^d$ and the transformation $\mathbf{v} \mapsto \mathbf{w}$ takes the form

$$w_1 = v_1 - (v_2 - v_1, e)e, \quad w_2 = v_2 + (v_2 - v_1, e)e.$$

If additionally $B(v_1 - v_2, \mathbf{n})$ depends only on $\|v_1 - v_2\|$ and $|(v_1 - v_2, e)|$, then $\tilde{B} = B$, $\tilde{\sigma} = \sigma$ and equation (A24) becomes the standard spatially homogeneous Boltzmann equation. In section 6 a spatially non-trivial version of equation (A24) is discussed.

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